THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2024-2025 Term 1
Suggested Solutions of Homework Assignment 3
Due Date: December 2, 2024

1. Let $f(x) = \dfrac{x-1}{x-3}$.

   (a) Is Lagrange's mean value theorem applicable to f on the interval $[4, 5]$?

   (b) If your answer to part (a) is yes, find all possible values $c \in (4, 5)$, at which point(s) the tangent line to the graph is parallel to the secant line connecting the two end points $(4, f(4))$ and $(5, f(5))$.

   Solution

   (a) Note that the function has a discontinuity at $x = 3$, while on the interval $[4, 5]$ it is continuous and differentiable, so the mean value theorem is applicable here. The derivative is
   $$f'(x) = -\frac{2}{(x-3)^2}.$$

   (b) By the Lagrange's mean value theorem, let
   $$f'(c) = \frac{f(5) - f(4)}{5 - 4} = \frac{2 - 3}{1} = -1,$$
   choosing the root that lies in $(4, 5)$. Then $\boxed{c = 3 + \sqrt{2}}$.

2. By using Lagrange's mean value theorem, or otherwise, show that

   (a) $\sin x \leq x$ for all $x \in [0, +\infty)$.

   (b) $(1 + x)^p \geq 1 + px$ for any $p \geq 1$ and $x \geq 0$.

   Solution

   (a) Let $f(x) = x - \sin x$. We want to show that $f(x) \geq 0$ for all $x \in [0, +\infty)$. Since $f(t)$ is continuous on $[0, x]$ and differentiable on $(0, x)$, one can apply Lagrange's MVT, then
   $$1 - \cos c = f'(c) = \frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x}.$$
   Since $1 - \cos c \geq 0$ for any c, and $x \geq 0$, thus $f(x) \geq 0$ for all $x \in [0, +\infty)$.

   (b) The equality holds when $p = 1$ or $x = 0$. When $p > 1$, $x > 0$, let
   $$f(x) = (1 + x)^p - 1 - px.$$
   It suffices to show that $f(x) > 0$ for all $x > 0$. For any $x > 0$, since f is continuous and differentiable on $[0, x]$, by Lagrange's MVT, there exists $c \in (0, x)$, such that
   $$\frac{f(x) - f(0)}{x - 0} = \frac{f(x)}{x} = f'(c) = p(1 + c)^{p-1} - p = p((1 + c)^{p-1} - 1^{p-1}) > 0,$$

where the last inequality is due to that the function x^a for $a > 0$, $x > 0$ is a strictly increasing function. Therefore, $f(x) > 0$ for all $x \geq 0$.

3. Let $0 < a < b < \frac{\pi}{2}$. Prove that there exists $a < \xi < b$ such that

$$\ln\left(\frac{\cos a}{\cos b}\right) = (b-a)\tan\xi.$$

Solution

Fixed any $0 < a < b < \frac{\pi}{2}$, define $f : [a,b] \to \mathbb{R}$ by

$$f(x) = \ln\cos x$$

for any $x \in [a,b]$. Note that $\cos x$ is continuous on $[a,b]$, differentiable on (a,b) and

$$\cos b < \cos x < \cos a$$

for $0 < a < x < b < \frac{\pi}{2}$. Moreover, $\ln x$ is continuous on $[\cos b, \cos a]$ and differentiable on $(\cos b, \cos a)$.

Hence, f is continuous on $[a,b]$ and differentiable on (a,b) with

$$f'(x) = \frac{(\cos x)'}{\cos x} = -\tan x$$

for any $x \in (a,b)$.

Using Lagrange's mean value theorem, there exist some $\xi \in (a,b)$, such that

$$f'(\xi) = \frac{f(a) - f(b)}{a-b},$$

that is,

$$-\tan\xi = \frac{\ln\cos a - \ln\cos b}{a-b}.$$

Since $\ln x - \ln y = \ln(\frac{x}{y})$, we have $\ln\cos a - \ln\cos b = \ln\left(\frac{\cos a}{\cos b}\right)$, and hence

$$\ln\left(\frac{\cos a}{\cos b}\right) = (b-a)\tan\xi.$$

4. Show that for all $0 < a < b \leq 1$,

$$(b-a)(1+\ln a) < \ln(\frac{b^b}{a^a}) < (b-a)(1+\ln b).$$

Solution

(a) Let $f(x) = x\ln x$ for $x > 0$. Consider $0 < a < b$, we know that f is continuous on $[a, b]$ and differentiable on (a, b). By the Lagrange's mean value theorem, there exists $c \in (a, b)$ such that

$$\frac{f(b)-f(a)}{b-a} = \frac{b\ln b - a\ln a}{b-a} = f^{'}(c) = 1 + \ln c.$$

Since $a < c < b$, $\ln a < \ln c < \ln b$. Thus

$$1 + \ln a < \frac{b\ln b - a\ln a}{b-a} < 1 + \ln b.$$

That is,

$$(b-a)(1+\ln a) < \ln(\frac{b^b}{a^a}) < (b-a)(1+\ln b).$$

5. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 0}\frac{\sin^{-1} x - \tan^{-1} x}{x^3}$

(b) $\displaystyle\lim_{x\to 0^+}\log_{\tan x}(\tan 2x)$

(c) $\displaystyle\lim_{x\to 0^+}\tan x \ln\sin x$

(d) $\displaystyle\lim_{x\to 1}\left(\frac{1}{\ln x} - \frac{x}{x-1}\right)$

(e) $\displaystyle\lim_{x\to +\infty}\frac{e^{1+\ln x}}{\ln(1+e^x)}$

Solution

(a) We compute the Taylor series of $\sin^{-1} x$ and $\tan^{-1} x$ at $x = 0$ to the third order:

$$\begin{aligned}
(\sin^{-1})'(x) &= (1-x^2)^{\frac{-1}{2}} \\
(\sin^{-1})''(x) &= x(1-x^2)^{\frac{-3}{2}} \\
(\sin^{-1})'''(x) &= (1+2x^2)(1-x^2)^{\frac{-5}{2}}
\end{aligned}$$

$$\begin{aligned}
(\tan^{-1})'(x) &= (1+x^2)^{-1} \\
(\tan^{-1})''(x) &= -2x(1+x^2)^{-2} \\
(\tan^{-1})'''(x) &= -2(1-x^2)(1+x^2)^{-3}
\end{aligned}$$

