THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2024-2025 Term 1
Suggested Solutions of WeBWork Coursework 4

If you find any typos or errors, please send an email to math1010@math.cuhk.edu.hk

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1. (1 point) Let

$$f(x) = \begin{cases} -5x, & x < 3, \\ 1, & x = 3, \\ 5x, & x > 3. \end{cases}$$

Find the indicated one-sided limits of f, and determine the continuity of f at the indicated point. You should also sketch a graph of $y = f(x)$, including hollow and solid circles in the appropriate places.

NOTE: Type DNE if a limit does not exist.

$\lim_{x\to 3^-} f(x) =$ ____________

$\lim_{x\to 3^+} f(x) =$ ____________

$\lim_{x\to 3} f(x) =$ ____________

$f(3) =$ ____________

Is f continuous at $x = 3$? (YES/NO) __________

Solution:

- -15

  $\lim_{x\to 3^-} f(x) = \lim_{x\to 3^-} (-5x) = -5 \cdot 3 = -15.$

- 15

  $\lim_{x\to 3^+} f(x) = \lim_{x\to 3^+} (5x) = 5 \cdot 3 = 15.$

- DNE

  $\lim_{x\to 3^-} f(x) \neq \lim_{x\to 3^+} f(x)$, So the limit does not exist.

- 1

  According to the definition of $f(x)$, we have $f(3) = 1$.

- No

  f is not continuous at $x = 3$ for the limit $\lim_{x\to 3} f(x)$ does not exist.

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2. (1 point) Let $f(x) = |x-2|$. Evaluate the following limits.

$$\lim_{x\to 2^-} \frac{f(x)-f(2)}{x-2} = \_\_\_\_$$

$$\lim_{x\to 2^+} \frac{f(x)-f(2)}{x-2} = \_\_\_\_$$

Thus the function $f(x)$ is not differentiable at 2.

Solution: Recall that $|x-2| = \begin{cases} -(x-2), & \text{if } x < 2 \\ x-2 & \text{if } x \geq 2 \end{cases}$

Hence

- -1

$$\lim_{x\to 2^-}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^-}\frac{-(x-2)-0}{x-2}=\lim_{x\to 2^-}(-1)=-1.$$

- 1

$$\lim_{x\to 2^+}\frac{f(x)-f(2)}{x-2}=\lim_{x\to 2^+}\frac{(x-2)-0}{x-2}=\lim_{x\to 2^-}(1)=1$$

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3. (1 point) Find $f'(x)$ and $f'(0)$, where:

$$f(x)=\begin{cases} x^2\sin\left(\frac{1}{x}\right) & x\neq 0\\ 0 & x=0\end{cases}$$

(a) Find the derivative of $f(x)$ for x not equal 0.
$f'(x) =$ ____________
(b) If the derivative does not exist enter DNE.
$f'(0) =$ ____________

Solution: Using the definition of the derivative we find that:

- $2x\sin(\frac{1}{x})-\cos(\frac{1}{x})$
  For $x\neq 0$, we have $f(x)=x^2\sin\left(\frac{1}{x}\right)$ and

$$f'(x)=\sin(\frac{1}{x})\frac{d}{dx}x^2+x^2\frac{d}{dx}\sin(\frac{1}{x})=2x\sin(\frac{1}{x})-\cos(\frac{1}{x}).$$

- 0

$$\begin{aligned}f'(0)&=\lim_{h\to 0}\frac{f(h)-f(0)}{h-0}\\&=\lim_{h\to 0}h^2\sin\left(\frac{1}{h}\right)\frac{1}{h}\\&=\lim_{h\to 0}h\sin(\frac{1}{h})\\&=0,\qquad\text{by the squeeze theorem.}\end{aligned}$$

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4. (1 point) Let

$$f(x)=\begin{cases}-9x^2+5x & \text{for } x<0,\\ 7x^2-2 & \text{for } x\geqslant 0.\end{cases}$$

According to the definition of the derivative, to compute $f'(0)$, we need to compute the left-hand limit

$\lim_{h\to 0^-}$ ____________, which is _____,

and the right-hand limit

$\lim_{h\to 0^+}$ ____________, which is _____.

We conclude that $f'(0)$ is _____.

Note: If a limit or derivative does not exist, and is not $\pm\infty$, enter 'DNE' as your answer. Enter 'inf' for ∞, '-inf' for $-\infty$.

