**THE CHINESE UNIVERSITY OF HONG KONG**
**Department of Mathematics**
**MATH1010 UNIVERSITY MATHEMATICS 2024-2025 Term 1**
**Suggested Solutions of WeBWork Coursework 10**

If you find any errors or typos, please email us at
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**1.** (1 point)

Use the Fundamental Theorem of Calculus to evaluate (if it exists)

$$\int_{-\pi}^{\pi} f(x)\,dx,$$

where

$$f(x) = \begin{cases} -9x & \text{if } -\pi \le x \le 0 \\ -5\sin(x) & \text{if } 0 < x \le \pi \end{cases}$$

If the integral does not exist, type "DNE" as your answer.

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**Solution:**

$\int_{-\pi}^{\pi} f(x)dx = \int_{-\pi}^{0} -9xdx - \int_{0}^{\pi} 5\sin(x)dx$

$= \frac{9x^2}{2}\Big|_{-\pi}^{0} + 5\cos(x)|_0^\pi$

$= -10 + \frac{9\pi^2}{2}$

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**2.** (1 point)

Evaluate the limit $\lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \frac{7j^3}{n^4}$.

$\lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \frac{7j^3}{n^4} =$ ________

**Solution:**

Let $S_n = \sum\limits_{j=1}^{n} \frac{7j^3}{n^4} = \frac{7}{n^4}\sum\limits_{j=1}^{n} j^3 = \frac{7}{n^4}\left(\frac{n(n+1)}{2}\right)^2 = \frac{7}{n^4}\cdot\frac{n^2(n+1)^2}{4} = \frac{7n^2(n^2+2n+1)}{4n^4} = \frac{7(n^2+2n+1)}{4n^2})$

$Then \lim\limits_{n\to\infty} \sum\limits_{j=1}^{n} \frac{7j^3}{n^4} = \lim\limits_{n\to\infty} S_n = \frac{7}{4}.$

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**3.** (1 point) The following sum

$$\sqrt{16 - \left(\frac{4}{n}\right)^2}\cdot\frac{4}{n} + \sqrt{16 - \left(\frac{8}{n}\right)^2}\cdot\frac{4}{n} + \ldots + \sqrt{16 - \left(\frac{4n}{n}\right)^2}\cdot\frac{4}{n}$$

is a right Riemann sum with $n$ subintervals of equal length for the definite integral

$$\int_0^b f(x)\,dx$$

where $b =$ ________
and $f(x) =$ ________

**Solution:** It is clear

$$\sqrt{16-\left(\frac{4}{n}\right)^2}\cdot\frac{4}{n}+\sqrt{16-\left(\frac{8}{n}\right)^2}\cdot\frac{4}{n}+\ldots+\sqrt{16-\left(\frac{4n}{n}\right)^2}\cdot\frac{4}{n} \;=\frac{4}{n}\sum_{i=1}^{n}\sqrt{16-(\frac{4i}{n})^2}$$

thus if one compare the form of Riemann sum, we know the interval $[0,4]$ is equally divided into $n$ subintervals and the intergrand function is $\sqrt{16-x^2}$
thus $b=4, f(x)=\sqrt{16-x^2}$

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**4.** (1 point) Compute the following limit. Use INF to denote $\infty$ and MINF to denote $-\infty$.
$\lim_{x\to 0}\frac{x}{\int_x^{x^2}\sqrt[3]{729-5t^3}\,dt} =$ ________

**Solution:** Let $f(x)=\int_x^{x^2}\sqrt[3]{729-5t^3}\,dt$ then $f'(x)=2x\sqrt[3]{729-5x^6}-\sqrt[3]{729-5x^3}$
thus it is clear $\lim_{x\to 0}f'(x)=-9$.By using L'Hopital's rule, we know

$$\lim_{x\to 0}\frac{x}{\int_x^{x^2}\sqrt[3]{729-5t^3}\,dt}=\lim_{x\to 0}\frac{x}{f(x)}=\lim_{x\to 0}\frac{1}{f'(x)}=-\frac{1}{9}$$

