THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 UNIVERSITY MATHEMATICS 2024-2025 Term 1
Suggested Solutions of WeBWork Coursework 1

1. (1 point)

In each part, find a formula for the general term of the sequence, starting with $n = 1$.

Enter the following information for $a_n =$.

(a)

$$\tfrac{1}{3}, \tfrac{1}{9}, \tfrac{1}{27}, \tfrac{1}{81}, \cdots$$

(b)

$$\tfrac{1}{3}, -\tfrac{1}{9}, \tfrac{1}{27}, -\tfrac{1}{81}, \cdots$$

(c)

$$\tfrac{2}{3}, \tfrac{8}{9}, \tfrac{26}{27}, \tfrac{80}{81}, \cdots$$

(d)

$$0, \frac{1}{\sqrt{\pi}}, \frac{4}{\sqrt[3]{\pi}}, \frac{9}{\sqrt[4]{\pi}}, \cdots$$

The general term of the sequence is $a_n = \dfrac{1}{3^n}$.

The general term of the sequence is $a_n = (-1)^{n+1}\dfrac{1}{3^n}$.

The general term of the sequence is $a_n = 1 - \dfrac{1}{3^n}$.

The general term of the sequence is $a_n = \dfrac{(n-1)^2}{\sqrt[n]{\pi}}$.

2. (1 point)

Determine whether the sequence $a_n = \dfrac{n^{19} + \sin(23n + 15)}{n^{23} + 15}$ converges or diverges. If it converges, find the limit.

Solution: Consider the sequence $b_n = \dfrac{n^{19} + 1}{n^{23} + 15}$ and $c_n = \dfrac{n^{19} - 1}{n^{23} + 15}$. Note that $c_n \leq a_n \leq b_n$ and $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = 0$. By Sandwich theorem, we have $\lim\limits_{n\to\infty} a_n = 0$.

3. (1 point) Use algebra to simplify the expression before evaluating the limit. In particular, factor the highest power of n from the numerator and denominator, then cancel as many factors of n as possible.

$\lim\limits_{n\to\infty} \dfrac{5n}{(4n^3+3)^{1/3}} = \lim\limits_{n\to\infty} \left(\dfrac{\_\_\_\_}{\_\_\_\_} \right) = \_\_\_\_\_\_.$

Solution:

$$\lim_{n\to\infty} \frac{5n}{(4n^3+3)^{1/3}} = \lim_{n\to\infty} \frac{5n}{n(4+3/n^3)^{1/3}} = \lim_{n\to\infty} \frac{5}{(4+3/n^3)^{1/3}} = \frac{5}{4^{1/3}}$$

4. (1 point)

Part 1: Evaluating a series

Consider the sequence $\{a_n\} = \left\{\frac{2}{n^2+2n}\right\}$.

a. The limit of this sequence is $\lim_{n\to\infty} a_n =$ -
b. The sum of all terms in this sequence is defined as the limit of the partial sums, which means

$$\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty}(\square) =$$

Enter infinity or -infinity if the limit diverges to ∞ or $-\infty$; otherwise, enter DNE if the limit does not exist.

Part 2: Evaluating another series

Consider the sequence $\{b_n\} = \left\{\ln\left(\frac{n+1}{n}\right)\right\}$.

a. The limit of this sequence is $\lim_{n\to\infty} b_n =$
b. The sum of all terms in this sequence is defined as the limit of the partial sums, which means

$$\sum_{n=1}^{\infty} b_n = \lim_{n\to\infty}(\square) =$$

Enter infinity or -infinity if the limit diverges to ∞ or $-\infty$; otherwise, enter DNE if the limit does not exist.

Part 3: Developing conceptual understanding

Suppose $\{c_n\}$ is a sequence.

a. If $\lim_{n\to\infty} c_n = 0$, then the series $\sum_{n=1}^{\infty} c_n$
   - choose
   - must
   - may or may not
   - cannot

   converge. Hint: look back at parts 1 and 2.
b. If $\lim_{n\to\infty} c_n \neq 0$, then the series $\sum_{n=1}^{\infty} c_n$
   - choose
   - must
   - may or may not
   - cannot

   converge.
c. If the series $\sum_{n=1}^{\infty} c_n$ converges, then $\lim_{n\to\infty} c_n$
   - choose
   - must
   - may or may not
   - cannot

   be equal to 0.