Solution:

- $\frac{-9h^2+5h+2}{h}$; $\quad -\infty$

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h-0}=\lim_{h\to 0^-}\frac{(-9h^2+5h)-(-2)}{h}=-\infty$$

- $7h$; $\quad 0$

$$\lim_{h\to 0^+}\frac{f(h)-f(0)}{h-0}=\lim_{h\to 0^+}\frac{(7h^2-2)-(-2)}{h}=\lim_{h\to 0^+}\frac{7h^2}{h}=0$$

- DNE

  Since $\lim_{h\to 0^-}\frac{f(h)-f(0)}{h-0}\neq\lim_{h\to 0^+}\frac{f(h)-f(0)}{h-0}$, we know that $f'(0)$ does not exist.

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5. (1 point)

Evaluate the following limits. If needed, enter 'INF' for ∞ and '-INF for $-\infty$.

(a) $\lim_{x\to\infty}\frac{\sqrt{4+2x^2}}{2+6x}=$______

(b) $\lim_{x\to-\infty}\frac{\sqrt{4+2x^2}}{2+6x}=$ ______

Solution:

- 0.235702260395516

$$\lim_{x\to\infty}\frac{\sqrt{4+2x^2}}{2+6x}=\lim_{x\to\infty}\frac{\sqrt{\frac{4}{x^2}+2}}{\frac{2}{x}+6}=\frac{\sqrt{2}}{6}$$

- -0.235702260395516

$$\lim_{x\to-\infty}\frac{\sqrt{4+2x^2}}{2+6x}=\lim_{x\to-\infty}\frac{\sqrt{\frac{4}{x^2}+2}}{-\frac{2}{x}-6}=-\frac{\sqrt{2}}{6}$$

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6. (1 point) Find a and b so that the function

$$f(x)=\begin{cases}6x^3-2x^2+2, & x<-2,\\ ax+b, & x\geq -2\end{cases}$$

is both continuous and differentiable.

$a=$ __________

$b=$ __________

Solution:

- 80

  To make $f(x)$ continuous, we must have $\lim_{x\to-2^-}f(x)=\lim_{x\to-2^+}f(x)$.

  Hence, $6\cdot(-2)^3-2\cdot(-2)^2+2=-2a+b$, $\quad$ i.e. $-54=-2a+b$.

  And we also have $f(-2)=-54=-2a+b$.

  To make $f(x)$ differentiable, we must have $Lf'(-2)=Rf'(-2)$, $\quad$ i.e.

$$\lim_{x\to-2^-}\frac{f(x)-f(-2)}{x-(-2)}=\lim_{x\to-2^+}\frac{f(x)-f(-2)}{x-(-2)}$$

  that is

$$\lim_{x\to -2^-} \frac{\left(6x^3-2x^2+2\right)-(-54)}{x-(-2)} = \lim_{x\to -2^+} \frac{(ax+b)-(-2a+b)}{x-(-2)}$$

which is reduced to

$$\lim_{x\to -2^-} \frac{(x+2)(6x^2-14x+28)}{x+2} = \lim_{x\to -2^+} a.$$

Hence $a = 6(-2)^2 - 14(-2) + 28 = 80$.

- 106
  $-54 = -2(80) + b \Rightarrow b = 106$

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7. (1 point) Suppose $f'(x)$ exists for all x in (a,b).

Mark all true items with a check. There may be more than one correct answer.

- A. $f(x)$ is continuous on (a,b).
- B. $f(x)$ is continuous at $x = a$.
- C. $f(x)$ is defined for all x in (a,b).
- D. $f'(x)$ is differentiable on (a,b).

Solution:

- AC
  A is true. The continuity of $f(x)$ can be deduced from the fact that $f'(x)$ exists for all x in (a, b).
  B is not always be true. Since (a,b) is an open interval, and $f'(x)$ is defined locally in this interval, we can't know the value of $f(x)$ at the endpoint x = a.
  C is true.
  D is false. Maybe $f'(x)$ is not continuous.

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8. (1 point) If $f'(a)$ exists, then $\lim\limits_{x\to a} f(x)$

- A. must exist, but there is not enough information to determine its value.
- B. is equal to $f(a)$.
- C. is equal to $f'(a)$.
- D. might not exist.
- E. does not exist.

Solution:

- B
  A is incorrect. Since $f'(a)$ exists, then we deduce that $f(x)$ is continuous at $x = a$, hence $\lim_{x\to a} f(x)$ is equal to $f(a)$.

B is correct. Since $f'(a)$ exists, then we deduce that $f(x)$ is continuous at $x = a$, hence $\lim_{x\to a} f(x)$ is equal to $f(a)$.