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**5.** (1 point)
Evaluate the integral

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} 6t^3\cos(t^2)\,dt$$

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**Solution:** We use change of variable $t^2=x$ and integration by part

$$\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} 6t^3\cos(t^2)\,dt=\int_{\sqrt{\pi/2}}^{\sqrt{\pi}} 3t^2\cos(t^2)\,dt^2=\int_{\pi/2}^{\pi}3x\cos(x)dx$$

$$=3x\sin(x)|_{\pi/2}^{\pi}-\int_{\pi/2}^{\pi}3\sin(x)dx=-\frac{3\pi}{2}+3\cos(x)|_{\pi/2}^{\pi}=-3-\frac{3\pi}{2}$$

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**6.** (1 point)

Evaluate the integral

$$\int_0^4 \left|\sqrt{x+2} - x\right| dx$$

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**Solution:** Note that by simple computation we know $\sqrt{x+2} \geq x$ if $x \in (0,2)$ and $\sqrt{x+2} \leq x$ if $\in (2,4)$ thus we have

$$\int_0^4 \left|\sqrt{x+2} - x\right| dx = \int_0^2 \sqrt{x+2} - x\,dx + \int_2^4 x - \sqrt{x+2}\,dx$$

$$= \left[\frac{2}{3}(x+2)^{\frac{3}{2}} - \frac{x^2}{2}\right]\Bigg|_0^2 + \left[\frac{x^2}{2} - \frac{2}{3}(x+2)^{\frac{3}{2}}\right]\Bigg|_2^4 = \frac{44}{3} - \frac{4\sqrt{2}}{3} - 4\sqrt{6}$$

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**7.** (1 point)

The interval $[0,3]$ is partitioned into $n$ equal subintervals, and a number $x_i$ is arbitrarily chosen in the $i^{th}$ subinterval for each $i$. Then:

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6x_i+2}{n} =$ ______________

**Solution:**

**Solution:**
Let's interpret the sum as a Riemann sum.
Recall that the Riemann sum for a function $f(x)$ on the interval $[0,3]$ has the form $\sum_{i=1}^{n} f(x_i)\frac{3}{n}$ since the length of each subinterval is $\Delta x = \frac{3}{n}$.
$\sum_{i=1}^{n} \frac{6x_i+2}{n} = \sum_{i=1}^{n} \frac{6x_i+2}{3} \cdot \frac{3}{n}$, therefore the given sum is the Riemann sum for $f(x) = \frac{6x+2}{3}$.

The limit of the Riemann sum as $n$ approaches infinity is the integral of the function $f(x)$ from 0 to 3, thus

$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6x+2}{3} \cdot \frac{3}{n} = \int_0^3 \frac{6x+2}{3} dx = \frac{1}{3}\int_0^3 (6x+2)dx = \left[\frac{1}{3}\left(3x^2+2x\right)\right]\Bigg|_0^3 = \frac{1}{3}\left(3\cdot 3^2 + 2\cdot 3\right) =$ 11

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**8.** (1 point)

(a) Consider the integral $\int_0^{\pi} \sin(5x)\,dx$. Which of the following expressions represents the integral as a limit of Riemann sums?

- A. $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\pi + \frac{5\pi i}{n}\right)$

- B. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin\left(\frac{\pi i}{n}\right)$
- C. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin\left(\frac{5\pi i}{n}\right)$
- D. $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{\pi i}{n}\right)$
- E. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin\left(\pi + \frac{5\pi i}{n}\right)$
- F. $\lim_{n\to\infty} \sum_{i=1}^{n} \sin\left(\frac{5\pi i}{n}\right)$

(b) Limit in the correct answer to (a) = ______________

**Solution:** (a)let $f(x) = \sin(5x)$ then devide interval $[0, \pi]$ into $n$ equal size subintervals, and use the language of Riemann sum, we have

$$\int_0^{\pi} \sin(5x)dx = \int_0^{\pi} f(x)dx = \frac{\pi}{n} \lim_{n\to\infty} \sum_{i=1}^{n} f(\frac{\pi i}{n}) = \lim_{n\to\infty} \sum_{i=1}^{n} \frac{\pi}{n} \sin(\frac{5\pi i}{n})$$

thus C is the right expression.

