THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2024-2025 Term 1
Suggested Solutions of WeBWork Coursework 3

If you find any typos or errors, please send an email to math1010@math.cuhk.edu.hk

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1. (1 point) Evaluate the limit

$$\lim_{x\to 6}\left(\sqrt{x^2+5}-\frac{x^2+6x}{x}\right)$$

If the limit does not exist enter DNE.
Limit $=\sqrt{41}-12$

Solution:

$$\begin{aligned}\lim_{x\to 6}\left(\sqrt{x^2+5}-\frac{x^2+6x}{x}\right) &= \lim_{x\to 6}\sqrt{x^2+5}-\lim_{x\to 6}\frac{x^2+6x}{x}\\ &= \sqrt{41}-12\end{aligned}$$

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2. (1 point)

Let $f(x)=\begin{cases}\sqrt{-1-x}+5, & \text{if } x<-2\\ 5, & \text{if } x=-2\\ 2x+10, & \text{if } x>-2\end{cases}$

Calculate the following limits. Enter **DNE** if the limit does not exist.

$\lim_{x\to -2^-} f(x) = 6$

$\lim_{x\to -2^+} f(x) = 6$

$\lim_{x\to -2} f(x) = 6$

Solution:

(a)

$$\begin{aligned}\lim_{x\to -2^-} f(x) &= \lim_{x\to -2^-}(\sqrt{-1-x}+5)\\ &= 6\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to -2^+} f(x) &= \lim_{x\to -2^+}(2x+10)\\ &= 6\end{aligned}$$

(c)

Because $\lim_{x\to -2^+} f(x) = \lim_{x\to -2^-} f(x)$, $\lim_{x\to -2} f(x)$ exists.

$$\lim_{x\to -2} f(x) = \lim_{x\to -2^+} f(x) = \lim_{x\to -2^-} f(x) = 6.$$

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3. (1 point) Evaluate the limits.

$$g(x) = \begin{cases} 6x+2 & x<2 \\ 12 & x=2 \\ 6x-2 & x>2 \end{cases}$$

Enter **DNE** if the limit does not exist.
a) $\lim_{x\to 2^-} g(x) = 14$
b) $\lim_{x\to 2^+} g(x) = 10$
c) $\lim_{x\to 2} g(x) = \text{DNE}$
d) $g(2) = 12$

Solution:

(a)

$$\begin{aligned}\lim_{x\to 2^-} g(x) &= \lim_{x\to 2^-} (6x+2) \\ &= 14\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 2^+} g(x) &= \lim_{x\to 2^+} (6x-2) \\ &= 10\end{aligned}$$

(c)

$$\lim_{x\to 2^+} g(x) \neq \lim_{x\to 2^-} g(x)$$

So the limit does not exist.
(d)

$$g(2) = 12$$

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4. (1 point) Evaluate the limit

$$\lim_{x\to 1^-} \left(\frac{1}{x-1} - \frac{1}{|x-1|} \right)$$

Enter **INF** for ∞, **-INF** for $-\infty$, or **DNE** if the limit does not exist (i.e., there is no finite limit and neither ∞ nor $-\infty$ is the limit).

Limit $= -\infty$

Solution:

$$\begin{aligned}\lim_{x\to 1^-}\left(\frac{1}{x-1}-\frac{1}{|x-1|}\right) &= \lim_{x\to 1^-}\left(\frac{1}{x-1}-\frac{1}{1-x}\right)\\ &= \lim_{x\to 1^-}\left(\frac{2}{x-1}\right)\\ &= -\infty\end{aligned}$$

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5. (1 point) Use the Squeeze Theorem to evaluate the limit

$$\lim_{x\to 0}\sin x\cos\left(\frac{1}{x^2}\right)$$

Enter **DNE** if the limit does not exist.

Limit = 0

Solution:

Regardless of the value of $x \neq 0$,

$$-1 \leq \cos\left(\frac{1}{x^2}\right) \leq 1$$

Assume first that $x > 0$, and x is small enough so that $\sin x > 0$. Multiply the inequality by $\sin x$.

$$-\sin x \leq \sin x\cos\left(\frac{1}{x^2}\right) \leq \sin x$$

By the Squeeze Theorem, since $\lim_{x\to 0^+}(-\sin x) = \lim_{x\to 0^+}\sin x = 0$, we must have

$$\lim_{x\to 0^+}\sin x\cos\left(\frac{1}{x^2}\right) = 0.$$

The argument works in a similar way when $x < 0$ but close enough so that $\sin x < 0$.

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6. (1 point) Let

$$f(x) = \frac{x^2+8}{x^2-25}.$$

Find the indicated one-sided limits of f.

NOTE: Remember that you use **INF** for ∞ and **-INF** for $-\infty$.

