Practice exercise 1

1. Suppose a $3 \times 3$ colour image sensor records the energies of RGB bands as:

$$
E_R = \begin{pmatrix}
5.074 & 5.567 & 2.393 \\
5.529 & 4.162 & 3.734 \\
1.635 & 1.488 & 5.788
\end{pmatrix}
$$

$$
E_G = \begin{pmatrix}
3.395 & 3.372 & 0.926 \\
0.973 & 1.956 & 1.765 \\
3.412 & 2.901 & 3.242
\end{pmatrix}
$$

$$
E_B = \begin{pmatrix}
3.169 & 0.143 & 2.715 \\
3.838 & 3.397 & 3.031 \\
2.623 & 3.736 & 2.973
\end{pmatrix}
$$

We want to store the record as 4-bit image, i.e. the pixels take value in the range of $[0, 2^4 - 1]$.

(a) By considering the red, green and blue bands as three separate channels, compute the grey images of the three bands such that they attain their maximum contrast respectively.

(b) By considering the red, green and blue bands as a single channel, compute the grey images of the three bands such that the single channel attains its maximum contrast.

2. Programming exercise: given badcontrast.jpg as the record from colour image sensor, provide programs and corresponding outputs in order to satisfy the given requirement:

(a) By considering the red, green and blue bands as three separate channels, output the resultant colour image such that the grey images of the three bands attain their maximum contrast respectively.

(b) By considering the red, green and blue bands as a single channel, output the resultant colour image such that the grey image of the single channel attains its maximum contrast.

(c) By converting the RGB format to HSV (hue, saturation, and value) format, output the resultant RGB image such that the grey image of the V channel in HSV format attains its maximum contrast. (Hint: Use the rgb2hsv and hsv2rgb commands in MATLAB).

Compare the resultant images. Which method is the best for maximizing the contrast of colour image?
3. For the following PSFs, determine whether it is (i) shift-invariant; (ii) separable. Prove your answer or provide a counterexample.

(a) \( h(x, \alpha, y, \beta) = \begin{cases} |(\alpha - x)(\beta - y)| & \text{if } |\alpha - x| \leq 1, |\beta - y| \leq 1 \\ 0 & \text{otherwise} \end{cases} \)

(b) \( h(x, \alpha, y, \beta) = \frac{e^{\alpha-x}}{\alpha} e^{\alpha-x} \)

(c) \( h(x, \alpha, y, \beta) = \frac{1}{\sqrt{(\alpha - x)^2 + (\beta - y)^2}} \sqrt{(\alpha - x)^2 + (\beta - y)^2} \)

(d) \( h(x, \alpha, y, \beta) = \frac{e^{(\alpha-x)-(\beta-y)}}{(\alpha - x)(\beta - y)} e^{(\alpha-x)-(\beta-y)} \)

(e) \( h(x, \alpha, y, \beta) = \begin{cases} 4 - (\alpha - x)^2 - (\beta - y)^2 & \text{if } |\alpha - x| \leq 2, |\beta - y| \leq 2 \\ 0 & \text{otherwise} \end{cases} \)

4. For some operators, the output image \( g \) can be represented as the weighted sum of some entries of \( f \). In this case, the operator can be represented as a matrix with entries the weight for each entry of \( f \). For example, the horizontal Sobel filter can be represented as

\[
\begin{bmatrix}
-1 & 0 & +1 \\
-2 & 0 & +2 \\
-1 & 0 & +1
\end{bmatrix}
\]

which actually means

\[ g(\alpha, \beta) = -f(\alpha-1, \beta-1) - 2f(\alpha-1, \beta-1) + f(\alpha+1, \beta-1) + f(\alpha-1, \beta+1) + 2f(\alpha, \beta+1) + f(\alpha+1, \beta+1) \]

We apply this operator to a \( 3 \times 3 \) image \( f \), which is repeated ad infinitum in all direction. In the other words,

\[
\begin{bmatrix}
\mathcal{f}_{13} & \mathcal{f}_{11} & \mathcal{f}_{13} \\
\mathcal{f}_{23} & \mathcal{f}_{21} & \mathcal{f}_{23} \\
\mathcal{f}_{33} & \mathcal{f}_{31} & \mathcal{f}_{33}
\end{bmatrix}
\cdot
\begin{bmatrix}
\mathcal{f}_{33} & \mathcal{f}_{31} & \mathcal{f}_{32} & \mathcal{f}_{33} & \mathcal{f}_{31} \\
\mathcal{f}_{13} & \mathcal{f}_{11} & \mathcal{f}_{13} & \mathcal{f}_{11} & \mathcal{f}_{13} \\
\mathcal{f}_{23} & \mathcal{f}_{21} & \mathcal{f}_{23} & \mathcal{f}_{21} & \mathcal{f}_{23} \\
\mathcal{f}_{33} & \mathcal{f}_{31} & \mathcal{f}_{32} & \mathcal{f}_{33} & \mathcal{f}_{31} \\
\mathcal{f}_{13} & \mathcal{f}_{11} & \mathcal{f}_{13} & \mathcal{f}_{11} & \mathcal{f}_{13}
\end{bmatrix}
\]

(a) Compute the resultant image \( g \).

