

**THE CHINESE UNIVERSITY OF HONG KONG**  
Department of Mathematics  
**MATH1010 University Mathematics 2024-2025 Term 1**  
**Homework Assignment 4**  
Due Date: 11 December 2024

1. Evaluate the following integrals:

(a)  $\int x \sec^2 x dx$

(d)  $\int \frac{x^5}{(1+x^3)^3} dx$

(b)  $\int \sec^2 x \ln \tan x dx$

(e)  $\int x^2 \ln \frac{1+x}{1-x} dx$

(c)  $\int e^{-x} \sin 3x dx$

(f)  $\int e^x \left( \frac{1+\sin x}{1+\cos x} \right) dx$

**Solution:**

(a)  $\int x \frac{1}{\cos^2 x} = \int x d \tan x = x \tan x - \int \tan x dx = \boxed{x \tan x + \ln |\cos x| + C}$

(b)  $\int \sec^2 x \ln \tan x dx = \int \ln \tan x d \tan x = \boxed{\tan x \ln \tan x - \tan x + C}$

(c) Note that

$$\int e^{-x} \sin 3x dx = - \int \sin 3x d e^{-x} = -e^{-x} \sin 3x + 3 \int e^{-x} \cos 3x dx$$

On the other hand,

$$\int e^{-x} \cos 3x dx = - \int \cos 3x d e^{-x} = -e^{-x} \cos 3x - 3 \int e^{-x} \sin 3x dx$$

Therefore,  $\int e^{-x} \sin 3x dx = -e^{-x} \sin 3x - 3e^{-x} \cos 3x - 9 \int e^{-x} \sin 3x dx$

$$\implies \int e^{-x} \sin 3x dx = \boxed{-\frac{1}{10}(e^{-x} \sin 3x + 3e^{-x} \cos 3x)} + C$$

(d)

$$\begin{aligned} \int \frac{x^5}{(1+x^3)^3} dx &= \frac{1}{3} \int \frac{x^3}{(1+x^3)^3} dx^3 \\ &= -\frac{1}{6} \int t d \frac{1}{(1+t)^2} \\ &= -\frac{1}{6} \left( t \frac{1}{(1+t)^2} - \int \frac{1}{(1+t)^2} dt \right) \\ &= \boxed{-\frac{1}{6} \frac{2x^3+1}{(1+x^3)^2} + C} \end{aligned}$$

(e)

$$\begin{aligned}
\int x^2 \ln \frac{1+x}{1-x} dx &= \frac{1}{3} \int \ln \frac{1+x}{1-x} dx^3 \\
&= \frac{x^3}{3} \ln \frac{1+x}{1-x} - \frac{1}{3} \int x^3 d \ln \frac{1+x}{1-x} \\
&= \frac{x^3}{3} \ln \frac{1+x}{1-x} - \frac{2}{3} \int \frac{x^3}{1-x^2} dx \\
&= \frac{x^3}{3} \ln \frac{1+x}{1-x} - \frac{2}{3} \int \frac{x(x^2-1)+x}{1-x^2} dx \\
&= \frac{x^3}{3} \ln \frac{1+x}{1-x} + \frac{2}{3} \int x dx - \frac{2}{3} \int \frac{x}{1-x^2} dx \\
&= \frac{x^3}{3} \ln \frac{1+x}{1-x} + \frac{x^2}{3} + \frac{1}{3} \int \frac{1}{1-x^2} d(1-x^2) \\
&= \boxed{\frac{x^3}{3} \ln \frac{1+x}{1-x} + \frac{x^2}{3} + \frac{1}{3} \ln(1-x^2) + C}
\end{aligned}$$

(f)

$$\begin{aligned}
&\int e^x \left( \frac{1+\sin x}{1+\cos x} \right) dx \\
&= \int \left( \frac{1+\sin x}{1+\cos x} \right) de^x \\
&= e^x \left( \frac{1+\sin x}{1+\cos x} \right) - \int e^x d \left( \frac{1+\sin x}{1+\cos x} \right) \\
&= e^x \left( \frac{1+\sin x}{1+\cos x} \right) - \int e^x \left( \frac{1+\sin x + \cos x}{(1+\cos x)^2} \right) dx \\
&= e^x \left( \frac{1+\sin x}{1+\cos x} \right) - \int e^x \left( \frac{1}{1+\cos x} \right) dx - \int e^x \left( \frac{\sin x}{(1+\cos x)^2} \right) dx \\
&= e^x \left( \frac{1+\sin x}{1+\cos x} \right) - \int e^x \left( \frac{1}{1+\cos x} \right) dx - \int e^x d \left( \frac{1}{1+\cos x} \right) \\
&= e^x \left( \frac{1+\sin x}{1+\cos x} \right) - \int e^x \left( \frac{1}{1+\cos x} \right) dx - e^x \left( \frac{1}{1+\cos x} \right) + \int e^x \left( \frac{1}{1+\cos x} \right) dx \\
&= \boxed{e^x \left( \frac{\sin x}{1+\cos x} \right) + C}
\end{aligned}$$