So the Taylor series are

$$\sin^{-1}(x) = x + \frac{x^3}{6} + O(x^4)$$

and

$$\tan^{-1}(x) = x - \frac{x^3}{3} + O(x^4)$$

Hence the limit is

$$\begin{aligned}\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3} &= \lim_{x\to 0}\frac{(x+\frac{1}{6}x^3+O(x^4))-(x-\frac{1}{3}x^3+O(x^4))}{x^3}\\ &= \lim_{x\to 0}\frac{\frac{1}{2}x^3+O(x^4)}{x^3}\\ &= \boxed{\frac{1}{2}}\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 0^+}\log_{\tan x}(\tan 2x) &= \lim_{x\to 0}\frac{\ln\tan 2x}{\ln\tan x}\\ &= \lim_{x\to 0^+}\frac{(\ln\tan 2x)'}{(\ln\tan x)'}\\ &= \lim_{x\to 0^+} 2\frac{\tan x\cos^2 x}{\tan 2x\cos^2 2x}\\ &= \lim_{x\to 0^+} 2\frac{\sin 2x}{\sin 4x}\\ &= \lim_{x\to 0^+}\frac{\sin 2x}{2x}\lim_{x\to 0^+}\frac{4x}{\sin 4x}\\ &= \boxed{1}\end{aligned}$$

(c)

$$\begin{aligned}\lim_{x\to 0^+}\tan x\ln\sin x &= \lim_{x\to 0^+}\frac{\ln\sin x}{\tan x}\\ &= \lim_{x\to 0^+}\frac{(\ln\sin x)'}{(\tan x)'}\\ &= \lim_{x\to 0^+}\frac{\frac{1}{\sin x}\cos x}{\frac{-1}{\tan^2 x}\sec^2 x}\\ &= \lim_{x\to 0^+} -\sin x\cos x = \boxed{0}\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{\ln x}-\frac{x}{x-1}\right) &= \lim_{x\to 1}\frac{x-1-\ln x}{(x-1)\ln x}-1\\ &= \lim_{x\to 1}\frac{(x-1-\ln x)'}{((x-1)\ln x)'}-1\\ &= \lim_{x\to 1}\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}-1\\ &= \lim_{x\to 1}\frac{x-1}{x\ln x+x-1}-1\\ &= \lim_{x\to 1}\frac{(x-1)'}{(x\ln x+x-1)'}-1\\ &= \lim_{x\to 1}\frac{1}{\ln x+1}-1 = \boxed{0}\end{aligned}$$

(e)

$$\begin{aligned}
\lim_{x\to+\infty}\frac{e^{1+\ln x}}{\ln(1+e^x)} &= \lim_{x\to+\infty}\frac{xe}{\ln(1+e^x)}\\
&= \lim_{x\to+\infty}\frac{(xe)'}{(\ln(1+e^x))'}\\
&= \lim_{x\to+\infty}\frac{e}{\frac{1}{1+e^x}e^x}\\
&= \lim_{x\to+\infty} e(1+e^{-x}) = \boxed{e}
\end{aligned}$$

6. Evaluate the following limits.

(a) $\displaystyle\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

(b) $\displaystyle\lim_{x\to 1} x^{\frac{2x}{x-1}}$

(c) $\displaystyle\lim_{x\to 0}\frac{(1+x)^x-1}{x^2}$

(d) $\displaystyle\lim_{x\to+\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$

Solution

(a)

$$\begin{aligned}
\lim_{x\to 0}\frac{1}{x^2}\ln\frac{\sin x}{x} &= \lim_{x\to 0}\frac{(\ln\frac{\sin x}{x})'}{(x^2)'}\\
&= \lim_{x\to 0}\frac{\frac{x}{\sin x}(\frac{\cos x}{x}-\frac{\sin x}{x^2})}{2x}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{x\cos x-\sin x}{x^2\sin x}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{(x\cos x-\sin x)'}{(x^2\sin x)'}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{-x\sin x}{2x\sin x+x^2\cos x}\\
&= \frac{1}{2}\lim_{x\to 0}\frac{-1}{2+\frac{x}{\tan x}}\\
&= \frac{1}{2}\frac{-1}{2+1} = -\frac{1}{6}
\end{aligned}$$

So

$$\begin{aligned}
\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}} &= e^{\lim_{x\to 0}\frac{1}{x^2}\ln\frac{\sin x}{x}}\\
&= \boxed{e^{\frac{-1}{6}}}
\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x\to 1}\frac{2x}{x-1}\ln x &= 2\lim_{x\to 1}\frac{\ln x}{1-\frac{1}{x}}\\ &= 2\lim_{x\to 1}\frac{(\ln x)'}{(1-\frac{1}{x})'}\\ &= 2\lim_{x\to 1}\frac{\frac{1}{x}}{\frac{1}{x^2}} = 2\end{aligned}$$

So

$$\begin{aligned}\lim_{x\to 1} x^{\frac{2x}{x-1}} &= e^{\lim\limits_{x\to 1}\frac{2x}{x-1}\ln x}\\ &= \boxed{e^2}\end{aligned}$$

(c) We compute the Taylor series of $f(x) = (1+x)^x = e^{x\ln(1+x)}$ at $x = 0$ up to x^2:

$$f'(x) = e^{x\ln(1+x)}(\ln(1+x) + 1 - \frac{1}{1+x})$$

$$f''(x) = e^{x\ln(1+x)}(\ln(1+x) + 1 - \frac{1}{1+x})^2 + e^{x\ln(1+x)}\frac{x+2}{(1+x)^2}$$