Solution:

Part 1:

a. $\lim_{n\to\infty} a_n = 0$
b. $\sum_{n=1}^{\infty} a_n = \frac{3}{2}$

Explanation:

For part (a), we begin by analyzing the sequence $a_n = \frac{2}{n^2+2n}$. Factoring the denominator, we get:

$$a_n = \frac{2}{n(n+2)}$$

Dividing both the numerator and the denominator by n^2, we simplify it to:

$$a_n = \frac{2}{n^2(1+\frac{2}{n})}$$

As $n \to \infty$, the term $\frac{2}{n} \to 0$, which gives:

$$\lim_{n\to\infty} a_n = 0$$

For part (b), the sum of the series is the limit of the partial sums:

$$S_N = \sum_{n=1}^{N} \frac{2}{n^2+2n}$$

Using partial fraction decomposition, we write:

$$\frac{2}{n(n+2)} = \frac{1}{n} - \frac{1}{n+2}$$

Thus, the partial sum becomes:

$$S_N = \sum_{n=1}^{N} \left(\frac{1}{n} - \frac{1}{n+2}\right)$$

This is a telescoping series, and most terms cancel out, leaving:

$$S_N = 1 + \frac{1}{2} - \frac{1}{N+1} - \frac{1}{N+2}$$

As $N \to \infty$, the remaining terms $\frac{1}{N+1}$ and $\frac{1}{N+2}$ tend to zero, so:

$$\sum_{n=1}^{\infty} a_n = 1 + \frac{1}{2} = \frac{3}{2}$$

Part 2:

a. $\lim_{n\to\infty} b_n = 0$
b. $\sum_{n=1}^{\infty} b_n = \infty$

Explanation:

For part (a), the sequence b_n is given by:

$$b_n = \ln\left(\frac{n+1}{n}\right)$$

We can simplify this as:

$$b_n = \ln\left(1 + \frac{1}{n}\right)$$

As $n \to \infty$, the term $\frac{1}{n} \to 0$, so:

$$\lim_{n\to\infty} b_n = \ln(1) = 0$$

For part (b), the sum of the series is the limit of the partial sums:

$$S_N = \sum_{n=1}^{N} \ln\left(\frac{n+1}{n}\right)$$

Using the properties of logarithms, we can express the sum as:

$$S_N = \ln\left(\frac{2}{1} \times \frac{3}{2} \times \cdots \times \frac{N+1}{N}\right)$$

This is a telescoping product, and after cancellation, we obtain:

$$S_N = \ln(N+1)$$

As $N \to \infty$, $\ln(N+1) \to \infty$, so:

$$\sum_{n=1}^{\infty} b_n = \infty$$

Part 3:

a. may or may not
b. cannot
c. must

Explanation. For **Part 1**, the sequence $\{a_n\} = \frac{2}{n^2+2n}$ has a limit of 0 as n approaches infinity. The series converges to 1 due to the telescoping nature of partial fractions.

For **Part 2**, the sequence $\{b_n\} = \ln\left(\frac{n+1}{n}\right)$ also approaches 0, but the series diverges to infinity as it represents the harmonic series in logarithmic form.

In **Part 3**, if $\lim_{n\to\infty} c_n = 0$, the series may or may not converge. If $\lim_{n\to\infty} c_n \neq 0$, the series cannot converge. If the series converges, $\lim_{n\to\infty} c_n$ must be 0.

□

5. (1 point) Consider the recursively defined sequence:

$$a_1 = 4$$
$$a_{n+1} = \frac{n+1}{n^2} a_n, \qquad \text{for } n \geq 1$$

The sequence is
- Eventually monotone increasing
- Eventually monotone decreasing
- Neither

The sequence is bounded below by _____

The sequence is bounded above by _____

The limit of the sequence is: _____
(If the sequence does not converge, enter "DNE")

Solution: The first few terms of the sequence are:

$a_1 = 4$,
$a_2 = (1+1)/1^2 \times 4 = 8$,
$a_3 = (2+1)/2^2 \times 8 = 6$,

$a_4 = (3+1)/3^2 \times 6 = 8/9$,
$a_5 = (4+1)/4^2 \times 8/9 = 5/18$,
...

