THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2024-2025 Term 1
Homework Assignment 4
Due Date: 9 December 2024

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

Signature ______________________ Date ______________________

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests here as well. For submitting your PDF homework on Gradescope, here are a few tips.

  Where is Gradescope?

  Do the following:

  1. Go to **2024R1 University Mathematics (MATH1010ABCDEF)**
  2. Go to **Course Contents**
  3. The green **Gradescope** icon will be there

- Late assignments will receive a grade of 0.

- For the declaration sheet:

  Either

  Print out the cover sheet (i.e. the first page of this document), and sign and date the statement of Academic Honesty. Use the attached file, sign and date the statement of Academic Honesty, convert it into a PDF and submit it with your homework assignments via Gradescope.

Or

Write your name on the first page of your submitted homework, and simply write out the sentence "I have read the university regulations."

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.

- Write your solutions on A4 white paper or use an iPad or other similar device to present your answers and submit a digital form via Gradescope. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Please be aware that you can only use a ball-point pen to write your answers for any exams.

- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. Evaluate the following integrals:

(a) $\displaystyle\int x\sec^2 x dx$

(b) $\displaystyle\int \sec^2 x \ln\tan x dx$

(c) $\displaystyle\int e^{-x}\sin 3x\, dx$

(d) $\displaystyle\int \frac{x^5}{(1+x^3)^3}dx$

(e) $\displaystyle\int x^2\ln\frac{1+x}{1-x}dx$

(f) $\displaystyle\int e^x\left(\frac{1+\sin x}{1+\cos x}\right)dx$

2. Evaluate the following integrals:

(a) $\displaystyle\int \sin^4 x\ \cos^3 x dx$

(b) $\displaystyle\int \cos^4 x\ \sin^2 x dx$

(c) $\displaystyle\int \sec x\ \tan^3 x dx$

(d) $\displaystyle\int \sec^4 x\ \tan^6 x dx$

(e) $\displaystyle\int \frac{2e^x}{e^{2x}-4}dx$

(f) $\displaystyle\int \sec x dx$

(g) $\displaystyle\int \frac{1}{\sqrt{1-x^2}}dx$

(h) $\displaystyle\int \frac{1}{1+x^2}dx$

3. Evaluate the following integrals by trigonometric substitutions:

(a) $\displaystyle\int \frac{x^2dx}{(1-x^2)^{\frac{3}{2}}}$

(b) $\displaystyle\int \frac{dx}{\sqrt{4+x^2}}$

(c) $\displaystyle\int \sqrt{4-x^2}\, dx$

(d) $\displaystyle\int \frac{1}{(x^2+1)^2}\, dx$

4. Prove the following reduction formulas.

(a) $\displaystyle I_n = \int \frac{x^n dx}{\sqrt{x+a}};\quad I_n = \frac{2x^n\sqrt{x+a}}{2n+1} - \frac{2an}{2n+1}I_{n-1},\ n \geq 1.$

(b) $\displaystyle I_n = \int \sin^n x dx;\quad I_n = -\frac{\cos x\sin^{n-1}x}{n} + \frac{n-1}{n}I_{n-2},\ n \geq 2.$

(c) $\displaystyle I_n = \int \frac{1}{x^n(x+1)}dx;\quad I_n = \frac{1}{-n+1}x^{-n+1} - I_{n-1},\ n \geq 2.$

(d) $\displaystyle I_n = \int_0^\pi x^n\sin x\, dx;\quad I_n = \pi^n - n(n-1)I_{n-2},\ n \geq 2$, then find I_6.

5. Find $F'(x)$ for the following functions:

(a) $\displaystyle F(x) = \int_\pi^x \frac{\cos y}{y}\, dy$

(b) $\displaystyle F(x) = \int_0^{x^3} e^{u^2}\, du$

(c) $\displaystyle F(x) = \int_x^{2x} (\ln t)^2\, dt$

(d) $\displaystyle F(x) = \int_{x^2}^{x^3} e^{\cos u}\, du$

(e) $\displaystyle F(x) = \int_1^x \frac{e^x+e^t}{t}dt$

(f) $\displaystyle F(x) = \int_{-\sqrt{\ln x}}^{\sqrt{\ln x}} \frac{\sin t}{t}dt$

6. Evaluate the following integrals of rational functions:

(a) $\displaystyle\int \frac{x^2}{1-x^2}dx$

(b) $\displaystyle\int \frac{4x+1}{x^2-6x+13}dx$

(c) $\displaystyle\int \frac{2x^3-x^2+3}{x^2-2x-3}dx$

(d) $\displaystyle\int \frac{x^2+1}{(x+1)^2(x-1)}dx$

(e) $\displaystyle\int \frac{2x^2-2}{2x^2-5x+2}dx$

(f) $\displaystyle\int \frac{-x+1}{2x^2+4x+5}dx$

7. Evaluate the following definite integrals:

(a) $\displaystyle\int_0^1 x^3\sqrt{1+3x^2}\,dx$

(b) $\displaystyle\int_0^{\pi} x\sin 2x\,dx$

(c) $\displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$

(d) $\displaystyle\int_0^5 |x^2-4x+3|\,dx$

(e) $\displaystyle\int_0^4 \ln(x^2+4)dx$

(f) $\displaystyle\int_0^{\frac{\pi}{3}} \tan^4(x)dx$

(g) $\displaystyle\int_0^1 \frac{x}{x^2+4x+5}dx$

(h) $\displaystyle\int_0^5 \frac{x}{\sqrt{9-x}}dx$

8. Evaluate the following indefinite integrals by using the t-substitution:

(a) $\displaystyle\int \frac{1}{3+2\sin x+\cos x}\,dx$

(b) $\displaystyle\int \frac{1}{2+\cos x}\,dx$

9. Suppose that $f : [-1,1] \to (0,\infty)$ is an even, continuous function such that

- $\displaystyle\int_{-1}^1 f(x)\,dx = 1$;
- $f(x)$ is strictly increasing over $[-1,0]$.

(a) Show that $f(x)$ attains its global maximum at 0.

(b) Let

$$G(r) = \int_{-r}^{r} f(x)\,dx.$$

Use first principles to show that $G(r)$ is differentiable over $[-1,1]$ and find its derivative.

10. By considering a suitable integral, evaluate the following limits:

(a) $\displaystyle\lim_{n\to\infty} \sum_{k=0}^{n-1} \ln\left(\frac{n+k}{n}\right)^{\frac{1}{n}}$

(b) $\displaystyle\lim_{n\to\infty} \sum_{k=1}^{n} \frac{n^2+k^2}{n^3+k^3}$

11. Let f_0 be a continuous function on $\mathbb{R}$. For each positive integer n, define the function f_n on $\mathbb{R}$ by $f_n(x) = \displaystyle\int_0^x f_{n-1}(t)dt$ for any $x \in \mathbb{R}$.

(a) Show that whenever m, n are positive integers,

$$\int_0^x (x-t)^{m-1} f_n(t)dt = \frac{1}{m}\int_0^x (x-t)^m f_{n-1}(t)dt$$

for any $x \in \mathbb{R}$.

(b) Show that whenever n is a positive integer, $f_n(x) = \dfrac{1}{(n-1)!}\displaystyle\int_0^x (x-t)^{n-1} f_0(t)dt$ for any $x \in \mathbb{R}$.