As $f(0) = e^{0\ln 1} = 1, f'(0) = e^{0\ln 1}(\ln 1 + 1 - \frac{0}{1+0}) = 0,\ f''(0) = e^{0\ln 1}(\ln 1 + 1 - \frac{0}{1+0})^2 + e^{0\ln 1}\frac{0+2}{(1+0)^2} = 2$, we have $(1+x)^x = 1 + x^2 + O(x^3)$, so

$$\lim_{x\to 0}\frac{(1+x)^x - 1}{x^2} = \lim_{x\to 0}\frac{x^2 + O(x^3)}{x^2} = \boxed{1}$$

(d)

$$\begin{aligned}\lim_{x\to+\infty} x\ln\frac{(x-1)^2}{x^2-4x+2} &= \lim_{x\to+\infty}\frac{\ln\frac{(x-1)^2}{x^2-4x+2}}{x^{-1}}\\ &= \lim_{x\to+\infty}\frac{(\ln\frac{(x-1)^2}{x^2-4x+2})'}{(x^{-1})'}\\ &= \lim_{x\to+\infty}\frac{2}{-x^{-2}}\frac{-x}{(x-1)(x^2-4x+2)} = 2\end{aligned}$$

So

$$\lim_{x\to+\infty}\left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x = e^{\lim\limits_{x\to+\infty} x\ln\frac{(x-1)^2}{x^2-4x+2}} = \boxed{e^2}$$

7. Find the x-intercepts, y-intercepts, asymptotes if there is any and sketch the graphs of the following functions.

(a) $y = \dfrac{x+5}{x-2}$

(b) $y = \dfrac{x^2-2}{x-1}$

(c) $y = |4 + 3x - x^2|$

(d) $y = x|x+2|$

(e) $y = \left|\dfrac{7-2x}{x+3}\right|$

(f) $y = \dfrac{1}{|x^2-4|}$

Solution

(See next page for the graphs.)

(a) The x-intercept is at where $y = \frac{x+5}{x-2} = 0$, so the x-intercept is $\boxed{(-5,0)}$.

The y-intercept is at where $x = 0$, so the y-intercept is $(0, \frac{0+5}{0-2}) = \boxed{(0, -\dfrac{5}{2})}$.

At $x = 2$, the denominator becomes 0, so $\boxed{x=2}$ is a vertical asymptote.
Since $\lim_{x\to\pm\infty} \frac{y(x)}{x} = 0$ and $\lim_{x\to\pm\infty} y(x) = 1$, $\boxed{y=1}$ is a horizontal asymptote.

(b) The x-intercept is at where $y = \frac{x^2-2}{x-1} = 0$, so the x-intercepts are $\boxed{(\sqrt{2},0)}$ and $\boxed{(-\sqrt{2},0)}$.

The y-intercept is at where $x = 0$, so the y-intercept is $(0, \frac{0^2-2}{0-1}) = \boxed{(0,2)}$.
At $x = 1$, the denominator becomes 0, so $\boxed{x=1}$ is a vertical asymptote.
Since $\lim_{x\to\pm\infty} \frac{y(x)}{x} = 1$ and $\lim_{x\to\pm\infty} y(x) - x = 1$, $\boxed{y = x+1}$ is an asymptote.

(c) The x-intercept is at where $y = |4+3x-x^2| = 0$, so the x-intercepts are $\boxed{(-1,0)}$ and $\boxed{(4,0)}$.
The y-intercept is at where $x = 0$, so the y-intercept is $(0, |4+3\cdot 0 - 0^2|) = \boxed{(0,4)}$.
Since the function has no singularity and $\lim_{x\to\pm\infty} \frac{y(x)}{x} = \pm\infty$, the function has no asymptote.

(d) The x-intercept is at where $y = x|x+2| = 0$, so the x-intercepts are $\boxed{(0,0)}$ and $\boxed{(-2,0)}$.
The y-intercept is at where $x = 0$, so the y-intercept is $(0, 0\cdot|0+2|) = \boxed{(0,0)}$.
Since the function has no singularity and $\lim_{x\to\pm\infty} \frac{y(x)}{x} = +\infty$, the function has no asymptote.

(e) The x-intercept is at where $y = \left|\frac{7-2x}{x+3}\right| = 0$, so the x-intercept is $\boxed{(\dfrac{7}{2},0)}$.

The y-intercept is at where $x = 0$, so the y-intercept is $(0, \left|\frac{7-2\cdot 0}{0+3}\right|) = \boxed{(0, \dfrac{7}{3})}$.

At $x = -3$, the denominator becomes 0, so $\boxed{x=-3}$ is a vertical asymptote.
Since $\lim_{x\to\pm\infty} \frac{y(x)}{x} = 0$ and $\lim_{x\to\pm\infty} y(x) = 2$, $\boxed{y=2}$ is an asymptote.

(f) The x-intercept is at where $y = \frac{1}{|x^2-4|} = 0$, so the function has no x-intercept.

The y-intercept is at where $x = 0$, so the y-intercept is $(0, \frac{1}{|0^2-4|}) = \boxed{(0, \dfrac{1}{4})}$.

At $x = -2$ and at $x = 2$, the denominator becomes 0, so $\boxed{x=2}$ and $\boxed{x=-2}$ are vertical asymptotes.
Since $\lim_{x\to\pm\infty} \frac{y(x)}{x} = 0$ and $\lim_{x\to\pm\infty} y(x) = 0$, $\boxed{y=0}$ is an asymptote.

