THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2024-2025 Term 1
Suggested Solutions of WeBWork Coursework 5

If you find any errors or typos, please email us at
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(1) (1 point)
Differentiate the following function:

$$f(t) = \sqrt[4]{t} - \frac{1}{\sqrt[4]{t}}$$

$f'(t) =$ ________________

Solution:

$$\begin{aligned} f'(t) &= \left(t^{\frac{1}{4}}\right)' - \left(t^{-\frac{1}{4}}\right)' \\ &= \frac{1}{4}t^{-\frac{3}{4}} - \left(-\frac{1}{4}t^{-\frac{5}{4}}\right) \\ &= \frac{1}{4}t^{-\frac{3}{4}} + \frac{1}{4}t^{-\frac{5}{4}} \\ &= \frac{1}{4}\left(t^{-\frac{3}{4}} + t^{-\frac{5}{4}}\right). \end{aligned}$$

Correct Answers:

$$\frac{1}{4}\left(t^{-\frac{3}{4}} + t^{-\frac{5}{4}}\right)$$

(2) (1 point)
Calculate the derivative of the following function.

$$f(x) = \frac{e^x}{(e^x+2)(x+3)}$$

$f'(x) =$ ________________

Solution:
To compute $f'(x)$ we begin with quotient rule

$$f'(x) = \frac{[e^x]' \cdot (e^x+2)(x+3) - e^x \cdot [(e^x+2)(x+3)]'}{[(e^x+2)(x+3)]^2}.$$

Next, recall that $[e^x]' = e^x$, and use the product rule to compute

$$[(e^x+2)(x+3)]' = [e^x+2]' \cdot (x+3) + (e^x+2) \cdot [x+3]'$$

which equals

$$e^x \cdot (x+3) + (e^x+2) \cdot 1.$$

Therefore

$$f'(x) = \frac{e^x \cdot (e^x+2)(x+3) - e^x \cdot [e^x(x+3) + (e^x+2)]}{[(e^x+2)(x+3)]^2}$$

and after factoring out e^x in the numerator, expanding

$$(e^x+2)(x+3) = xe^x + 3e^x + 2x + 6,$$

and distributing the minus sign, we get

$$f'(x) = \frac{e^x(xe^x + 3e^x + 2x + 6 - xe^x - 3e^x - e^x - 2)}{[(e^x+2)(x+3)]^2}$$

which simplifies to

$$f'(x) = \frac{e^x(2x - e^x + 4)}{[(e^x+2)(x+3)]^2}.$$

Correct Answers:

$$\frac{e^x(2x - e^x + 4)}{[(e^x+2)(x+3)]^2}$$

(3) (1 point)

Differentiate $g(x) = \ln\left(\frac{7-x}{7+x}\right)$.

Solution:

$$\begin{aligned} g'(x) &= \left(\frac{7-x}{7+x}\right)^{-1} \cdot \left(\frac{7-x}{7+x}\right)' \\ &= \frac{7+x}{7-x} \cdot \frac{(-1)\cdot(7+x) - (7-x)\cdot 1}{(7+x)^2} \\ &= \frac{1}{7-x} \cdot \left(\frac{-14}{7+x}\right) \\ &= \frac{14}{x^2-49}. \end{aligned}$$

Correct Answers:

$$\frac{14}{x^2-49}$$

(4) (1 point)

Let $f(x) = |x|\ln(2-x)$. Find $f'(x)$.

$$f'(x) = \begin{cases} \\ \\ \end{cases}$$

________ if $x < c$
________ if $x = c$
________ if $c < x < d$

where $c =$ ___ and $d =$ ___.

Enter 'DNE' if the derivative does not exist.

Solution: One can find that $c = 0$, $d = 2$.