2. Evaluate the following integrals:

(a)  $\int \sin^4 x \cos^3 x dx$

(d)  $\int \sec^4 x \tan^6 x dx$

(b)  $\int \cos^4 x \sin^2 x dx$

(e)  $\int \frac{2e^x}{e^{2x}-4} dx$

(c)  $\int \sec x \tan^3 x dx$

(f)  $\int \sec x dx$

$$(g) \int \frac{1}{\sqrt{1-x^2}} dx$$

$$(h) \int \frac{1}{1+x^2} dx$$

**Solution:**

(a)

$$\begin{aligned} \int \sin^4(x) \cos^3(x) dx &= \int \sin^4(x) (1 - \sin^2(x)) \cos(x) dx \\ &= \int \sin^4(x) \cos(x) dx - \int \sin^6(x) \cos(x) dx \\ &= \int \frac{1}{5} \cdot 5 \sin^4(x) \sin'(x) dx - \int \frac{1}{7} \cdot 7 \sin^6(x) \sin'(x) dx \\ &= \boxed{\frac{\sin^5(x)}{5} - \frac{\sin^7(x)}{7} + C} \end{aligned}$$

(b)

$$\begin{aligned} &\int \cos^4(x) \sin^2(x) dx \\ &= \int \sin^2(x) \cos^2(x) \cdot \cos^2(x) dx \\ &= \int \frac{\sin^2(2x)}{4} \cdot \frac{\cos(2x) + 1}{2} dx \\ &= \int \frac{\sin^2(2x)}{8} dx + \int \frac{\sin^2(2x) \cos(2x)}{8} dx \\ &= \int \frac{1 - \cos(4x)}{16} dx + \int \frac{\sin^2(2x) \cdot 2 \cos(2x)}{16} dx \\ &= \boxed{\frac{x}{16} - \frac{\sin(4x)}{64} + \frac{\sin^3(2x)}{48} + C} \end{aligned}$$

(c)

$$\begin{aligned} \int \sec(x) \tan^3(x) dx &= \int \tan^2(x) \sec'(x) dx \\ &= \int (\sec^2(x) - 1) \sec'(x) dx = \boxed{\frac{\sec^3(x)}{3} - \sec(x) + C} \end{aligned}$$

(d)

$$\begin{aligned} &\int \sec^4(x) \tan^6(x) dx \\ &= \int \sec^2(x) \tan^6(x) \tan'(x) dx \\ &= \int (1 + \tan^2(x)) \tan^6(x) \tan'(x) dx \\ &= \int (\tan^6(x) + \tan^8(x)) \tan'(x) dx \\ &= \boxed{\frac{\tan^7(x)}{7} + \frac{\tan^9(x)}{9} + C}, \end{aligned}$$

(e)

$$\begin{aligned}
\int \frac{2e^x dx}{e^{2x} - 4} &= \int \frac{2du}{u^2 - 4} \quad (\text{substituting } u = e^x) \\
&= \int \frac{1}{2} \left( \frac{1}{u-2} - \frac{1}{u+2} \right) du \\
&= \frac{1}{2} (\ln(|u-2|) - \ln(|u+2|)) + C \\
&= \frac{1}{2} \ln \left( \left| \frac{u-2}{u+2} \right| \right) + C \\
&= \boxed{\frac{1}{2} \ln \left( \left| \frac{e^x - 2}{e^x + 2} \right| \right) + C}
\end{aligned}$$

(f)

$$\begin{aligned}
&\int \sec(x) dx \\
&= \int \frac{dx}{\cos(x)} = \int \frac{\cos(x) dx}{\cos^2(x)} = \int \frac{\cos(x) dx}{1 - \sin^2(x)} \\
&= \int \frac{1}{2} \left( \frac{1}{1 + \sin(x)} + \frac{1}{1 - \sin(x)} \right) \sin'(x) dx \\
&= \int \frac{1}{2} \cdot \frac{1}{1 + \sin(x)} \cdot \frac{d}{dx}(1 + \sin(x)) dx + \int -\frac{1}{2} \cdot \frac{1}{1 - \sin(x)} \cdot \frac{d}{dx}(1 - \sin(x)) dx \\
&= \frac{1}{2} \ln(|1 + \sin(x)|) - \frac{1}{2} \ln(|1 - \sin(x)|) + C \\
&= \frac{1}{2} \ln \left( \left| \frac{1 + \sin(x)}{1 - \sin(x)} \right| \right) + C \\
&= \frac{1}{2} \ln \left( \left| \frac{(1 + \sin(x))^2}{1 - \sin^2(x)} \right| \right) + C \\
&= \frac{1}{2} \ln \left( \left| \frac{(1 + \sin(x))^2}{\cos^2(x)} \right| \right) + C \\
&= \ln \left( \left| \frac{1 + \sin(x)}{\cos(x)} \right| \right) + C \\
&= \boxed{\ln(|\sec(x) + \tan(x)|) + C}
\end{aligned}$$

$$(g) \int \frac{dx}{\sqrt{1-x^2}} = \int \frac{\sin'(\theta) d\theta}{\sqrt{1-\sin^2(\theta)}} = \int d\theta = \theta + C = \boxed{\arcsin(x) + C}$$