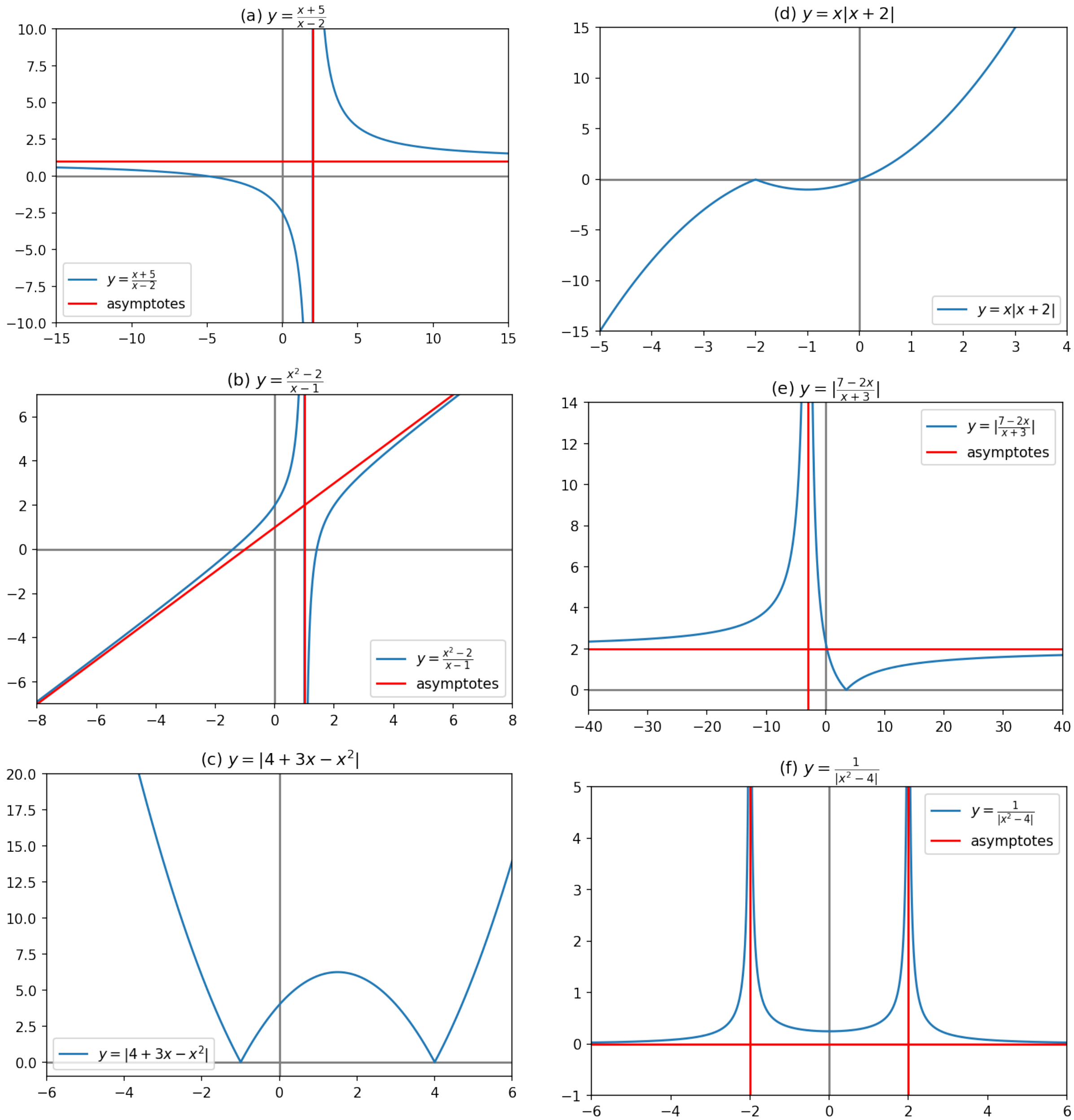

Figure 1: The graphs of the functions for question 7. Asymptotes, if they exist, are also drawn.

8. For each of the following functions $f(x)$, find
   - $f'(x)$ and $f''(x)$.
   - range of values of x for which $f(x)$ is increasing.
   - asymptotes of $y = f(x)$.
   - all relative extremum points

   Then sketch the graph of $y = f(x)$.

(a) $f(x) = \dfrac{x}{(x-2)^2}$

(b) $f(x) = \dfrac{x^2+5x+7}{x+2}$

(c) $f(x) = \dfrac{x^2}{x^2-2x+2}$

(d) $f(x) = x^{\frac{2}{3}} - 1$

Solution

(See next page for the graphs.)

(a)

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x}{(x-2)^2} = \frac{1}{(x-2)^2} - \frac{2x}{(x-2)^3} = \boxed{-\frac{x+2}{(x-2)^3}}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x} - \frac{x+2}{(x-2)^3} = -\left(\frac{1}{(x-2)^3} - \frac{3(x+2)}{(x-2)^4}\right) = \boxed{\frac{2x+8}{(x-2)^4}}$$

f is differentiable on the domain $(-\infty, 2) \cup (2, \infty)$, and $f'(x) > 0$ if and only if $-2 < x < 2$. So f is increasing on $\boxed{[-2, 2)}$.

Since when $x = 2$, the denominator becomes 0, so $\boxed{x = 2}$ is a vertical asymptote.

As $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$ and $\lim_{x\to\pm\infty} f(x) = 0$, $\boxed{y = 0}$ is an asymptote.

The only critical point of $f(x)$ is $x = -2$, at which $f''(-2) = \frac{1}{64} > 0$, so $\boxed{x = -2}$ is the only relative extremum and is a relative minimum.

(b)

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+5x+7}{x+2} = \frac{2x+5}{x+2} - \frac{x^2+5x+7}{(x+2)^2} = \frac{x^2+4x+3}{(x+2)^2} = \boxed{\frac{(x+1)(x+3)}{(x+2)^2}}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2+4x+3}{(x+2)^2} = \frac{2x+4}{(x+2)^2} - (x^2+4x+3)\frac{-2}{(x+2)^3} = \boxed{\frac{2}{(x+2)^3}}$$

f is differentiable on the domain $(-\infty, -2) \cup (-2, \infty)$, and $f'(x) > 0$ if and only if $x < -3$ or $-1 < x$. Also, $f(-3) = -1 < 3 = f(-1)$. So f is increasing on $\boxed{(-\infty, -3] \cup [-1, \infty)}$

Since when $x = -2$, the denominator becomes 0, so $\boxed{x = -2}$ is a vertical asymptote.

As $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 1$ and $\lim_{x\to\pm\infty} f(x) - x = 3$, so $\boxed{y = x + 3}$ is an asymptote.

The only critical points are $x = -1$ and $x = -3$. Since $f''(-1) = 2 > 0$ and $f''(-3) = -2 < 0$, so the only relative extrema are at $x = -1$ and $x = -3$, where $\boxed{x = -1}$ is a relative minimum and $\boxed{x = -3}$ is a relative maximum.