The sequence is eventually monotonic: **decreasing**

Explanation: For $n \geq 3$, we have $(n+1)/n^2 < 1$, so $a_{n+1} < a_n$. Therefore, the sequence is decreasing starting from a_3.

The sequence is bounded below by **0**

Explanation: Since the sequence is decreasing for $n \geq 3$, all terms are greater than or equal to the limit value. We can prove that the limit value is 0 (see point 4), so the lower bound is 0.

The sequence is bounded above by **8**

Explanation: Since the sequence is decreasing starting from a_3, all terms are less than or equal to a_2, which is 8.

The limit of the sequence is: **0**

Explanation:

Consider the recursively defined sequence:

$$a_1 = 4$$
$$a_{n+1} = \frac{n+1}{n^2} a_n, \quad \text{for } n \geq 1$$

The sequence is **eventually monotone decreasing**. We analyze the recursive relation. The ratio between consecutive terms is:

$$\frac{a_{n+1}}{a_n} = \frac{n+1}{n^2}.$$

For large n,

$$\frac{n+1}{n^2} \to 0,$$

which suggests that the sequence will eventually decrease.

To confirm that the sequence is eventually monotone decreasing, we observe that:

$$\frac{n+1}{n^2} < 1 \quad \text{for all } n \geq 2.$$

Thus, for $n \geq 2$, we have $a_{n+1} < a_n$, implying that the sequence is eventually decreasing.

The sequence is bounded below by 0, since each term is positive:

$$a_n > 0 \quad \text{for all } n.$$

Furthermore, the second term $a_2 = 8$ is the maximum, so the sequence is bounded above by 8.

The sequence $\{a_n\}$ is eventually monotone decreasing and bounded below. By the **Monotone Convergence Theorem**, a bounded and monotone sequence must converge. Therefore, the sequence $\{a_n\}$ converges to some limit L.

Taking the limit in the recursive relation:

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \frac{n+1}{n^2} a_n.$$

Since $\frac{n+1}{n^2} \to 0$ as $n \to \infty$, it follows that:

$$L = 0.$$

Thus, the sequence converges to **0**.

6. (1 point) Consider the recursively defined sequence:

$$a_1 = 1, \quad a_2 = 1$$

$$a_{n+2} = \frac{a_{n+1} + a_n}{2}, \qquad \text{for } n \geq 1$$

The limit of the sequence is: ____
(If the sequence does not converge, enter "DNE")

OPTIONAL: Discuss the convergence of the sequence for different values of a_1 and a_2.

Solution: The limit of the sequence is: **1**

Explanation: We can calculate the first few terms of the sequence:

$a_1 = 1$, $a_2 = 1$, $a_3 = (a_2 + a_1)/2 = (1+1)/2 = 1$, $a_4 = (a_3 + a_2)/2 = (1+1)/2 = 1$.

We can observe that starting from a_3, all terms are equal to 1. Therefore, the sequence converges to 1.

OPTIONAL: Discuss the convergence of the sequence for different values of a_1 and a_2.

Explanation: Let's analyze the convergence of the sequence for different values of a_1 and a_2.

Case 1: $a_1 = a_2$

If $a_1 = a_2 = c$, then:

$$\begin{aligned}
a_3 &= (a_2 + a_1)/2 = (c + c)/2 = c, \\
a_4 &= (a_3 + a_2)/2 = (c + c)/2 = c, \\
&\vdots \\
a_n &= c \text{ for all } n \geq 1.
\end{aligned}$$

In this case, the sequence is constant and converges to c.

Case 2: $a_1 \neq a_2$

Let's define $d_n = a_{n+1} - a_n$. Then:

$$\begin{aligned}
d_1 &= a_2 - a_1, \\
d_2 &= a_3 - a_2 = (a_2 + a_1)/2 - a_2, \\
&= (a_1 - a_2)/2 = -d_1/2, \\
d_3 &= a_4 - a_3 = (a_3 + a_2)/2 - a_3, \\
&= (a_2 - a_3)/2 = -d_2/2 = d_1/4, \\
&\vdots \\
d_n &= (-1)^{n-1} \cdot d_1/2^{n-1}.
\end{aligned}$$