Here we have used the substitution  $x = \sin(\theta)$ .

$$(h) \int \frac{dx}{1+x^2} = \int \frac{\sec^2(\theta) d\theta}{1+\tan^2(\theta)} = \int d\theta = \theta + C = \boxed{\arctan(x) + C}$$

Here we have used the substitution  $x = \tan(\theta)$ .

3. Evaluate the following integrals by trigonometric substitutions:

$$(a) \int \frac{x^2 dx}{(1-x^2)^{\frac{3}{2}}}$$

$$(c) \int \sqrt{4-x^2} dx$$

$$(b) \int \frac{dx}{\sqrt{4+x^2}}$$

$$(d) \int \frac{1}{(x^2+1)^2} dx$$

**Solution:**

(a) Put  $x = \sin t$ , we have

$$\begin{aligned} \int \frac{x^2 dx}{(1-x^2)^{\frac{3}{2}}} &= \int \frac{\sin^2 t}{\cos^3 t} \cos t dt = \int \tan^2 t dt = \int (\sec^2 t - 1) dt \\ &= \tan t - t + C = \boxed{\frac{x}{\sqrt{1-x^2}} - \arcsin x + C}. \end{aligned}$$

$$\begin{aligned} (b) \text{ Put } x &= 2 \tan t, t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ then } \sqrt{4+x^2} = 2 \sec t, dx = 2 \sec^2 t dt, \\ \int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{1}{2 \sec t} 2 \sec^2 t dt = \int \sec t dt = \int \frac{1}{\cos t} dt = \int \frac{\cos t}{\cos^2 t} dt \\ &= \int \frac{1}{1-\sin^2 t} d(\sin t) = \frac{1}{2} \int \left[ \frac{1}{1-\sin t} + \frac{1}{1+\sin t} \right] d \sin t \\ &= \frac{1}{2} \ln \left| \frac{1+\sin t}{1-\sin t} \right| + C = \frac{1}{2} \ln \left| \frac{1+\sin t}{\cos^2 x} \right| + C = \ln |\sec t + \tan t| + C \\ &= \boxed{\ln \left| \frac{\sqrt{4+x^2}}{2} + \frac{x}{2} \right| + C}. \end{aligned}$$

$$\begin{aligned} (c) \text{ Put } x &= 2 \sin t, \text{ then } dx = 2 \cos t dt \\ \int \sqrt{4-4\sin^2 t} (2 \cos t dt) &= \int 4 \cos^2 t dt = \int 2(\cos(2t) + 1) dt \\ &= \sin(2t) + 2t + C = \sin \left( 2 \arcsin \left( \frac{x}{2} \right) \right) + 2 \arcsin \left( \frac{x}{2} \right) + C \\ &= \boxed{\frac{1}{2} x \sqrt{4-x^2} + 2 \arcsin \left( \frac{x}{2} \right) + C}. \end{aligned}$$

$$\begin{aligned} (d) \text{ Put } x &= \tan t, \text{ with } t \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right), \text{ then } dx = \sec^2 t dt \\ \int \frac{1}{(\tan^2 t + 1)^2} \sec^2 t dt &= \int \frac{1}{\sec^2 t} dt = \frac{1}{2} \int (\cos(2t) + 1) dt \\ &= \frac{1}{4} \sin(2t) + \frac{t}{2} + C = \frac{1}{2} \sin t \cos t + \frac{t}{2} + C = \boxed{\frac{x}{2(x^2+1)} + \frac{1}{2} \arctan x + C}. \end{aligned}$$

4. Prove the following reduction formulas.

$$(a) I_n = \int \frac{x^n dx}{\sqrt{x+a}}; \quad I_n = \frac{2x^n \sqrt{x+a}}{2n+1} - \frac{2an}{2n+1} I_{n-1}, \quad n \geq 1.$$

$$(b) I_n = \int \sin^n x dx; \quad I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}, \quad n \geq 2.$$

$$(c) I_n = \int \frac{1}{x^n(x+1)} dx; \quad I_n = \frac{1}{-n+1} x^{-n+1} - I_{n-1}, \quad n \geq 2.$$

(d)  $I_n = \int_0^\pi x^n \sin x \, dx$ ;  $I_n = \pi^n - n(n-1)I_{n-2}$ ,  $n \geq 2$ , then find  $I_6$ .

