THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1010 University Mathematics 2024-2025 Term 1
Homework Assignment 3
Due Date: 27 November 2024

I declare that the assignment here submitted is original except for source material explicitly acknowledged, the piece of work, or a part of the piece of work has not been submitted for more than one purpose (i.e. to satisfy the requirements in two different courses) without declaration, and that the submitted soft copy with details listed in the "Submission Details" is identical to the hard copy, if any, which has been submitted. I also acknowledge that I am aware of University policy and regulations on honesty in academic work, and of the disciplinary guidelines and procedures applicable to breaches of such policy and regulations, as contained on the University website https://www.cuhk.edu.hk/policy/academichonesty/

It is also understood that assignments without a properly signed declaration by the student concerned will not be graded by the course teacher.

Signature	Date

General Regulations

- All assignments will be submitted and graded on Gradescope. You can view your grades and submit regrade requests here as well. For submitting your PDF homework on Gradescope, here are a few tips.

  Where is Gradescope?

  Do the following:

  1. Go to **2024R1 University Mathematics (MATH1010ABCDEF)**
  2. Go to **Course Contents**
  3. The green **Gradescope** icon will be there

- Late assignments will receive a grade of 0.

- For the declaration sheet:

  Either

  Print out the cover sheet (i.e. the first page of this document), and sign and date the statement of Academic Honesty. Use the attached file, sign and date the statement of Academic Honesty, convert it into a PDF and submit it with your homework assignments via Gradescope.

Or

Write your name on the first page of your submitted homework, and simply write out the sentence "I have read the university regulations."

- Write your COMPLETE name and student ID number legibly on the cover sheet (otherwise we will not take any responsibility for your assignments). Please write your answers using a black or blue pen, NOT any other color or a pencil.
- Write your solutions on A4 white paper or use an iPad or other similar device to present your answers and submit a digital form via Gradescope. Please do not use any colored paper and make sure that your written solutions are a suitable size (easily read). Please be aware that you can only use a ball-point pen to write your answers for any exams.
- Show all work for full credit. In most cases, a correct answer with no supporting work will NOT receive full credit. What you write down and how you write it are the most important means of your answers getting good marks on this homework. Neatness and organization are also essential.

1. Let $f(x) = \dfrac{x-1}{x-3}$.

   (a) Is Lagrange's mean value theorem applicable to f on the interval $[4, 5]$?

   (b) If your answer to part (a) is yes, find all possible values $c \in (4, 5)$, at which point(s) the tangent line to the graph is parallel to the secant line connecting the two end points $(4, f(4))$ and $(5, f(5))$.

2. By using Lagrange's mean value theorem, or otherwise, show that

   (a) $\sin x \leq x$ for all $x \in [0, +\infty)$.

   (b) $(1+x)^p \geq 1 + px$ for any $p \geq 1$ and $x \geq 0$.

3. Let $0 < a < b < \dfrac{\pi}{2}$. Prove that there exists $a < \xi < b$ such that

$$\ln\left(\frac{\cos a}{\cos b}\right) = (b-a)\tan\xi.$$

4. Show that for all $0 < a < b \leq 1$,

$$(b-a)(1+\ln a) < \ln\left(\frac{b^b}{a^a}\right) < (b-a)(1+\ln b).$$

5. Evaluate the following limits.

   (a) $\displaystyle\lim_{x\to 0} \frac{\sin^{-1} x - \tan^{-1} x}{x^3}$

   (b) $\displaystyle\lim_{x\to 0^+} \log_{\tan x}(\tan 2x)$

   (c) $\displaystyle\lim_{x\to 0^+} \tan x \ln \sin x$

   (d) $\displaystyle\lim_{x\to 1} \left(\frac{1}{\ln x} - \frac{x}{x-1}\right)$

   (e) $\displaystyle\lim_{x\to+\infty} \frac{e^{1+\ln x}}{\ln(1+e^x)}$

6. Evaluate the following limits.

   (a) $\displaystyle\lim_{x\to 0} \left(\frac{\sin x}{x}\right)^{\frac{1}{x^2}}$

   (b) $\displaystyle\lim_{x\to 1} x^{\frac{2x}{x-1}}$

   (c) $\displaystyle\lim_{x\to 0} \frac{(1+x)^x - 1}{x^2}$

   (d) $\displaystyle\lim_{x\to+\infty} \left(\frac{x^2-2x+1}{x^2-4x+2}\right)^x$

7. Find the x-intercepts, y-intercepts, asymptotes if there is any and sketch the graphs of the following functions.

   (a) $y = \dfrac{x+5}{x-2}$

   (b) $y = \dfrac{x^2-2}{x-1}$

   (c) $y = |4 + 3x - x^2|$

   (d) $y = x|x+2|$

   (e) $y = \left|\dfrac{7-2x}{x+3}\right|$

   (f) $y = \dfrac{1}{|x^2-4|}$

8. For each of the following functions $f(x)$, find

- $f'(x)$ and $f''(x)$.
- range of values of x for which $f(x)$ is increasing.
- asymptotes of $y = f(x)$.
- all relative extremum points

Then sketch the graph of $y = f(x)$.

(a) $f(x) = \dfrac{x}{(x-2)^2}$

(b) $f(x) = \dfrac{x^2+5x+7}{x+2}$

(c) $f(x) = \dfrac{x^2}{x^2-2x+2}$

(d) $f(x) = x^{\frac{2}{3}} - 1$

9. For each of the following functions $f(x)$, find $f(0)$, $f'(0)$, $f''(0)$ and $f'''(0)$ and the Taylor series up to the term in x^3 of $f(x)$ about the point $x = 0$.

(a) $f(x) = \ln \cos x$

(b) $f(x) = e^x \sin x$

10. Find the Taylor series up to the term in $(x-c)^3$ of the functions about $x = c$.

(a) $\dfrac{1}{1+x}$; $c = 1$.

(b) $\dfrac{2-x}{3+x}$; $c = 1$.

(c) $\dfrac{x}{(x-1)(x-2)}$; $c = 0$.

(d) $\cos x$; $c = \frac{\pi}{4}$.

(e) $\sin^2 x$; $c = 0$

(f) $\ln x$; $c = e$.

(g) 3^x; $c = 0$.

(h) $\sqrt{2+x}$; $c = 1$.

(i) $\dfrac{1}{\sqrt{7-3x}}$; $c = 1$.

11. (a) Find $\dfrac{d^2y}{dx^2}$ at $(1, 0)$, if

$$y^3 + y = x^3 - x.$$

(b) Find the Taylor polynomial of order 3 around $x = 0$ for $f(x) = e^{\cos x}$.

12. By considering appropriate Taylor series expansions, evaluate the limits below:

(a) $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{\ln(1+2x)}$

(b) $\lim\limits_{x\to 0} \left(\dfrac{1}{\ln(1+2x)} + \dfrac{1}{\ln(1-2x)} \right)$

(c) $\lim\limits_{x\to 0} \dfrac{x(1-\cos x)}{1-\sqrt{1-x^3}}$

(d) $\lim\limits_{x\to 0} \dfrac{e^{3x} - \sin x - \cos x + \ln(1-2x)}{-1+\cos(5x)}$