Now, $a_{n+1} = a_n + d_n$, so:

$$\begin{aligned}
a_1 &= a_1, \\
a_2 &= a_1 + d_1, \\
a_3 &= a_2 + d_2 = a_1 + d_1 - d_1/2, \\
&= a_1 + d_1/2, \\
a_4 &= a_3 + d_3 = a_1 + d_1/2 + d_1/4, \\
&= a_1 + 3d_1/4, \\
&\vdots \\
a_n &= a_1 + d_1(1 - 1/2 + 1/4 - 1/8 + \dots \\
&\quad + (-1)^{n-2}/2^{n-2}).
\end{aligned}$$

As $n \to \infty$, $(1 - 1/2 + 1/4 - 1/8 + \dots \quad + (-1)^{n-2}/2^{n-2}) \to \frac{1}{1-(-1/2)} = 2/3$,

$$\lim_{n\to\infty} a_n = a_1 + (2/3)d_1 = (1/3)a_1 + (2/3)a_2$$

7. (1 point)

Consider the sequence

$$a_n = \frac{n\cos(n\pi)}{2n-1}.$$

Write the first five terms of a_n, and find $\lim_{n\to\infty} a_n$. If the sequence diverges, enter "DNE" in the answer box for its limit.

a) First five terms: _____,_____,_____,_____,_____.

b) $\lim_{n\to\infty} a_n =$ ____________________.

Solution: The first five terms are

$$a_1 = -1, \quad a_2 = \frac{2}{3}, \quad a_3 = -\frac{3}{5}, \quad a_4 = \frac{4}{7}, \quad a_5 = -\frac{5}{9}$$

Note that

$$\lim_{n\to\infty} a_{2n} = \lim_{n\to\infty} \frac{2n\cos(2n\pi)}{4n-1} = \lim_{n\to\infty} \frac{1}{2 - 1/2n} = \frac{1}{2}$$

while

$$\lim_{n\to\infty} a_{2n+1} = \lim_{n\to\infty} \frac{(2n+1)\cos((2n+1)\pi)}{4n+1} = \lim_{n\to\infty} -\frac{1 + 1/2n}{2 + 1/2n} = -\frac{1}{2}$$

Since $\lim_{n\to\infty} a_{2n} \neq \lim_{n\to\infty} a_{2n+1}$, $\lim_{n\to\infty} a_n$ does not exist.

8. (1 point) The sequence $\{a_n\}$ is defined by $a_1 = 2$, and

$$a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right),$$

for $n \geqslant 1$. Assuming that $\{a_n\}$ converges, find its limit.

$\lim_{n\to\infty} a_n =$ ________.

Hint: Let $a = \lim_{n\to\infty} a_n$. Then, since $a_{n+1} = \frac{1}{2}(a_n + 2/a_n)$, we have $a = \frac{1}{2}(a + 2/a)$. Now solve for a.

Solution: Let $a = \lim_{n\to\infty} a_n$. Since $a_{n+1} = \frac{1}{2}\left(a_n + \frac{2}{a_n}\right)$, we have

$$\begin{aligned} a &= \frac{1}{2}\left(a + \frac{2}{a}\right) \\ 2a^2 &= a^2 + 2 \\ a^2 &= 2 \end{aligned}$$

So $a = \sqrt{2}$ or $a = -\sqrt{2}$, where the latter is rejected since $a_n \geq 0$ (rigorous proof by mathematical induction). Therefore, $\lim_{n\to\infty} a_n = a = \sqrt{2}$.

9. (1 point) Determine whether the sequence is divergent or convergent. If it is convergent, evaluate its limit.

(If it diverges to infinity, state your answer as *inf* . If it diverges to negative infinity, state your answer as *-inf* . If it diverges without being infinity or negative infinity, state your answer as *DNE*)

$$\lim_{n\to\infty} (-1)^n \sin(4/n)$$

Answer: ____________

Solution: Note that, for $n \geq 1$,

$$-|\sin(4/n)| \leq (-1)^n \sin(4/n) \leq |\sin(4/n)|$$

Moreover, $\lim_{n\to\infty} |\sin(4/n)| = |\sin(0)| = 0$, and similarly $\lim_{n\to\infty} -|\sin(4/n)| = 0$. Therefore $\lim_{n\to\infty}(-1)^n \sin(4/n) = 0$.