**Solution:**

$$\begin{aligned} \text{(a)} \quad I_n &= \int 2x^n d(\sqrt{x+a}) \\ &= 2x^n \sqrt{x+a} - \int 2nx^{n-1} \sqrt{x+a} \, dx \\ &= 2x^n \sqrt{x+a} - \int \frac{2nx^{n-1}}{\sqrt{x+a}} (x+a) \, dx \\ &= 2x^n \sqrt{x+a} - 2anI_{n-1} - \int \frac{2nx^n}{\sqrt{x+a}} \, dx. \end{aligned}$$

Therefore,  $(2n+1)I_n = 2x^n \sqrt{x+a} - 2anI_{n-1}$ , for  $n \geq 1$ .

$$\begin{aligned} \text{(b)} \quad I_n &= \int \sin^n x \, dx = - \int \sin^{n-1} x \, d \cos x = -(\cos x \sin^{n-1} x - (n-1) \int \cos^2 x \sin^{n-2} x \, dx) = \\ &= -\cos x \sin^{n-1} x + (n-1)I_{n-2} - (n-1)I_n \Rightarrow I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2} \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad \text{For } n \geq 2, \quad I_n &= \int \frac{1}{x^n(x+1)} \, dx = \int \left( \frac{1+x}{x^n(x+1)} - \frac{x}{x^n(x+1)} \right) \, dx \\ &= \int \frac{1}{x^n} \, dx - \int \frac{1}{x^{n-1}(x+1)} \, dx = \frac{1}{-n+1} x^{-n+1} - I_{n-1}. \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad I_n &= - \int_0^\pi x^n d(\cos x) = -x^n \cos x \Big|_0^\pi + \int_0^\pi \cos x \, d(x^n) = \pi^n + \int_0^\pi \cos x \, n x^{n-1} \, dx \\ &= \pi^n + n \int_0^\pi x^{n-1} d(\sin x) = \pi^n + n \left[ \sin x \, x^{n-1} \Big|_0^\pi - \int_0^\pi \sin x \, (n-1) x^{n-2} \, dx \right] \\ &= \pi^n - n(n-1)I_{n-2} \Rightarrow I_n = \pi^n - n(n-1)I_{n-2}, \text{ and } I_6 = \boxed{\pi^6 - 30\pi^4 + 360\pi^2 - 1440}. \end{aligned}$$

5. Find  $F'(x)$  for the following functions:

(a)  $F(x) = \int_\pi^x \frac{\cos y}{y} \, dy$

(d)  $F(x) = \int_{x^2}^{x^3} e^{\cos u} \, du$

(b)  $F(x) = \int_0^{x^3} e^{u^2} \, du$

(e)  $F(x) = \int_1^x \frac{e^x + e^t}{t} \, dt$

(c)  $F(x) = \int_x^{2x} (\ln t)^2 \, dt$

(f)  $F(x) = \int_{-\sqrt{\ln x}}^{\sqrt{\ln x}} \frac{\sin t}{t} \, dt$

**Solution:**

(a)  $F'(x) = \boxed{\frac{\cos x}{x}}$

(b)  $F'(x) = \boxed{3x^2 e^{x^6}}$

(c)  $F(x) = \int_1^{2x} (\ln t)^2 \, dt + \int_x^1 (\ln t)^2 \, dt \Rightarrow F'(x) = \boxed{2(\ln(2x))^2 - (\ln x)^2}$

(d)  $F'(x) = \boxed{3x^2 e^{\cos x^3} - 2x e^{\cos x^2}}$

(e)  $F(x) = e^x \int_1^x \frac{1}{t} \, dt + \int_1^x \frac{e^t}{t} \, dt \Rightarrow F'(x) = \boxed{e^x \left( \ln x + \frac{2}{x} \right)}$

$$\begin{aligned}
 \text{(f)} \quad F(x) &= \int_0^{\sqrt{\ln x}} \frac{\sin t}{t} dt - \int_0^{-\sqrt{\ln x}} \frac{\sin t}{t} dt = \frac{\sin \sqrt{\ln x}}{\sqrt{\ln x}} \frac{1}{2x\sqrt{\ln x}} - \frac{\sin(-\sqrt{\ln x})}{\sqrt{\ln x}} \frac{1}{2x\sqrt{\ln x}} \\
 &= \boxed{\frac{\sin \sqrt{\ln x}}{x \ln x}}
 \end{aligned}$$

6. Evaluate the following integrals of rational functions:

$$\begin{array}{ll}
 \text{(a)} \int \frac{x^2}{1-x^2} dx & \text{(d)} \int \frac{x^2+1}{(x+1)^2(x-1)} dx \\
 \text{(b)} \int \frac{4x+1}{x^2-6x+13} dx & \text{(e)} \int \frac{2x^2-2}{2x^2-5x+2} dx \\
 \text{(c)} \int \frac{2x^3-x^2+3}{x^2-2x-3} dx & \text{(f)} \int \frac{-x+1}{2x^2+4x+5} dx
 \end{array}$$