(c)

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{x^2}{x^2-2x+2} = \frac{2x}{x^2-2x+2} - \frac{x^2(2x-2)}{(x^2-2x+2)^2} = \boxed{-\frac{2x(x-2)}{(x^2-2x+2)^2}}$$

$$f''(x) = \frac{-4x+4}{(x^2-2x+2)^2} - \frac{2(-2x^2+4x)(2x-2)}{(x^2-2x+2)^3} = \boxed{\frac{4(x-1)(x^2-2x-2)}{(x^2-2x+2)^3}}$$

f is differential on the domain $(-\infty, \infty)$, and $f'(x) > 0$ if and only if $0 < x < 2$, so f is increasing on $\boxed{[0,2]}$.

As $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$ and $\lim_{x\to\pm\infty} f(x) = 1$, $\boxed{y = 1}$ is an asymptote.

The critical points of f are $x = 0$ and $x = 2$. Since $f''(0) = 1 > 0$ and $f''(2) = -1 < 0$, so the only relative extrema are $x = 0$ and $x = 2$, where $\boxed{x = 0}$ is a relative minimum and $\boxed{x = 2}$ is a relative maximum.

(d)

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}(x^{\frac{2}{3}} - 1) = \frac{2}{3}x^{\frac{-1}{3}} = \boxed{\frac{2}{3\sqrt[3]{x}}}$$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x}\frac{2}{3}x^{\frac{-1}{3}} = -\frac{2}{9}x^{\frac{-4}{3}} = \boxed{-\frac{2}{9\sqrt[3]{x^4}}}$$

f is differentiable on $(-\infty, 0) \cup (0, \infty)$, and $f'(x) > 0$ if and only if $x > 0$. So f is increasing on $\boxed{[0, \infty)}$.

Since $\lim_{x\to\pm\infty} \frac{f(x)}{x} = 0$ but $\lim_{x\to\pm\infty} f(x)$ does not exist. So f has no asymptote.

The only critical points of f are $x = 0$ as f is not differentiable at $x = 0$ and $f'(x) \neq 0$ on $(-\infty, 0) \cup (0, \infty)$. Since for $x \neq 0$, $f(x) = -1 + \sqrt[3]{x^2} \geq -1 = f(0)$, $\boxed{x = 0}$ is the only relative extremum and is a relative minimum.

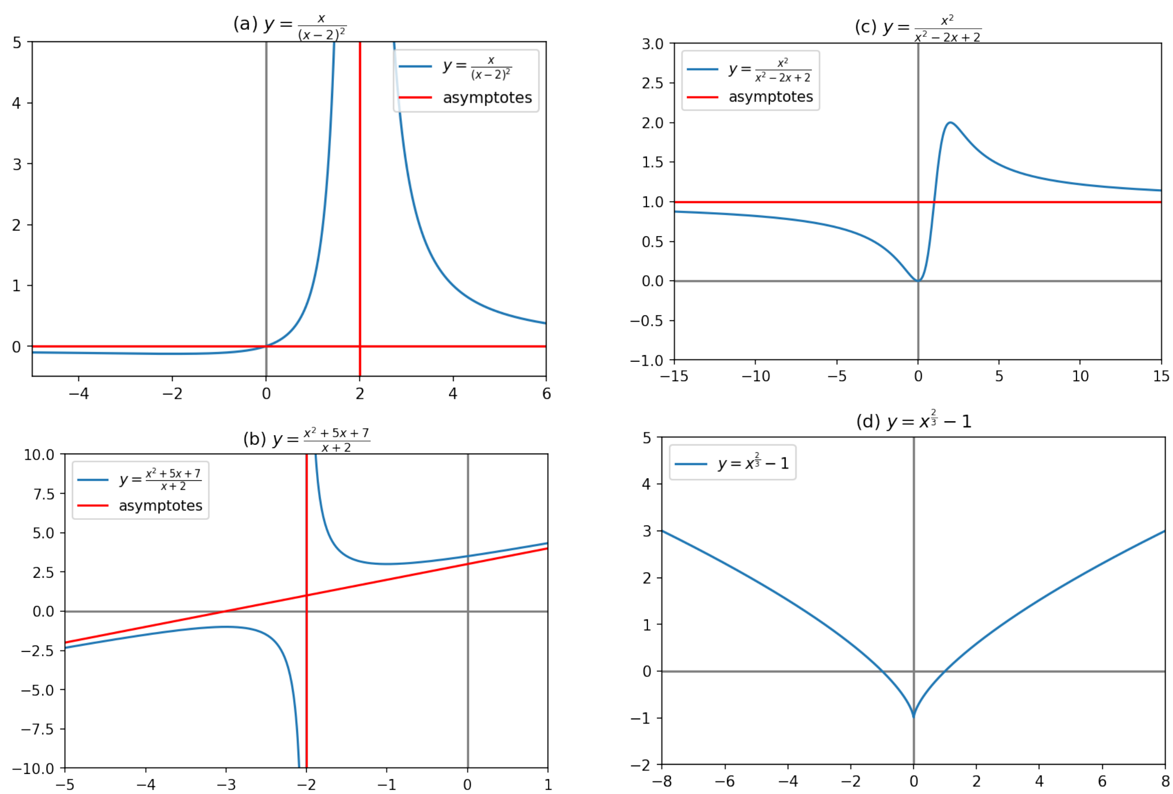

Figure 2: The graphs for question 8. Asymptotes, if they exist, are also drawn.

9. For each of the following functions $f(x)$, find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ and the Taylor series up to the term in x^3 of $f(x)$ about the point $x = 0$.