In fact for $N = \left\lfloor \frac{4}{\pi/2} \right\rfloor + 1$, the tail terms $n \geq N$ satisfy

$$-4/n \leq (-1)^n \sin(4/n) \leq 4/n$$

this is because when $n \geq N$, we have $0 < 4/n < \pi/2$ and for $0 < x < \pi/2$, the inequality $\sin(x) < x$ holds. By squeeze theorem, $\lim_{n\to\infty}(-1)^n \sin(4/n) = \lim_{n\to\infty} 4/n = 0$.

10. (1 point) Consider the sequence $a_n = \left\{\dfrac{2n+1}{2n} - \dfrac{2n}{2n+1}\right\}$. Graph this sequence and use your graph to help you answer the following questions.
Part 1: Is the sequence bounded?

(1) Is the sequence a_n bounded above by a number?

(2) Is the sequence a_n bounded below by a number?
(3) Select all that apply: The sequence a_n is
A. bounded.
B. bounded below.
C. bounded above.
D. unbounded.

Part 2: Is the sequence monotonic?
The sequence a_n is
A. decreasing.
B. alternating
C. increasing.
D. none of the above
Part 3: Does the sequence converge?

(1) The sequence a_n is
- convergent
- divergent

(2) The limit of the sequence a_n is

Part 4: Conceptual follow up questions

(1) Select all that apply: The sequence $\left\{(-1)^n \dfrac{10n^2+1}{n^2+n}\right\}$ is
A. monotonic
B. divergent
C. convergent
D. not monotonic
E. unbounded
F. bounded

(2) Select all that apply: The sequence $\left\{\dfrac{10n^3+1}{n^2+n}\right\}$ is
A. unbounded
B. not monotonic
C. divergent
D. monotonic
E. convergent
F. bounded

(3) If a sequence is bounded, it ....
- must
- may or may not
- cannot

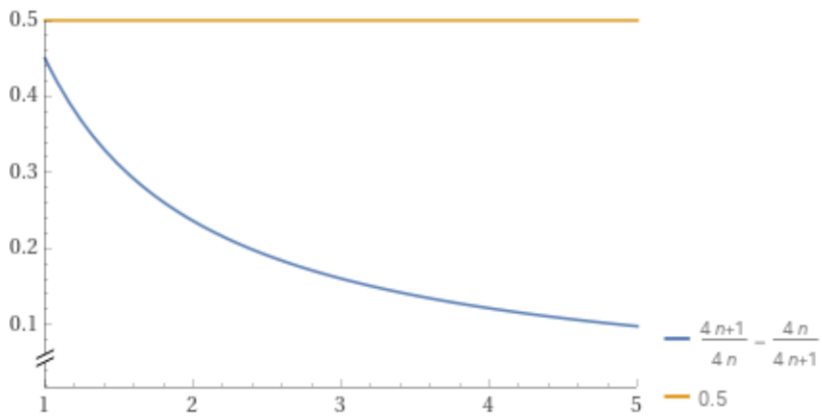
converge.
(4) If a sequence is monotonic, it ....
- must
- may or may not
- cannot

converge.

(5) If a sequence is bounded and monotonic, it ....
- must
- may or may not
- cannot

converge.

Solution:



Part 1:

(1) Yes, the sequence is bounded above by 1.
Explanation:
$a_n = \frac{2n+1}{2n} - \frac{2n}{2n+1} = \frac{(2n+1)^2-(2n)^2}{2n(2n+1)} = \frac{(4n+1)}{2n(2n+1)} < \frac{(4n+2)}{2n(2n+1)} = \frac{2}{2n} \leq 1.$

(2) Yes, the sequence is bounded below by 0. *Explanation:*
$a_n = \frac{2n+1}{2n} - \frac{2n}{2n+1} > \frac{2n}{2n} - \frac{2n}{2n+1} > 0.$

(3) The sequence is bounded, bounded below and bounded above (i.e A, B and C are the correct answers).

Part 2: The sequence a_n is monotonic decreasing
Explanation: Let's first simplify the given sequence:

$$a_n = \left(\frac{2n+1}{2n}\right) - \left(\frac{2n}{2n+1}\right)$$

Step-by-Step Simplification:

1. The first term is:

$$\frac{2n+1}{2n} = 1 + \frac{1}{2n}$$

2. The second term is:

$$\frac{2n}{2n+1} = 1 - \frac{1}{2n+1}$$

So, the sequence becomes:

$$a_n = \left(1 + \frac{1}{2n}\right) - \left(1 - \frac{1}{2n+1}\right)$$

$$a_n = \frac{1}{2n} + \frac{1}{2n+1}$$

This expression is positive for all n and decreases as n increases because both $\frac{1}{2n}$ and $\frac{1}{2n+1}$ decrease as n grows.