**Solution:**

$$\begin{aligned}
 \text{(a)} \quad \int \frac{x^2 dx}{1-x^2} &= \int \left( -1 + \frac{1}{1-x^2} \right) dx = -x + \frac{1}{2} \int \left( \frac{1}{1-x} + \frac{1}{1+x} \right) dx \\
 &= \boxed{-x + \frac{-\ln|1-x| + \ln|1+x|}{2} + C}
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad \int \frac{4x+1}{x^2-6x+13} dx &= \int \frac{4x-12+13}{x^2-6x+13} dx = 2 \int \frac{2x-6}{x^2-6x+13} dx + 13 \int \frac{1}{x^2-6x+13} dx \\
 &= 2 \int \frac{d(x^2-6x+13)}{x^2-6x+13} dx + 13 \int \frac{1}{(x-3)^2+4} dx \\
 &= 2 \ln(x^2-6x+13) + 13 \cdot \frac{2}{4} \int \frac{1}{\left(\frac{x-3}{2}\right)^2+1} d\left(\frac{x-3}{2}\right) \\
 &= \boxed{2 \ln(x^2-6x+13) + \frac{13}{2} \arctan\left(\frac{x-3}{2}\right) + C}
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad \int \frac{2x^3-x^2+3}{x^2-2x-3} dx &= \int (2x+3+12 \frac{x+1}{(x-3)(x+1)}) dx = x^2+3x+12 \int \frac{1}{x-3} dx \\
 &= \boxed{x^2+3x+12 \ln|x-3| + C}
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad \int \frac{x^2+1}{(x+1)^2(x-1)} dx &= \int \left[ \frac{1}{2(x+1)} + \frac{1}{2(x-1)} - \frac{1}{(x+1)^2} \right] dx \\
 &= \boxed{\frac{1}{2} \ln|x+1| + \frac{1}{2} \ln|x-1| + \frac{1}{x+1} + C}
 \end{aligned}$$

$$\begin{aligned}
 \text{(e)} \quad \int \frac{2x^2-2}{2x^2-5x+2} dx &= \int \frac{2x^2-5x+2+5x-4}{(x-2)(2x-1)} dx = \int \left( 1 + \frac{5x-4}{(x-2)(2x-1)} \right) dx = \\
 &= \int \left( 1 + \frac{2(2x-1)+x-2}{(x-2)(2x-1)} \right) dx = \int \left( 1 + \frac{2}{x-2} + \frac{1}{2x-1} \right) dx \\
 &= \boxed{x+2 \ln|x-2| + \frac{1}{2} \ln|2x-1| + C}
 \end{aligned}$$

$$\begin{aligned}
\text{(f)} \quad & \int \frac{-x+1}{2x^2+4x+5} dx = \int \frac{-(x+1)+2}{2x^2+4x+5} dx \\
& = -\frac{1}{4} \ln|2x^2+4x+5| + C + \int \frac{2}{2x^2+4x+5} dx \\
& = -\frac{1}{4} \ln|2x^2+4x+5| + C + 2 \int \frac{1}{2(x+1)^2+3} dx \\
& = -\frac{1}{4} \ln|2x^2+4x+5| + C + \frac{2}{3} \int \frac{1}{\left(\frac{\sqrt{2}(1+x)}{\sqrt{3}}\right)^2+1} dx \\
& = -\frac{1}{4} \ln|2x^2+4x+5| + C + \frac{2}{3} \int \frac{\sqrt{\frac{3}{2}}}{\left(\frac{\sqrt{2}(1+x)}{\sqrt{3}}\right)^2+1} d\left(\sqrt{\frac{2}{3}}(x+1)\right) \\
& = \boxed{-\frac{1}{4} \ln|2x^2+4x+5| + C + \sqrt{\frac{2}{3}} \arctan\left(\sqrt{\frac{2}{3}}(x+1)\right)}
\end{aligned}$$

7. Evaluate the following definite integrals:

$$\text{(a)} \quad \int_0^1 x^3 \sqrt{1+3x^2} dx$$

$$\text{(e)} \quad \int_0^4 \ln(x^2+4) dx$$

$$\text{(b)} \quad \int_0^\pi x \sin 2x dx$$

$$\text{(f)} \quad \int_0^{\frac{\pi}{3}} \tan^4(x) dx$$

$$\text{(c)} \quad \int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx$$

$$\text{(g)} \quad \int_0^1 \frac{x}{x^2+4x+5} dx$$

$$\text{(d)} \quad \int_0^5 |x^2-4x+3| dx$$

$$\text{(h)} \quad \int_0^5 \frac{x}{\sqrt{9-x}} dx$$

**Solution:**

$$\begin{aligned}
\text{(a)} \quad & \int_0^1 x^3 \sqrt{1+3x^2} dx = \int_0^1 \frac{1}{2} x^2 \sqrt{1+3x^2} dx^2 = \frac{1}{2} \int_0^1 t \sqrt{1+3t} dt = \frac{1}{2} \int_0^1 t d\left(\frac{2}{9}(1+3t)^{\frac{3}{2}}\right) \\
& = \frac{1}{9} \left[ t(1+3t)^{\frac{3}{2}} \right]_0^1 - \frac{1}{9} \int_0^1 (1+3t)^{\frac{3}{2}} dt \\
& = \frac{8}{9} - \frac{1}{9} \left[ \frac{2}{15} (1+3t)^{\frac{5}{2}} \right]_0^1 = \boxed{\frac{58}{135}}
\end{aligned}$$