   (a) $f(x) = \ln \cos x$ (b) $f(x) = e^x \sin x$

Solution

(a) $f(0) = \ln \cos 0 = 0$

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln \cos x = \frac{1}{\cos x}(-\sin x) = -\tan x$$

So $f'(0) = -\tan 0 = 0$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x} - \tan x = -\sec^2 x$$

So $f''(0) = -\sec^2 0 = -1$

$$f'''(x) = \frac{\mathrm{d}}{\mathrm{d}x} - \sec^2 x = \frac{2}{\cos^3}(-\sin x) = -2 \tan x \sec^2 x$$

So $f'''(0) = 0$

So the Taylor series of $f(x) = \ln \cos x$ about $x = 0$ up to x^3 is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + O(x^4) = \boxed{-\frac{1}{2}x^2 + O(x^4)}$$

(b) $f(0) = e^0 \sin 0 = 0$

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x} e^x \sin x = e^x \sin x + e^x \cos x = e^x(\sin x + \cos x)$$

So $f'(0) = e^0(\sin 0 + \cos 0) = 1$

$$f''(x) = \frac{\mathrm{d}}{\mathrm{d}x} e^x(\sin x + \cos x) = e^x(\sin x + \cos x) + e^x(\cos x - \sin x) = 2e^x \cos x$$

So $f''(0) = 2e^0 \cos 0 = 2$

$$f'''(0) = \frac{\mathrm{d}}{\mathrm{d}x} 2e^x \cos x = 2(e^x \cos x - e^x \sin x) = 2e^x(\cos x - \sin x)$$

So $f'''(0) = 2e^0(\cos 0 - \sin 0) = 2$

So the Taylor series of $f(x) = e^x \sin x$ about $x = 0$ up to x^3 is

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + O(x^4) = \boxed{x + x^2 + \frac{1}{3}x^3 + O(x^4)}$$

10. Find the Taylor series up to the term in $(x-c)^3$ of the functions about $x = c$.

(a) $\dfrac{1}{1+x}$; $c = 1$.

(b) $\dfrac{2-x}{3+x}$; $c = 1$.

(c) $\dfrac{x}{(x-1)(x-2)}$; $c = 0$.

(d) $\cos x$; $c = \frac{\pi}{4}$.

(e) $\sin^2 x$; $c = 0$

(f) $\ln x$; $c = e$.

(g) 3^x; $c = 0$.

(h) $\sqrt{2+x}$; $c = 1$.

(i) $\dfrac{1}{\sqrt{7-3x}}$; $c = 1$.

Solution

(a) Let $f(x) = \frac{1}{1+x}$. Then $f(c) = \frac{1}{1+c} = \frac{1}{2}$, $f'(c) = \frac{-1}{(1+c)^2} = -\frac{1}{4}$, $f''(c) = \frac{2}{(1+c)^3} = \frac{1}{4}$, $f'''(c) = \frac{-6}{(1+c)^4} = -\frac{3}{8}$.

So $\frac{1}{1+x} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$

$= \boxed{\frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-c)^3 + O((x-c)^4)}$

(b) Let $f(x) = \frac{2-x}{3+x} = -1 + \frac{5}{3+x}$. Then $f(c) = -1 + \frac{5}{3+c} = \frac{1}{4}$, $f'(c) = \frac{-5}{(3+c)^2} = -\frac{5}{16}$, $f''(c) = \frac{10}{(3+c)^3} = \frac{5}{32}$, $f'''(c) = \frac{-30}{(3+c)^4} = -\frac{15}{128}$.

So $\frac{2-x}{3+x} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$

$= \boxed{\frac{1}{4} - \frac{5}{16}(x-1) + \frac{5}{64}(x-1)^2 - \frac{5}{256}(x-1)^3 + O((x-1)^4)}$

(c) Let $f(x) = \frac{x}{(x-1)(x-2)}$. Then $f(c) = \frac{0}{(0-1)(0-2)} = 0$, $f'(c) = -\frac{c^2-2}{(c-1)^2(c-2)^2} = \frac{1}{2}$, $f''(c) = \frac{2(c^3-6c+6)}{(c-1)^3(c-2)^3} = \frac{3}{2}$, $f'''(c) = -\frac{6(c^4-12c^2+24c-14)}{(c-1)^4(c-2)^4} = \frac{21}{4}$.

So $\frac{x}{(x-1)(x-2)} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$

$= \boxed{\frac{1}{2}x + \frac{3}{4}x^2 + \frac{7}{8}x^3 + O(x^4)}$

(d) Let $f(x) = \cos x$. Then $f(c) = \cos c = \frac{\sqrt{2}}{2}$, $f'(c) = -\sin c = -\frac{\sqrt{2}}{2}$, $f''(c) = -\cos c = -\frac{\sqrt{2}}{2}$, $f'''(c) = \sin c = \frac{\sqrt{2}}{2}$.

So $\cos x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$

$= \boxed{\frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x-\frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2 + \frac{\sqrt{2}}{12}(x-\frac{\pi}{4})^3 + O((x-\frac{\pi}{4})^4)}$

(e) Let $f(x) = \sin^2 x$. Then $f(c) = \sin^2 c = 0$, $f'(c) = \sin(2c) = 0$, $f''(c) = 2\cos(2c) = 2$, $f'''(c) = -4\sin(2c) = 0$.

So $\sin^2 x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$

$= \boxed{x^2 + O(x^4)}$

(f) Let $f(x) = \ln x$. Then $f(c) = \ln c = 1$, $f'(c) = \frac{1}{c} = \frac{1}{e}$, $f''(c) = -\frac{1}{c^2} = -\frac{1}{e^2}$, $f'''(c) = \frac{2}{c^3} = \frac{2}{e^3}$.