Answer:

Thus, the sequence a_n is **decreasing**.

Part 3:

(1) The sequence a_n is convergent because it's bounded and monotonic.
(2) The limit of the sequence a_n is 0. *Proof:*

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty}\left(\frac{2n+1}{2n} - \frac{2n}{2n+1}\right) = \lim_{n\to\infty}\frac{2n+1}{2n} - \lim_{n\to\infty}\frac{2n}{2n+1} = 1 - 1 = 0.$$

Part 4:

(1) For n is even the sequence becomes $\left\{\frac{10n^2+1}{n^2+n}\right\}$ and $\frac{10n^2+1}{n^2+n} \leq \frac{10n^2+1}{n^2+1} < \frac{10n^2+10}{n^2+1} \leq 10$.

For n is odd the sequence becomes $\left\{-\frac{10n^2+1}{n^2+n}\right\}$ and $-\frac{10n^2+1}{n^2+n} \geq -\frac{10n^2+1}{n^2+1} > -\frac{10n^2+10}{n^2+1} \geq -10$. Thus the sequence is bounded above by 10 and bounded below by -10.

Therefore, the sequence is bounded but not monotonic because it changes sign.

For even $n = 2k$, $\frac{10n^2+1}{n^2+n} = \frac{10+1/n^2}{1+1/n}$, we have

$$\lim_{k\to\infty} a_{2k} = \lim_{n\to\infty}\frac{10+1/n^2}{1+1/n} = \frac{\lim_{n\to\infty} 10+1/n^2}{\lim_{n\to\infty} 1+1/n} = 10,$$

while for odd $n = 2k-1$, $-\frac{10n^2+1}{n^2+n} = -\frac{10+1/n^2}{1+1/n}$, we have

$$\lim_{k\to\infty} a_{2k-1} = \lim_{n\to\infty} -\frac{10+1/n^2}{1+1/n} = -\frac{\lim_{n\to\infty} 10+1/n^2}{\lim_{n\to\infty} 1+1/n} = -10,$$

The limits of even subsequence and odd subsequence do not match, therefore the sequence is divergent.

So the correct answers are B, D, and F.

(2) We denote the sequence by $a_n = \dfrac{10n^3+1}{n^2+n}$. Then for arbitrary n, we have

$$\begin{aligned} a_{n+1} - a_n &= \frac{10(n+1)^3+1}{(n+1)^2+(n+1)} - \frac{10n^3+1}{n^2+n} \\ &= \frac{10(n+1)^3+1}{(n+2)(n+1)} - \frac{10n^3+1}{(n+1)n} \\ &= \frac{[10(n+1)^3+1]n - (10n^3+1)(n+2)}{(n+2)(n+1)n} \\ &= \frac{10n^3+30n^2+10n-2}{(n+2)(n+1)n} \end{aligned}$$

The numerator $10n^3+30n^2+10n-2 > 10n-2 \geq 8 > 0$ for $n \geq 1$, so $a_{n+1} - a_n > 0$ for arbitrary $n \geq 1$, $n \in \mathbb{N}$, hence the sequence is monotonic increasing.

Note that the following inequality holds for $n \geq 1$:

$$\frac{10n^3+1}{n^2+n} > \frac{10n^3}{n^2+n} \geq \frac{10n^3}{n^2+n^2} = 5n$$

so the sequence is unbounded, hence it's divergent.

So the correct answers are A, C, D.

(3) If a sequence is bounded, it may or may not converge.

A bounded sequence may jump up and down indefinitely. Part 4 (a) is an example. The sequence $\left\{(-1)^n \dfrac{10n^2+1}{n^2+n}\right\}$ is bounded but not monotonic and not convergent.

(4) If a sequence is monotonic, it may or may not converge.

A sequence may monotonically tend to $+\infty$ or $-\infty$. Part 4 (b) is an example. The sequence $\left\{\dfrac{10n^3+1}{n^2+n}\right\}$ is monotonically increasing but unbounded, hence it is not convergent.

(5) If a sequence is bounded and monotonic, it must converge. [This is the monotonic convergence theorem.]