(b) Write down the matrix \( H \) representing the operator.

5. Consider the following operator.

\[ g(\alpha, \beta) = -f(\alpha - 1, \beta - 1) + f(\alpha + 1, \beta + 1) \]

Assume all the settings follow those in Q4.

(a) Show that the operator is linear and shift-invariant.

(b) Compute the resultant image \( g \).

(c) Write down the matrix \( H \) representing the operator.
6. A simple discrete Gaussian filter, which is usually used to blur the image, can be represented as
\[
\begin{pmatrix}
0 & 1 & 0 \\
1 & 4 & 1 \\
0 & 1 & 0
\end{pmatrix}
\]
Assume all the settings follow those in Q4.

(a) Compute the resultant image \( g \).

(b) Write down the matrix \( H \) representing the operator.

(c) A student argues that since \( g = Hf, \ f = H^{-1}g \). Therefore, by finding the inverse of \( H \), it will be very simple to de-blur the images. Do you agree? Why or why not?

7. Consider a general operator represented by
\[
W = \begin{pmatrix}
w_{11} & w_{12} & w_{13} \\
w_{21} & w_{22} & w_{23} \\
w_{31} & w_{32} & w_{33}
\end{pmatrix}
\]
Assume all the settings follow those in Q4.

(a) Compute the matrix \( H \) representing the operator.

(b) Show that if the matrix \( W \) can be expressed in the Kronecker product of 2 matrices \( w_r^T, w_c^T \), then the operator is separable.

(c) Express \( H \) as the Kronecker product of 2 matrices \( H_r, H_c \).

8. Let \( A, B \) be circulant matrix of size \( N \times N \), i.e.
\[
A = \begin{pmatrix}
a_1 & a_2 & \cdots & a_N \\
\vdots & \vdots & \ddots & \vdots \\
a_2 & a_3 & \cdots & a_1
\end{pmatrix}, \quad B = \begin{pmatrix}
b_1 & b_2 & \cdots & b_N \\
\vdots & \vdots & \ddots & \vdots \\
b_N & b_1 & \cdots & b_{N-1}
\end{pmatrix}
\]

(a) Prove that \( AB \) and \( BA \) are still circulant matrices.

(b) For 2 matrices \( X, Y \) of size \( N \times N \), represented as sub-matrices shown below:
\[
X = \begin{pmatrix}
x_{11} & x_{12} \\
x_{21} & x_{22}
\end{pmatrix}, \quad Y = \begin{pmatrix}
y_{11} & y_{12} \\
y_{21} & y_{22}
\end{pmatrix}
\]
where \( X_{11}, Y_{11} \) are matrices of size \( n \times n \),
\( X_{21}, Y_{21} \) are matrices of size \( (N - n) \times n \),
\( X_{12}, Y_{12} \) are matrices of size \( n \times (N - n) \),
\( X_{22}, Y_{22} \) are matrices of size \( (N - n) \times (N - n) \).
Show that
\[
XY = \begin{pmatrix}
x_{11}y_{11} + x_{12}y_{21} & x_{11}y_{12} + x_{12}y_{22} \\
x_{21}y_{11} + x_{22}y_{21} & x_{21}y_{12} + x_{22}y_{22}
\end{pmatrix}
\]
(c) Using (a) and (b), show that for $\tilde{A}, \tilde{B}$ block circulant matrix, $\tilde{A}\tilde{B}$ is still a block circulant matrix.

(d) In order to save memory space, a circulant matrix is usually represented as the vector that circulates through the matrix. For example, if

$$A = \begin{pmatrix}
a_1 & a_2 & \cdots & a_N \\
a_N & a_1 & \cdots & a_{N-1} \\
\vdots & \vdots & \ddots & \vdots \\
a_2 & a_3 & \cdots & a_1
\end{pmatrix},$$

then $A$ is represented as $[a_1, a_2, \cdots, a_N]$.

Compute the vector used to represent $AB$. What is its relation with those representing $A$ and $B$?

9. Prove that the multiplication of circulant matrices of same size is commutative. (i.e. if $A, B$ are circulant matrices of size $N \times N$, then $AB = BA$.)

10. Programming exercise: Given saltandpepper.png and gaussian.png, which have some noise on the images, provide programs and corresponding outputs in order to satisfy the given requirement:

(a) Apply $3 \times 3$ mean filter on the images.

(b) Apply $3 \times 3$ median filter on the images.

Compare the effect of different filter on different images.