If $x < 0$,

$$f(x) = -x\ln(2-x),$$

so

$$f'(x) = -\ln(2-x) + (-x)\cdot\left(\frac{-1}{2-x}\right) = -\ln(2-x) + \frac{x}{2-x}.$$

If $0 < x < 2$,

$$f(x) = x\ln(2-x),$$

so

$$f'(x) = \ln(2-x) - \frac{x}{2-x}.$$

But if $x = 0$, we must use the definition of $f'(0)$. Let's consider the left and right derivatives of f at $x = 0$.

$$\lim_{h\to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^+} \frac{h\ln(2-h) - 0}{h} = \lim_{h\to 0^+} \ln(2-h) = \ln 2.$$

$$\lim_{h\to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h\to 0^-} \frac{-h\ln(2-h) - 0}{h} = \lim_{h\to 0^-} -\ln(2-h) = -\ln 2.$$

Since $\ln 2 \neq -\ln 2$, the derivative doesn't exist at $x = 0$.

Correct Answers:
- $x/(2-x) - \ln(2-x)$
- DNE
- $\ln(2-x) - x/(2-x)$
- 0
- 2

(5) (1 point)

Find $\dfrac{dy}{dx}$ if

$$5x^3y^2 - 4x^2y = 3.$$

Express your answer in terms of x, y if necessary.

$\dfrac{dy}{dx} =$ ____________

Solution: Taking the derivative with respect to x we get

$$0 = 15x^2y^2 + 10x^3y\frac{dy}{dx} - 8xy - 4x^2\frac{dy}{dx},$$

so

$$8xy - 15x^2y^2 = \left(10x^3y - 4x^2\right)\frac{dy}{dx}.$$

Therefore,

$$\frac{dy}{dx} = \frac{8xy - 15x^2y^2}{10x^3y - 4x^2}.$$

Correct Answers:

$$\frac{dy}{dx} = \frac{8xy - 15x^2y^2}{10x^3y - 4x^2}$$

(6) (1 point)

Consider the following function: $y = x^{x^2}$.

$\dfrac{dy}{dx} =$ ________________

(you will lose 25% of your points if you request a hint.)

Solution:

After taking the log of both sides, you should get: $\ln y = x^2 \ln x$.

Taking the derivative: $\dfrac{1}{y}\dfrac{dy}{dx} = 2x\ln x + x$.

Therefore, $\dfrac{dy}{dx} = y(2x\ln x + x) = x^{x^2+1}(2\ln x + 1)$.

Correct Answers:

$$\frac{dy}{dx} = x^{x^2+1}(2\ln x + 1)$$

(7) (1 point)

Let $f(x) = \dfrac{5x^2}{(3-4x)^3}$.

Find the equation of the line tangent to the graph of f at $x = 1$.

Tangent line: $y =$ ____________________

Solution: Differentiating gives

$$f'(x) = \frac{10x(3-4x)^3 - 5x^2 \cdot 3(3-4x)^2(-4)}{(3-4x)^6} = \frac{10x(3-4x) - 5x^2 \cdot 3 \cdot (-4)}{(3-4x)^4} = \frac{20x^2+30x}{(3-4x)^4}.$$

And hence the slope of the tangent line of the graph at $x = 1$ is $f'(1) = 50$. Since $f(1) = -5$ and the point $(1, 5)$ is also on this line, we know the tangent line $y - (-5) = 50(x-1)$, that is, $y = 50x - 55$.

Correct Answers:

$$y = 50x - 55$$

(8) (1 point)

If the equation of motion of a particle is given by $s(t) = A\cos(wt + d)$, the particle is said to undergo *simple harmonic motion* . Assume $0 \le d < \pi$.

(a) Find the velocity of the particle at time t.
(b) What is the smallest positive value of t for which the velocity is 0? Assume that w and d are positive.

(a) $v(t) =$ ____________
(b) $t =$ ____________

Solution:

(a) Differentiating with respect to t gives: $v(t) = s'(t) = -Aw\sin(wt + d)$.
(b) By (a), $v(t) = 0$ implies $\sin(wt + d) = 0$, then $wt + d = n\pi$, where n is any integer. So $t = \dfrac{n\pi - d}{w}$. Since $0 \le d < \pi$ and $w > 0$, the smallest positive vale of t is given by taking $n = 1$ and we get

$$t = \frac{\pi - d}{w}.$$

Correct Answers:

$$-Aw\sin(wt + d)$$

$$(\pi - d)/w$$

(9) (1 point)

A parabola is defined by the equation

$$x^2 - 2xy + y^2 - 2x - 2y + 13 = 0$$

The parabola has horizontal tangent lines at the **point(s)** ______.