You should also sketch a graph of $y = f(x)$, including vertical and horizontal asymptotes.

$\lim_{x\to -5^-} f(x) = \infty$

$\lim_{x\to -5^+} f(x) = -\infty$

$\lim_{x\to 5^-} f(x) = -\infty$

$\lim_{x\to 5^+} f(x) = \infty$

$\lim_{x\to -\infty} f(x) = 1$

$\lim_{x\to \infty} f(x) = 1$

Solution:

(a)

$$\lim_{x\to -5^-} f(x) = \lim_{x\to -5^-} \frac{x^2+8}{x^2-25} = \lim_{x\to -5^-} \frac{x^2+8}{(x-5)(x+5)}$$
$$= \infty$$

(b)

$$\lim_{x\to -5^+} f(x) = \lim_{x\to -5^+} \frac{x^2+8}{x^2-25} = \lim_{x\to -5^+} \frac{x^2+8}{(x-5)(x+5)}$$
$$= -\infty$$

(c)

$$\lim_{x\to 5^-} f(x) = \lim_{x\to 5^-} \frac{x^2+8}{x^2-25} = \lim_{x\to 5^-} \frac{x^2+8}{(x-5)(x+5)}$$
$$= -\infty$$

(d)

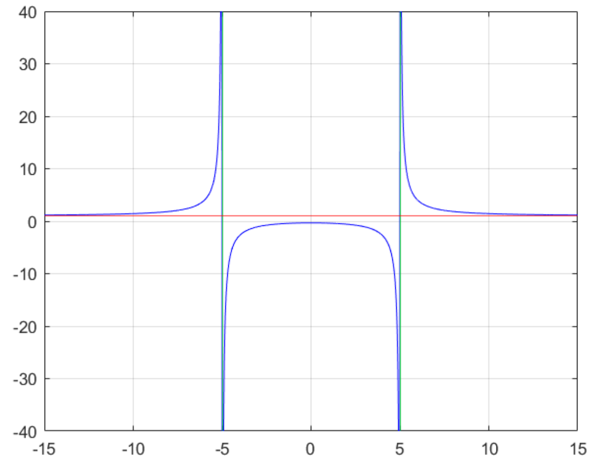
$$\lim_{x\to 5^+} f(x) = \lim_{x\to 5^+} \frac{x^2+8}{x^2-25} = \lim_{x\to 5^+} \frac{x^2+8}{(x-5)(x+5)}$$
$$= \infty$$

(e)

$$\lim_{x\to -\infty} f(x) = \lim_{x\to -\infty} \frac{x^2+8}{x^2-25} = \lim_{x\to -\infty} \frac{1+\frac{8}{x^2}}{1-\frac{25}{x^2}}$$
$$= 1$$

(f)

$$\lim_{x\to \infty} f(x) = \lim_{x\to \infty} \frac{x^2+8}{x^2-25} = \lim_{x\to \infty} \frac{1+\frac{8}{x^2}}{1-\frac{25}{x^2}}$$
$$= 1$$



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7. (1 point)

a. [choose/true/false] If $\lim_{x\to 1^-} f(x) = 5$, then $\lim_{x\to 1} f(x) = 5$.

b. [choose/true/false] If $\lim_{x\to 1^-} f(x) = 5$, then $\lim_{x\to 1^+} f(x) = 5$.

c. [choose/true/false] If $\lim_{x\to 1} f(x) = 5$, then $\lim_{x\to 1^-} f(x) = 5$.

d. [choose/true/false] If $\lim_{x\to 1} f(x) = 5$, then $\lim_{x\to 1^+} f(x) = 5$.

e. Select all true statements. Assume that all the limits are all taken at the same point.

- A. If the right-hand limit exists, then the two-sided limit exists.
- B. If the left- and right-hand limits both exist and are equal, then the two-sided limit exists.
- C. If the two-sided limit exists, then the left- and right-hand limits both exist and are equal.
- D. If the left-hand limit exists, then the two-sided limit exists.