$$\text{(b)} \quad \int_0^\pi x \sin 2x dx = \int_0^\pi -\frac{1}{2} x d \cos 2x = -\frac{1}{2} \left( [x \cos 2x]_0^\pi - \int_0^\pi \cos 2x dx \right) = \boxed{-\frac{\pi}{2}}$$

$$\begin{aligned}
\text{(c)} \quad & \int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx = -\int_0^1 x d\sqrt{4-x^2} = -(x\sqrt{4-x^2})_0^1 - \int_0^1 \sqrt{4-x^2} dx \\
& = -\sqrt{3} + \int_0^1 \frac{4}{\sqrt{4-x^2}} dx - \int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx,
\end{aligned}$$

so we have

$$\begin{aligned}
& \int_0^1 \frac{x^2}{\sqrt{4-x^2}} dx = \frac{1}{2} \left( -\sqrt{3} + 4 \int_0^1 \frac{1}{\sqrt{1-(x/2)^2}} d(x/2) \right) \\
& = -\frac{\sqrt{3}}{2} + 2 \arcsin\left(\frac{x}{2}\right) \Big|_0^1 = \boxed{-\frac{\sqrt{3}}{2} + \frac{\pi}{3}}.
\end{aligned}$$



$$\begin{aligned}
\text{(d)} \quad \int_0^5 |x^2 - 4x + 3| dx &= \int_0^5 |(x-1)(x-3)| dx \\
&= \int_0^1 (x^2 - 4x + 3) dx - \int_1^3 (x^2 - 4x + 3) dx + \int_3^5 (x^2 - 4x + 3) dx \\
&= \left(\frac{1}{3} - 2 + 3\right) \times 2 - \left(\frac{1}{3} \cdot 3^3 - 2 \cdot 3^2 + 3 \cdot 3\right) \times 2 + \left(\frac{1}{3} \cdot 5^3 - 2 \cdot 5^2 + 3 \cdot 5\right) \\
&= \boxed{\frac{28}{3}}
\end{aligned}$$

$$\begin{aligned}
\text{(e)} \quad \int_0^4 \ln(x^2 + 4) dx &= \int_0^4 \ln(x^2 + 4) \cdot 1 dx \\
&= x \ln(x^2 + 4) \Big|_0^4 - \int_0^4 \frac{2x^2}{x^2 + 4} dx \\
&= x \ln(x^2 + 4) \Big|_0^4 - \int_0^4 \left(2 - \frac{8}{x^2 + 4}\right) dx \\
&= x \ln(x^2 + 4) \Big|_0^4 - \int_0^4 2 dx + \int_0^4 4 \cdot \frac{1/2}{1 + (x/2)^2} dx \\
&= x \ln(x^2 + 4) \Big|_0^4 - 2x \Big|_0^4 + 4 \arctan\left(\frac{x}{2}\right) \Big|_0^4 \\
&= \boxed{4 \ln(20) - 8 + 4 \arctan(2)}
\end{aligned}$$

$$\begin{aligned}
\text{(f)} \quad \int_0^{\pi/3} \tan^4(x) dx &= \int_0^{\pi/3} \tan^2(x) (\sec^2(x) - 1) dx \\
&= \int_0^{\pi/3} (\tan^2(x) \sec^2(x) - \tan^2(x)) dx \\
&= \int_0^{\pi/3} (\tan^2(x) \sec^2(x) - \sec^2(x) + 1) dx \\
&= \left(\frac{\tan^3(x)}{3} - \tan(x) + x\right) \Big|_0^{\pi/3} \\
&= \frac{(\sqrt{3})^3}{3} - \sqrt{3} + \frac{\pi}{3} = \boxed{\frac{\pi}{3}}.
\end{aligned}$$

$$\begin{aligned}
\text{(g)} \quad \int_0^1 \frac{x}{x^2 + 4x + 5} dx &= \int_0^1 \frac{(x+2)}{(x+2)^2 + 1} dx - \int_0^1 \frac{2}{(x+2)^2 + 1} dx \\
&= \left(\frac{1}{2} \ln[(x+2)^2 + 1]\right) \Big|_0^1 - \left(2 \arctan(x+2)\right) \Big|_0^1 \\
&= \frac{1}{2} \ln(10) - \frac{1}{2} \ln(5) - 2 \arctan(3) + 2 \arctan(2) \\
&= \boxed{\frac{1}{2} \ln(2) - 2 \arctan(3) + 2 \arctan(2)}
\end{aligned}$$

$$\begin{aligned}
\text{(h)} \quad \int_0^5 \frac{x}{\sqrt{9-x}} dx &= - \int_9^4 \frac{9-u}{\sqrt{u}} du = \int_4^9 \left(9u^{-\frac{1}{2}} - u^{\frac{1}{2}}\right) du \\
&= \left(18u^{\frac{1}{2}} - \frac{2}{3}u^{\frac{3}{2}}\right) \Big|_4^9 = 18 \times (9^{\frac{1}{2}} - 4^{\frac{1}{2}}) - \frac{2}{3} \times (9^{\frac{3}{2}} - 4^{\frac{3}{2}}) = \boxed{\frac{16}{3}}
\end{aligned}$$