So $\ln x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$

$= \boxed{1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 + O((x-e)^4)}$

(g) Let $f(x) = 3^x$. Then $f(c) = 3^c = 1$, $f'(c) = 3^c \ln 3 = \ln 3$, $f''(c) = 3^c(\ln 3)^2 = (\ln 3)^2$, $f'''(c) = 3^c(\ln 3)^3 = (\ln 3)^3$.
So $3^x = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$$= \boxed{1 + x\ln 3 + \frac{(\ln 3)^2}{2}x^2 + \frac{(\ln 3)^3}{6}x^3 + O(x^4)}$$

(h) Let $f(x) = \sqrt{2+x}$. Then $f(c) = \sqrt{2+c} = \sqrt{3}$, $f'(c) = \frac{1}{2}(2+c)^{\frac{-1}{2}} = \frac{\sqrt{3}}{6}$, $f''(c) = -\frac{1}{4}(2+c)^{\frac{-3}{2}} = -\frac{\sqrt{3}}{36}$, $f'''(c) = \frac{3}{8}(2+c)^{\frac{-5}{2}} = \frac{\sqrt{3}}{72}$.
So $\sqrt{2+x} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$$= \boxed{\sqrt{3} + \frac{\sqrt{3}}{6}(x-1) - \frac{\sqrt{3}}{72}(x-1)^2 + \frac{\sqrt{3}}{432}(x-1)^3 + O((x-1)^4)}$$

(i) Let $f(x) = \frac{1}{\sqrt{7-3x}}$. Then $f(c) = \frac{1}{\sqrt{7-3c}} = \frac{1}{2}$, $f'(c) = -\frac{1}{2}(7-3c)^{\frac{-3}{2}} = \frac{3}{16}$, $f''(c) = \frac{27}{3}(7-3x)^{\frac{-5}{2}} = \frac{27}{128}$, $f'''(c) = \frac{405}{8}(7-3x)^{\frac{-7}{2}} = \frac{405}{1024}$.
So $\frac{1}{\sqrt{7-3c}} = f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \frac{f'''(c)}{6}(x-c)^3 + O((x-c)^4)$
$$= \boxed{\frac{1}{2} + \frac{3}{16}(x-1) + \frac{27}{256}(x-1)^2 + \frac{135}{2048}(x-1)^3 + O((x-1)^4)}$$

Alternatively, by using the Taylor series of the elementary functions,

(a) $\frac{1}{x+1} = \frac{1}{2}\frac{1}{1+\frac{x-1}{2}} = \frac{1}{2}\left(1 - \frac{x-1}{2} + (\frac{x-1}{2})^2 - (\frac{x-1}{2})^3 + O((x-1)^4)\right)$
$= \frac{1}{2} - \frac{1}{4}(x-1) + \frac{1}{8}(x-1)^2 - \frac{1}{16}(x-1)^3 + O((x-1)^4)$

(b) $\frac{2-x}{3+x} = -1 + \frac{5}{4}\frac{1}{1+\frac{x-1}{4}} = -1 + \frac{5}{4}\left(1 - \frac{x-1}{4} + (\frac{x-1}{4})^2 - (\frac{x-1}{4})^3 + O((x-1)^4)\right)$
$= \frac{1}{4} - \frac{5}{16}(x-1) + \frac{5}{64}(x-1)^2 - \frac{5}{256}(x-1)^3 + O((x-4)^2)$

(c) $\frac{x}{(x-1)(x-2)} = -\frac{1}{1-\frac{x}{2}} + \frac{1}{1-x}$
$= -\left(1 + \frac{x}{2} + (\frac{x}{2})^2 + (\frac{x}{2})^3 + O(x^4)\right) + (1 + x + x^2 + x^3 + O(x^4))$
$= \frac{1}{2}x + \frac{3}{4}x^2 + \frac{7}{8}x^3 + O(x^4)$

(d) $\cos x = \cos(x - \frac{\pi}{4} + \frac{\pi}{4}) = \frac{\sqrt{2}}{2}\left(\cos(x-\frac{\pi}{4}) - \sin(x-\frac{\pi}{4})\right)$
$= \frac{\sqrt{2}}{2}\left((1 - \frac{(x-\frac{\pi}{4})^2}{2} + O((x-\frac{\pi}{4})^4)) - ((x-\frac{\pi}{4}) - \frac{(x-\frac{\pi}{4})^3}{6} + O((x-\frac{\pi}{4})^4))\right)$
$= \frac{\sqrt{2}}{2} - \frac{\sqrt{2}}{2}(x-\frac{\pi}{4}) - \frac{\sqrt{2}}{4}(x-\frac{\pi}{4})^2 + \frac{\sqrt{2}}{12}(x-\frac{\pi}{4})^3 + O((x-\frac{\pi}{4})^4)$

(e) $\sin^2 x = \frac{1}{2}(1-\cos(2x)) = \frac{1}{2}(1 - (1 - \frac{(2x)^2}{2} + O(x^4)))$
$= x^2 + O(x^4)$

(f) $\ln x = 1 + \ln(1 + \frac{x-e}{e}) = 1 + \left(\frac{x-e}{e} - \frac{1}{2}(\frac{x-e}{e})^2 + \frac{1}{3}(\frac{x-e}{e})^3 + O((x-e)^4)\right)$
$= 1 + \frac{1}{e}(x-e) - \frac{1}{2e^2}(x-e)^2 + \frac{1}{3e^3}(x-e)^3 + O((x-e)^4)$

(g) $3^x = e^{x\ln 3} = 1 + x\ln 3 + \frac{1}{2}(x\ln 3)^2 + \frac{1}{6}(x\ln 3)^3 + O(x^4)$
$= 1 + x\ln 3 + \frac{(\ln)^3}{2}x^2 + \frac{(\ln 3)^3}{6}x^3 + O(x^4)$

(h) $\sqrt{2+x} = \sqrt{3}(1 + \frac{x-1}{3})^{\frac{1}{2}}$
$= \sqrt{3}\left(1 + \frac{1}{2}\frac{x-1}{3} + \frac{\frac{1}{2}(\frac{1}{2}-1)}{2}(\frac{x-1}{3})^2 + \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)}{6}(\frac{x-1}{3})^3 + O((x-1)^4)\right)$
$= \sqrt{3} + \frac{\sqrt{3}}{6}(x-1) - \frac{\sqrt{3}}{72}(x-1)^2 + \frac{\sqrt{3}}{432}(x-1)^3 + O((x-1)^4)$

(i) $\frac{1}{\sqrt{7-3x}} = \frac{1}{2}(1-\frac{x-1}{4/3})^{\frac{-1}{2}}$
$= \frac{1}{2}(1 - \frac{-1}{2}\frac{x-1}{4/3} + \frac{\frac{-1}{2}(\frac{-1}{2}-1)}{2}(\frac{x-1}{4/3})^2 - \frac{\frac{-1}{2}(\frac{-1}{2}-1)(\frac{-1}{2}-2)}{6})(\frac{x-1}{4/3})^3 + O((x-1)^4)$
$= \frac{1}{2} + \frac{3}{16}(x-1) + \frac{27}{256}(x-1)^2 + \frac{135}{2048}(x-1)^3 + O((x-1)^4)$

11. (a) Find $\frac{d^2y}{dx^2}$ at $(1,0)$, if

$$y^3 + y = x^3 - x.$$

(b) Find the Taylor polynomial of order 3 around $x=0$ for $f(x) = e^{\cos x}$.