Solution:
(a.) False
(b.) False
(c.) True
(d.) True
(e.) BC

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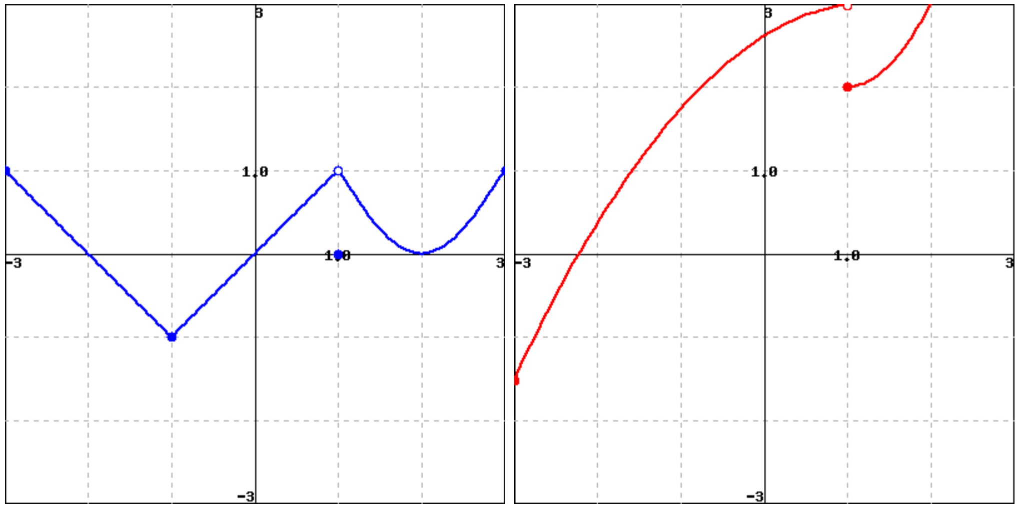
8. (1 point) Use the given graphs of the function f (left, in blue) and g (right, in red) to find the following limits:



1. $\lim\limits_{x\to 1}[f(x)+g(x)] = \text{DNE}$ help (limits)
2. $\lim\limits_{x\to 2}[f(x)+g(x)] = 3$
3. $\lim\limits_{x\to 0} f(x)g(x) = 0$
4. $\lim\limits_{x\to 0} \dfrac{f(x)}{g(x)} = 0$
5. $\lim\limits_{x\to -1} \sqrt{3+f(x)} = \sqrt{2}$

Note: *You can click on the graphs to enlarge the images.*

Solution:
1. When x appoarching 1 from two sides, left and right limit of $g(x)$ are not equal, however in this case limit of $f(x)$ exists, so limit of $f(x)+g(x)$ does not exist.

2. Limit exist this time for both $f(x)$ and $g(x)$, simply add up these two limits we get 3.

3. Again limit of $f(x)$ and $g(x)$ exist, as $\lim\limits_{x\to 0} f(x) = 0$, the answer should be 0.

4. Notice limit of $g(x)$ exists and it is a nonzero real number, so limit of $\frac{f(x)}{g(x)}$ is totally controlled by the behavior of $f(x)$ near 0. Since around 0 you can invert $g(x)$ locally, i.e. $\frac{f(x)}{g(x)}=\frac{1}{g(x)}f(x)$, similar to 8.3 we get 0.

5. Routinely check that $\lim\limits_{x\to -1} f(x)$ exists, so does $\lim\limits_{x\to -1} \sqrt{3+f(x)}$, since $\lim\limits_{x\to -1} f(x) = -1$, the answer should be $\sqrt{2}$.

9. (1 point)

Find $\lim_{x \to 0^+} \sqrt{x}\, e^{\sin(\pi/x)}$

0

Solution:

Although $e^{\sin \frac{\pi}{x}}$ looks awful, it's actually bounded by positive real numbers e^{-1} and e as $\sin \frac{\pi}{x}$ is bounded by -1 and 1. So the really effective part is $\sqrt{x}$, as $\lim_{x \to 0^+} \sqrt{x} = 0$ (Since square root only defined on non negative real number!), and $\sqrt{x}e^{-1} \leq \sqrt{x}e^{\sin \frac{\pi}{x}} \leq \sqrt{x}e$,the answer should be 0 by Squeeze Theorem.

10. (1 point)

Evaluate the limit:

$$\lim_{x \to 0} \frac{x^2}{\sin^2(3x)} = \frac{1}{9}$$

Solution:

$$\lim_{x \to 0} \frac{x^2}{\sin^2(3x)} = \lim_{x \to 0} \frac{1}{\frac{\sin(3x)}{x} \cdot \frac{\sin(3x)}{x}} = \lim_{x \to 0} \frac{1}{\frac{3\sin(3x)}{3x}} \cdot \lim_{x \to 0} \frac{1}{\frac{3\sin(3x)}{3x}} = \frac{1}{3} \cdot \frac{1}{3} = \frac{1}{9}.$$

11. (1 point)

Evaluate the limit:

$$\lim_{x \to 0} \frac{\tan 5x}{\tan 8x} = \frac{5}{8}$$

Solution:

$$\lim_{x \to 0} \frac{\tan 5x}{\tan 8x} = \lim_{x \to 0} \frac{\cos 8x}{\cos 5x} \cdot \frac{\sin 5x}{5x} \cdot \frac{5}{8} \cdot \frac{8x}{\sin 8x} = \frac{5}{8}.$$