8. Evaluate the following indefinite integrals by using the  $t$ -substitution:

$$(a) \int \frac{1}{3 + 2 \sin x + \cos x} dx$$

$$(b) \int \frac{1}{2 + \cos x} dx$$

**Solution:** We use the substitution  $t = \tan\left(\frac{x}{2}\right)$  below. Then  $\sin(x) = \frac{2t}{1+t^2}$ ,  $\cos(x) = \frac{1-t^2}{1+t^2}$  and  $dx = \frac{2}{1+t^2} dt$ , and so we have the following results.

(a)

$$\begin{aligned} & \int \frac{1}{3 + 2 \sin(x) + \cos(x)} dx \\ &= \int \frac{1}{3 + 2 \cdot \frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\ &= \int \frac{2}{2t^2 + 4t + 4} dt = \int \frac{1}{t^2 + 2t + 2} dt \\ &= \int \frac{1}{1 + (t+1)^2} dt \\ &= \arctan(t+1) + C \\ &= \boxed{\arctan\left(\tan\left(\frac{x}{2}\right) + 1\right) + C}, \text{ where } C \text{ is an arbitrary constant.} \end{aligned}$$

(b)

$$\begin{aligned} & \int \frac{1}{2 + \cos(x)} dx \\ &= \int \frac{1}{2 + \frac{1-t^2}{1+t^2}} \cdot \frac{2}{1+t^2} dt \\ &= \int \frac{2}{3+t^2} dt = \int \frac{2}{\sqrt{3}} \cdot \frac{1/\sqrt{3}}{1 + (t/\sqrt{3})^2} dt \\ &= \frac{2}{\sqrt{3}} \arctan\left(\frac{t}{\sqrt{3}}\right) + C \\ &= \boxed{\frac{2}{\sqrt{3}} \arctan\left(\frac{\tan(x/2)}{\sqrt{3}}\right) + C}, \text{ where } C \text{ is an arbitrary constant.} \end{aligned}$$

9. Suppose that  $f : [-1, 1] \rightarrow (0, \infty)$  is an even, continuous function such that

- $\int_{-1}^1 f(x) dx = 1$ ;
- $f(x)$  is strictly increasing over  $[-1, 0]$ .

(a) Show that  $f(x)$  attains its global maximum at 0.

(b) Let

$$G(r) = \int_{-r}^r f(x) dx.$$

Use first principles to show that  $G(r)$  is differentiable over  $[-1, 1]$  and find its derivative.

**Solution:**

(a) Since  $f(x)$  is an even function on  $[-1, 1]$ , and it is strictly increasing over  $[-1, 0]$ , we know that  $f(x)$  is strictly decreasing over  $[0, 1]$ . Hence,  $f(x)$  attains its global maximum at  $x = 0$ .

(b) Since  $f(x)$  is even on  $[-1, 1]$ ,

$$f(-x) = f(x), \text{ for all } x \in [-1, 1].$$

We can get the result that, for all  $r \in [-1, 1]$ ,

$$\int_{-r}^0 f(x) dx = \int_0^r f(x) dx. \text{ (Substitute } -x \text{ for } x).$$

So we have

$$G(r) = \int_{-r}^r f(x) dx = 2 \int_0^r f(x) dx.$$

Hence, by the Fundamental Theorem of the Calculus, we know that  $G(r)$  is differentiable on  $[-1, 1]$ , since  $f(x)$  is continuous on  $[-1, 1]$ . We have

$$G'(r) = \boxed{2f(r)}, \text{ for all } r \in [-1, 1].$$

10. By considering a suitable integral, evaluate the following limits:

$$(a) \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \ln \left( \frac{n+k}{n} \right)^{\frac{1}{n}}$$

$$(b) \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n^2 + k^2}{n^3 + k^3}$$