Solution

(a) By implicit differentiation,

$$3y^2y' + y' = 3x^2 - 1.$$

Let $x=1$, $y=0$, solve for $y'=2$. Now that

$$3y^2y'' + 6y(y')^2 + y'' = 6x,$$

plug in $x=1$, $y=0$, then $\boxed{y''=6}$ at the point $(1,0)$.

(b) By Taylor series, since $\cos x = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \cdots$, and $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots$, thus

$$\begin{aligned} e^{\cos x} &= 1 + (1-\frac{x^2}{2}) + \frac{1}{2}(1-\frac{x^2}{2})^2 + \frac{1}{3!}(1-\frac{x^2}{2})^3 + \cdots \\ &= 1 + 1 + \frac{1}{2} + \frac{1}{3!} + \frac{1}{4!} + \cdots - \frac{1}{2}(x^2 + x^2 + \frac{x^2}{2} + \frac{x^2}{3!} + \frac{x^2}{4!} + \cdots) \\ &= \boxed{e - \frac{e}{2}x^2}. \end{aligned}$$

12. By considering appropriate Taylor series expansions, evaluate the limits below:

(a) $\lim_{x\to 0} \frac{e^{3x}-1}{\ln(1+2x)}$

(b) $\lim_{x\to 0} \left(\frac{1}{\ln(1+2x)} + \frac{1}{\ln(1-2x)}\right)$

(c) $\lim_{x\to 0} \frac{x(1-\cos x)}{1-\sqrt{1-x^3}}$

(d) $\lim_{x\to 0} \frac{e^{3x} - \sin x - \cos x + \ln(1-2x)}{-1+\cos(5x)}$

Solution

(a) Note $e^{2x} = 1 + 3x + \frac{(3x)^2}{2} + \ldots$
and $\ln(1+2x) = 2x - \frac{1}{2}(2x)^2 + \ldots$
Then

$$\begin{aligned} \lim_{x\to 0} \frac{e^{3x}-1}{\ln(1+2x)} &= \lim_{x\to 0} \frac{1 + 3x + \frac{9}{2}x^2 + \cdots - 1}{2x - 2x^2 + \ldots} \\ &= \lim_{x\to 0} \frac{3x + \frac{9}{2}x^2 + \ldots}{2x - 2x^2 + \ldots} = \lim_{x\to 0} \frac{3 + \frac{9}{2}x + \ldots}{2 - 2x + \ldots} \\ &= \boxed{\frac{3}{2}}. \end{aligned}$$

(b)

$$\lim_{x\to 0}\Big(\frac{1}{\ln(1+2x)}+\frac{1}{\ln(1-2x)}\Big) = \lim_{x\to 0}\frac{\ln(1-2x)+\ln(1+2x)}{\ln(1+2x)\ln(1-2x)}$$
$$= \lim_{x\to 0}\frac{\ln(1-4x^2)}{\ln(1+2x)\ln(1-2x)}$$

$$= \lim_{x\to 0}\frac{-4x^2-\dfrac{(2x)^4}{2}+\dots}{(x-\dfrac{(2x)^2}{2}+\dfrac{(2x)^3}{3})(-2x-\dfrac{(2x)^2}{2}-\dfrac{(2x)^3}{3}+\dots)} = \lim_{x\to 0}\frac{-4x^2+o(x^2)}{-2x^2+o(x^2)} = \boxed{2}.$$

(c) Note

$$1-\cos x = 1-(1-\frac{1}{2}x^2)+\frac{1}{4!}x^4+\dots$$
$$= \frac{1}{2}x^2-\frac{1}{4!}x^4+\dots$$

So

$$\lim_{x\to 0}\frac{x(1-\cos x)}{1-\sqrt{1-x^3}} = \lim_{x\to 0}\frac{x(1-\cos x)(1+\sqrt{1-x^3})}{1-(1-x^3)}$$
$$= \lim_{x\to 0}\frac{x(1-\cos x)(1+\sqrt{1-x^3})}{x^3}$$

Note

$$\lim_{x\to 0}(1+\sqrt{1-x^3}) = 2,$$
$$\text{and } \lim_{x\to 0}\frac{x(1-\cos x)}{x^3} = \lim_{x\to 0}\frac{\frac{1}{2}x^3-\frac{1}{4!}x^5+\dots}{x^3} = \lim_{x\to 0}\frac{\frac{1}{2}-\frac{1}{4!}x^2+\dots}{1} = \frac{1}{2}.$$

$$\implies \lim_{x\to 0}\frac{x(1-\cos x)}{1-\sqrt{1-x^3}} = 2\cdot\frac{1}{2} = \boxed{1}$$

(d) Note that

$$e^{3x} = 1+3x+\frac{9}{2}x^2+o(x^2),$$
$$\sin x = x+o(x^2),$$
$$\cos x = 1-\frac{1}{2}x^2+o(x^2),$$
$$\ln(1-2x) = -2x-2x^2+o(x^2),$$
$$\cos(5x) = 1-\frac{25}{2}x^2+o(x^2).$$

Thus

$$\begin{aligned}
&\lim_{x\to 0}\frac{e^{3x}-\sin x-\cos x+\ln(1-2x)}{-1+\cos(5x)}\\
&=\lim_{x\to 0}\frac{1+3x+\frac{9}{2}x^2-x-x+\frac{x^2}{2}-2x-2x^2+o(x^2)}{-1+1-\frac{25}{2}x^2+o(x^2)}\\
&=\boxed{-\frac{6}{25}}.
\end{aligned}$$