Supplementary note 3

Haar and Walsh function

Definition 3.1: (Haar functions)

The Haar functions are defined recursively as follows:

\[
H_0(t) \equiv 1 \quad \text{for } 0 \leq t < 1 \\
H_1(t) \equiv \begin{cases} 
1 & \text{if } 0 \leq t < 1/2 \\
-1 & \text{if } 1/2 \leq t < 1 
\end{cases}
\]

\[
H_{2^p+n}(t) \equiv \begin{cases} 
\sqrt{2^p} & \text{if } \frac{n}{2^p} \leq t < \frac{n+0.5}{2^p} \\
-\sqrt{2^p} & \text{if } \frac{n+0.5}{2^p} \leq t < \frac{n+1}{2^p} \\
0 & \text{elsewhere}
\end{cases}
\]

where \( p = 1, 2, \ldots; n = 0, 1, 2, \ldots, 2^p - 1 \)

\( H_{2^p+n}(t) \) is compactly supported a smaller region if \( p \) is bigger.

Example 3.1: Compute \( W_1(t) \).

Put \( j = 0, q = 1 \). Then:

\[
W_1(t) = (-1)^{[\frac{j}{2}]+q} \left\{ W_j(2t) + (-1)^{j+q}W_j(2t-1) \right\}
\]

where \([\frac{j}{2}]\) = smallest integer smaller or equal to \( \frac{j}{2} \); \( q = 0 \) or \( 1 \); \( j = 0, 1, 2, \ldots \) and

\[
W_0(t) \equiv \begin{cases} 
1 & 0 \leq t < 1 \\
0 & \text{elsewhere}
\end{cases}
\]
Definition 3.3: (Discrete Haar Transform)

The Haar Transform of a $N \times N$ image is performed as follows. Divide $t$ variable by the size of matrix $= N$. That’s:

\[
\begin{array}{cccc}
0 & 1 & 2 & i \\
\frac{1}{N} & \frac{1}{N} & \frac{i}{N} & \frac{N-1}{N}
\end{array}
\]

Let $H(k, i) \equiv H_k \left( \frac{i}{N} \right)$ where $k, i = 0, 1, 2, \cdots, N - 1$

We obtain the Haar Transform matrix:

\[
\tilde{H} \equiv \frac{1}{\sqrt{N}} H \quad \text{where } H \equiv (H(k, i))_{0 \leq k, i \leq N-1}
\]

(Then $\tilde{H}^T \tilde{H} = I$)

The Haar Transform of $f \in M_{n \times n}$ is defined as:

\[
g = \tilde{H} f \tilde{H}^T
\]
Definition 3.4: (Discrete Walsh Transform)
The Walsh Transform of a $N \times N$ image is defined similarly as Haar Transform.
Define $W(k, i) \equiv W_k \left( \frac{i}{N} \right)$ where $k, i = 0, 1, 2, \ldots, N - 1$
Then, the Walsh Transform matrix:
$$\tilde{W} \equiv \frac{1}{\sqrt{N}} W$$ where $W \equiv (W(k, i))_{0 \leq k, i \leq N-1}$
(Then $\tilde{W}^T \tilde{W} = I$)
The Walsh Transform of $f \in M_{n \times n}$ is defined as:
$$g = \tilde{W} f \tilde{W}^T$$

Example 3.2: Compute the Haar Transform matrix for a $4 \times 4$ image.
Solution: Divide $[0, 1]$ into 4 portions:

Check that:

$H_0$

$H_1$

$H_3$

$H_2$
We get that:

\[
H = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 1 & -1 & -1 \\
\sqrt{2} & -\sqrt{2} & 0 & 0 \\
0 & 0 & \sqrt{2} & -\sqrt{2}
\end{pmatrix}
\quad \text{and} \quad
\tilde{H} = \frac{1}{\sqrt{4}}H = \frac{1}{2}H
\]

Easy to check that \(\tilde{H}^T \tilde{H} = I\).

Example 3.3: Compute the Haar Transform of

\[
f = \begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
1 & 0 & 0 & 1 \\
0 & 1 & 1 & 0
\end{pmatrix}
\]

Solution:

\[
g = \tilde{H}f\tilde{H}^T = \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 \\
0 & 0 & -1 & 1 \\
0 & 0 & 1 & -1
\end{pmatrix}
\]

Example 3.4: Suppose \(g\) in Example 3.3 is changed to:

\[
g = \begin{pmatrix}
2 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 \\
0 & 0 & 1 & 0
\end{pmatrix}
\]

Reconstruct the original image.

Solution:

Remark:

- Haar Transform usually produces coefficient matrix with more zeroes (compression)
- Error in the coefficient matrix causes localized error in the reconstructed image
  (Assign detail of accuracy in compression)

Example 3.5: Compute the Walsh Transform matrix for a 4 \(\times\) 4 image.

Solution: Again, divide [0, 1] into 4 portions:
Check that:

\[ W_0 \]
\[ W_1 \]
\[ W_3 \]
\[ W_2 \]

So,

\( H^T H = I \)

Example 3.6: Compute the Walsh Transform of

\[
    f = \begin{pmatrix}
        0 & 1 & 1 & 0 \\
        1 & 0 & 0 & 1 \\
        1 & 0 & 0 & 1 \\
        0 & 1 & 1 & 0 
    \end{pmatrix}
\]

Solution:
Remark:

- The idea of Haar / Walsh Transform is to transform an image to "transformed image" with much more zeroes.
- The coefficient in the "transformed" image tells us information of frequency of image intensity changes.