**Solution:**

$$\begin{aligned} (a) \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \ln \left( \frac{n+k}{n} \right)^{\frac{1}{n}} &= \lim_{n \rightarrow \infty} \sum_{k=0}^{n-1} \frac{1}{n} \ln \left( 1 + \frac{k}{n} \right) \\ &= \int_0^1 \ln(1+x) dx \\ &= [x \ln(1+x)] \Big|_0^1 - \int_0^1 \frac{x}{1+x} dx \text{ (Integration by part)} \\ &= [x \ln(1+x)] \Big|_0^1 - \int_0^1 \left( 1 - \frac{1}{1+x} \right) dx \\ &= [x \ln(1+x)] \Big|_0^1 - (x - \ln|1+x|) \Big|_0^1 \\ &= \ln 2 - (1 - \ln 2) \\ &= \boxed{2 \ln 2 - 1} \end{aligned}$$

$$\begin{aligned}
\text{(b)} \quad & \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{n^2 + k^2}{n^3 + k^3} \\
&= \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{1 + \left(\frac{k}{n}\right)^2}{1 + \left(\frac{k}{n}\right)^3} \cdot \frac{1}{n} \\
&= \int_0^1 \frac{1 + x^2}{1 + x^3} dx \\
&= \int_0^1 \frac{1 + x^2}{(x+1)(x^2 - x + 1)} dx \\
&= \int_0^1 \left( \frac{2}{3} \cdot \frac{1}{x+1} + \frac{1}{3} \cdot \frac{x+1}{x^2 - x + 1} \right) dx \\
&= \frac{2}{3} \int_0^1 \frac{1}{x+1} dx + \frac{1}{3} \int_0^1 \frac{x+1}{x^2 - x + 1} dx \\
&= \frac{2}{3} \int_0^1 \frac{1}{x+1} dx + \frac{1}{6} \int_0^1 \frac{(2x-1) + 3}{x^2 - x + 1} dx \\
&= \frac{2}{3} \int_0^1 \frac{1}{x+1} dx + \frac{1}{6} \int_0^1 \frac{2x-1}{x^2 - x + 1} dx + \frac{1}{2} \int_0^1 \frac{1}{\left(x - \frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} dx \\
&= \frac{2}{3} (\ln|x+1|) \Big|_0^1 + \frac{1}{6} (\ln|x^2 - x + 1|) \Big|_0^1 + \frac{1}{2} \left( \frac{2}{\sqrt{3}} \arctan \frac{2x-1}{\sqrt{3}} \right) \Big|_0^1 \\
&= \boxed{\frac{2}{3} \ln 2 + \frac{\pi}{3\sqrt{3}}}
\end{aligned}$$

11. Let  $f_0$  be a continuous function on  $\mathbb{R}$ . For each positive integer  $n$ , define the function  $f_n$  on  $\mathbb{R}$  by  $f_n(x) = \int_0^x f_{n-1}(t) dt$  for any  $x \in \mathbb{R}$ .

(a) Show that whenever  $m, n$  are positive integers,

$$\int_0^x (x-t)^{m-1} f_n(t) dt = \frac{1}{m} \int_0^x (x-t)^m f_{n-1}(t) dt$$

for any  $x \in \mathbb{R}$ .

(b) Show that whenever  $n$  is a positive integer,  $f_n(x) = \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f_0(t) dt$  for any  $x \in \mathbb{R}$ .

**Solution:**

(a) Note that, for any positive integer  $n$ ,  $f_n(0) = 0$  and

$$f'_n(x) = \frac{d}{dx} \int_0^x f_{n-1}(t) dt = f_{n-1}(x) \quad \text{for all } x \in \mathbb{R},$$

by the Fundamental Theorem of Calculus.

Hence, for any  $x \in \mathbb{R}$ , integration by parts yields

$$\begin{aligned}
 & \int_0^x (x-t)^{m-1} f_n(t) dt \\
 &= \int_0^x -\frac{1}{m} f_n(t) \frac{d}{dt} ((x-t)^m) dt \\
 &= \left[ -\frac{1}{m} (x-t)^m \cdot f_n(t) \right]_0^x - \int_0^x -\frac{1}{m} (x-t)^m \cdot f_n'(t) dt \\
 &= \left[ -\frac{1}{m} (x-x)^m \cdot f_n(x) \right] - \left[ -\frac{1}{m} (x-0)^m \cdot f_n(0) \right] - \int_0^x -\frac{1}{m} (x-t)^m \cdot f_{n-1}(t) dt \\
 &= \frac{1}{m} \int_0^x (x-t)^m f_{n-1}(t) dt.
 \end{aligned}$$

(b) Let  $n$  be a positive integer and let  $x \in \mathbb{R}$ . By (a), we have

$$\begin{aligned}
 f_n(x) &= \int_0^x f_{n-1}(t) dt \\
 &= \int_0^x (x-t)^0 f_{n-1}(t) dt \\
 &= \frac{1}{1} \int_0^x (x-t)^1 f_{n-2}(t) dt \\
 &= \frac{1}{1 \cdot 2} \int_0^x (x-t)^2 f_{n-3}(t) dt \\
 &= \dots \\
 &= \frac{1}{k!} \int_0^x (x-t)^k f_{n-k-1}(t) dt \\
 &= \dots \\
 &= \frac{1}{(n-1)!} \int_0^x (x-t)^{n-1} f_0(t) dt.
 \end{aligned}$$