C is incorrect. The reason is as same as before.

D is incorrect. The reason is as same as before.

E is incorrect. The reason is as same as before.

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9. (1 point) Give the interval(s) on which the function is continuous.

$$h(k) = \sqrt{9-k} + \sqrt{k+5}$$

Solution:

- $[-5, 9]$

  Since the square root function is continuous on its domain, we just need to calculate the domain of this function. That is, $\begin{cases} 9-k \geq 0 \\ k+5 \geq 0 \end{cases}$

  The first radical implies $9-k \geq 0 \ \Rightarrow \ k \leq 9$.

  The second radical implies $k+5 \geq 0 \ \Rightarrow \ k \geq -5$.

  Together, these two restrictions imply $-5 \leq k \leq 9$.

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10. (1 point) Shown below are six statements about functions. Match each statement to one of the functions shown below which BEST matches that statement.

___1. $\lim\limits_{x\to 8^+} f(x)$ and $\lim\limits_{x\to 8^-} f(x)$ both exist and are finite, but they are not equal.
___2. The graph of $y = f(x)$ has vertical tangent line at $(8, f(8))$
___3. $\lim\limits_{x\to 8^-} f(x) = -\infty$.
___4. $\lim\limits_{x\to 8^+} f(x)$ exists but $\lim_{x\to 8^-} f(x)$ does not.
___5. $\lim\limits_{x\to 8} f(x) = \infty$.
___6. $\lim\limits_{x\to 8} f(x)$ exists but f is not continuous at 8.

A. $f(x) = \begin{cases} 4x & \text{if } x < 8 \\ 0 & \text{if } x = 8 \\ 4x-64 & \text{if } x > 8 \end{cases}$

B. $f(x) = \sqrt[3]{x-8}$

C. $f(x) = \frac{1}{x-8}$

D. $f(x) = \begin{cases} 4x & \text{if } x < 8 \\ 0 & \text{if } x = 8 \\ 64-4x & \text{if } x > 8 \end{cases}$

E. $f(x) = \begin{cases} \cos\left(\frac{1}{x-8}\right) & \text{if } x < 8 \\ 0 & \text{if } x = 8 \\ 4x+64 & \text{if } x > 8 \end{cases}$

F. $f(x) = \frac{1}{(x-8)^2}$

Solution:

- A

  $\lim\limits_{x\to 8^+} f(x) = 4\cdot 8 = 32$

  $\lim\limits_{x\to 8^-} f(x) = 4\cdot 8 - 64 = -32$

- B

  Since the derivative of $f(x)$ is $\frac{1}{3}(x-8)^{-\frac{2}{3}}$, and $f'(8) = \infty$, the graph of $y = f(x)$ has vertical tangent line at $(8, f(8))$.

- C

  $\lim_{x\to 8^-} f(x) = -\infty$ and $\lim_{x\to 8^+} f(x) = +\infty$
- E

  $\lim_{x\to 8^-} f(x)$ doesn't exist, $\lim_{x\to 8^+} f(x) = 32+64 = 96$, hence $\lim_{x\to 8^+} f(x)$ exists but $\lim_{x\to 8^-} f(x)$ doesn't.
- F

  $\lim_{x\to 8^-} f(x) = \infty$, $\lim_{x\to 8^+} f(x) = \infty$, hence $\lim_{x\to 8} f(x) = \infty$.
- D

  $\lim_{x\to 8^-} f(x) = 32$, $\lim_{x\to 8^+} f(x) = 32$, hence $\lim_{x\to 8} f(x)$ exists but $f(8) = 0 \neq 32$, so f is not continuous at 8 .

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11. (1 point)
Why is the following function discontinuous at $x = 0$?

$$f(x) = \begin{cases} e^x & \text{if } x < 0 \\ \\ x^2 & \text{if } x \geq 0 \end{cases}$$

(a) $f(0)$ does not exist.
(b) $\lim_{x\to 0} f(x)$ does not exist (or is infinite).
(c) Both (a) and (b).
(d) $f(0)$ and $\lim_{x\to 0} f(x)$ exist, but they are not equal.

Solution:

- (b)

  (a) is incorrect. Since $f(0) = 0^2 = 0$;
  (b) is correct. The function is discontinuous at $x = 0$ because $\lim_{x\to 0^+} f(x) = 0^2 = 0$ is not equal to $\lim_{x\to 0^-} f(x) = e^0 = 1$.
  (c) is incorrect.
  (d) is incorrect. This is because $\lim_{x\to 0} f(x)$ does not exist .