The parabola has vertical tangent lines at the **point(s)** ______.

Solution: Differentiating implicitly with respect to x gives

$$2x + (-2y - 2x\,\frac{dy}{dx}) + 2y\,\frac{dy}{dx} - 2 - 2\,\frac{dy}{dx} = 0,$$

so

$$(y - x - 1)\,\frac{dy}{dx} = y - x + 1,$$

and so

$$\frac{dy}{dx} = \frac{y - x + 1}{y - x - 1}.$$

The tangent line to the parabola is horizontal where $\frac{dy}{dx} = 0$, i.e., where $x - y = 1$. Observe that the equation of the parabola can be rewritten in the form

$$(x - y)^2 - 2(x - y) + 13 - 4y = 0,$$

and $x - y = 1$ gives $12 = 4y$, so $y = 3$, and $x = 4$. Hence, the tangent line to the parabola is horizontal at the point $(4, 3)$ and nowhere else.

The tangent line for the the parabola is vertical where

$$0 = \frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} = \frac{y - x - 1}{y - x + 1},$$

i.e., where $x - y = -1$. Together with the last displayed equation of the parabola, this gives $16 - 4y = 0$, so $y = 4$, and $x = 3$. Hence, the tangent line to the parabola is vertical at the point $(3, 4)$ and nowhere else.

Correct Answers:

- `(4,3)`
- `(3,4)`

(10) (1 point)

Let $f(x) = x^3 \tan^{-1}(6x)$

$f'(x) =$ ______

Solution: Using the product and chain rules, we see

$$f'(x) = 3x^2 \cdot \tan^{-1}(6x) + x^3 \cdot \frac{6}{1 + (6x)^2} = 3x^2 \tan^{-1}(6x) + \frac{6x^2}{1 + 36x^2}.$$

(Note: here $\tan^{-1}(6x)$ means that $\arctan(6x)$.)

Correct Answers:

$$3x^2 \tan^{-1}(6x) + \frac{6x^2}{1 + 36x^2}$$

(11) (1 point)

Suppose that

$$f(x) = \frac{3x^2}{\sqrt{2x^2 + 4}}.$$

Find $f'(x)$, and then evaluate f' at $x = 1$ and $x = -1$.

$f'(1) =$ ______

$f'(-1) =$ ______

Solution:

$$f'(x) = \frac{6x \cdot \sqrt{2x^2+4} - 3x^2 \cdot \frac{1}{2}(2x^2+4)^{-\frac{1}{2}}4x}{2x^2+4}$$
$$= \frac{6x(2x^2+4) - 6x^3}{(2x^2+4)^{\frac{3}{2}}}$$
$$= 6x \cdot \frac{x^2+4}{(2x^2+4)^{\frac{3}{2}}}$$

So $f'(1) = \frac{5}{\sqrt{6}} = 2.04124145231931508$ and $f'(-1) = -\frac{5}{\sqrt{6}} = -2.04124145231931508$.

Correct Answers:

- `2.0412414523193150`
- `-2.0412414523193150`

(12) (1 point)

The equation of the tangent line to the graph of $y = x\cos(3x)$ at $x = \pi$ is given by $y = mx + b$ for

$m =$ ____
and
$b =$ ____

Solution: Differentiating gives

$$\frac{dy}{dx} = \cos(3x) + x \cdot [-\sin(3x)3] = \cos(3x) - 3x\sin(3x).$$

And hence the slope of the tangent line of the graph at $x = \pi$ is

$$m = \cos(3\pi) - 3\pi\sin(3\pi) = -1.$$

Since when $x = \pi$, $y = \pi\cos(3\pi) = -\pi$, the point $(\pi, -\pi)$ is also on this line. Hence we know that $-\pi = y = mx + b = (-1)\pi + b$, so $b = 0$.

Correct Answers:

- `-1`
- `0`