(b)

$$\int_0^{\pi} \sin(5x)dx = -\frac{\cos(5x)}{5}\bigg|_0^{\pi} = \frac{2}{5}$$

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**9.** (1 point)

Consider the integral $\int_2^6 \frac{x}{1+x^5}\,dx$. Which of the following expressions represents the integral as a limit of Riemann sums?

- A. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{4}{n} \frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)}$
- B. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \frac{2+\frac{6i}{n}}{1+\left(2+\frac{6i}{n}\right)^5}$
- C. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{6}{n} \frac{2+\frac{6i}{n}}{1+\left(2+\frac{6i}{n}\right)}$
- D. $\lim_{n\to\infty} \sum_{i=1}^{n} \frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}$

- E. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}$
- F. $\lim_{n\to\infty}\sum_{i=1}^{n}\frac{2+\frac{6i}{n}}{1+\left(2+\frac{6i}{n}\right)^5}$

**Solution:** By dividing interval $[0,4]$ into $n$ equal-size subintervals,we have

$$\int_2^6 \frac{x}{1+x^5}\,dx = \int_0^4 \frac{x+2}{1+(x+2)^5}\,dx = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{4}{n}\frac{2+\frac{4i}{n}}{1+\left(2+\frac{4i}{n}\right)^5}$$

Thus E is the right expression.

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**10.** (1 point)

Let $F(x) = \displaystyle\int_4^x \frac{6}{\ln(5t)}\,dt$, for $x \geq 4$.

**A.** $F'(x) =$ ______________

**B.** On what interval or intervals is $F$ increasing?
$x \in$ ______________
*(Give your answer as an interval or a list of intervals, e.g.,* **(-infinity,8]** *or* **(1,5),(7,10)**, *or enter* **none***for no intervals.)*

**C.** On what interval or intervals is the graph of $F$ concave up?
$x \in$ ______________
*(Give your answer as an interval or a list of intervals, e.g.,* **(-infinity,8]** *or* **(1,5),(7,10)**, *or enter* **none***for no intervals.)*

**Solution:**

**A.** $F'(x) = \frac{6}{\ln(5x)}$.

**B.** For $x \geq 4$, $F'(x) > 0$, so $F(x)$ is increasing for all $x \in [4,\infty)$.

**C.** $F''(x) = -6\frac{1}{x\ln(5x)^2} < 0$ for $x \geq 4$, so the graph of $F(x)$ is concave down for all $x \in [4,\infty)$ (and is concave up for no intervals).

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**11.** (1 point)

Suppose that $F(x) = \displaystyle\int_1^x f(t)\,dt$, where

$$f(t) = \int_1^{t^2} \frac{\sqrt{7+u^4}}{u}\,du.$$

Find $F''(2)$.

$F''(2) =$ ______________

**Solution:** since $F(x) = \displaystyle\int_1^x f(t)\,dt$ and

$$f(x) = \int_1^{x^2} \frac{\sqrt{7+u^4}}{u}\,du.$$

we have $F'(x) = f(x)$ and

$$f'(x) = 2x \cdot \frac{\sqrt{7 + x^8}}{x^2} = \frac{2\sqrt{7 + x^8}}{x}$$

thus

$$F''(x) = f'(x) = \frac{2\sqrt{7 + x^8}}{x}$$

$$F''(2) = f'(x) = \frac{2\sqrt{7 + 2^8}}{2} \approxeq 16.2172747402$$