INFINITE DIMENSIONAL POLYTOPES

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Introduction.

In [10], Phelps defined the class of $\beta$-polytopes to be the family of finite codimensional slices of Choquet simplexes. He showed that the family of finite codimensional $\beta$-polytopes coincides with the usual finite dimensional polytopes. Thus the class of $\beta$-polytopes properly contains both the latter class and the Choquet simplexes, and they share a number of properties of both classes. An infinite dimensional $\beta$-polytope cannot, however, be centrally symmetric and this has been shown (in [10]) to be the basis for the fact that a number of “permanence properties” of finite dimensional polytopes are no longer valid for $\beta$-polytopes. In what follows, we define a larger class of polytopes: the compact convex sets which are affinely homeomorphic to closed finite codimensional slices of unit balls of the duals of Lindenstrauss spaces (a Banach space whose dual is an $L^1(\mu)$ space). The definition was originally suggested by J. Lindenstrauss and this class of polytopes contains centrally symmetric sets. We call this class of sets the class of $L$-polytopes. In section 1, we give some results concerning the unit ball of an $L^1(\mu)$ space. In section 2, we characterize the maximal faces of $L$-polytopes. Extension properties for affine continuous functions on closed faces also hold in the class of $L$-polytopes. In section 3, we show that every extreme point of an $L$-polytope is a polyhedral vertex and in section 4, we give examples that some properties for finite dimensional polytopes cannot be generalized.

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1. Basic results.

In this section, our main task is to show that all maximal faces of the unit ball of an $L^1(\mu)$ space are affinely isomorphic. We also give a characterization of the maximal faces of the unit ball of an $L^1(\mu)$ space where $\mu$ is $\sigma$-finite. For the sake of completeness we include some results which may be known but for which we know of no reference.

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**Lemma 1.1.** Let $K$ be a convex subset of a linear space. Suppose that $A$ is a convex subset of $K$, then the set

$$A' = \{ x \in K : \lambda x + (1 - \lambda)z \in A, \text{ for some } z \in K \text{ and } 0 < \lambda < 1 \}$$

is a face of $K$ containing $A$. Moreover, $A'$ equals the intersection of all faces of $K$ containing $A$.

Throughout the section, we consider the space $L^1(X, \mu)$ only. We let $B$ denote the unit ball of $L^1(X, \mu)$ and $F$ will denote a maximal face of $B$. (We make the convention that a maximal face is a maximal proper face.) For simplicity, we just take $\mu$ to be a positive measure. The propositions proved here can be easily generalized to an arbitrary measure (except Theorem 1.8). From Lemma 1.1 and also from the separation theorem, it follows that if $A$ is a convex subset of $\{ x \in B : \| x \| = 1 \}$, then there exists a proper face of $B$ containing $A$.

**Proposition 1.2.** Every maximal face of $B$ is norm closed.

**Proof.** Let $F$ be a maximal face and let $\overline{F}$ denote the closure of $F$; then $\overline{F}$ is convex and each element of $\overline{F}$ is of norm 1. Thus, there exists a proper face $F_1$ of $B$ containing $\overline{F}$. By maximality of $F$ we have $F \subseteq \overline{F} \subseteq F_1 \subseteq F$.

For each $x \in L^1(X, \mu)$ we define $\text{supp} x = \{ t \in X : x(t) \neq 0 \}$. This is defined to within a set of measure zero.

**Lemma 1.3.** Let $A$ be a convex subset of $\{ x \in B : \| x \| = 1 \}$. Then for any $x, y \in A$, the set $\text{supp} x^+ \cap \text{supp} y^-$ has measure zero.

**Proof.** Let $E = \text{supp} x^+ \cap \text{supp} y^-$. If $\mu(E) > 0$, we have $|x + y| < |x| + |y|$ a.e. on $E$, therefore

$$\int_E |x + y| < \int_E |x| + |y| .$$

Consequently, we have

$$\int_E 1 \leq \int_E (|x| + |y|) = 1 ,$$

contradicting the convexity of $A$.

**Lemma 1.4.** Let $F$ be a maximal face of $B$ and suppose $x \in F$. If $y$ is an integrable function of norm 1 such that
supp\(y^+\subseteq\text{supp}x^+\) and \(\text{supp}y^-\subseteq\text{supp}x^-\),
then \(y\in F\).

**Proof.** From Lemma 1.3, it follows that for any \(z\in F\), the sets

\[
\text{supp}x^+\cap\text{supp}z^- \quad \text{and} \quad \text{supp}x^-\cap\text{supp}z^+
\]

have measure zero. By hypothesis, we see that the same holds if we replace \(x\) by \(y\). Thus, we have for any \(\lambda\in[0,1],\)

\[
|\lambda y + (1-\lambda)z| = \lambda|y| + (1-\lambda)|z| \quad \text{a.e.}
\]

It follows that

\[
\int|\lambda y + (1-\lambda)z| = \int(\lambda|y| + (1-\lambda)|z|) = 1,
\]

so \(\text{conv}(\{y\}\cup F)\) is a convex subset of \(\{x\in B: \|x\|=1\}\) and there exists a proper face of \(B\) containing \(\text{conv}(\{y\}\cup F)\). By the maximality of \(F\), we have \(y\in F\).

**Lemma 1.5.** Let \(F\) be a maximal face of \(B\). Then for any \(\sigma\)-finite measurable set \(E\) of positive measure, there exists \(y\in F\) such that \(\text{supp}y=E\).

**Proof.** We assume first that \(\mu(E)<\infty\). It suffices to obtain \(x\in F\) such that \(\text{supp}x\supseteq E\). Indeed, if such a function \(x\) exists, let

\[
E_1 = \text{supp}x^+\cap E, \quad E_2 = \text{supp}x^-\cap E,
\]

then \(E_1\cap E_2\) is a null set and \(E_1\cup E_2=E\). Let

\[
y = (\mu(E))^{-1}(\chi_{E_1} - \chi_{E_2}).
\]

By Lemma 1.4, we see that \(y\in F\) and \(\text{supp}y=E\).

We obtain the function \(x\) as follows. Let

\[
\alpha = \sup\{\mu(\text{supp}z\cap E) : z\in F\}.
\]

We will first find an \(x\in F\) such that \(\mu(\text{supp}x\cap E) = \alpha\). To this end, for each positive integer \(n\), we choose \(x_n\in F\) such that

\[
\mu(\text{supp}x_n\cap E) > \alpha - n^{-1}
\]

and we let \(x=\sum 2^{-n}x_n\). Since \(F\) is closed, it contains \(x\) and by Lemma 1.3, the set \(\text{supp}x_n^+\cap\text{supp}x_m^-\) is a zero set for any \(m,n,\) hence \(\text{supp}x\supseteq\text{supp}x_n\) for each \(n\). Consequently

\[
\alpha \geq \mu(\text{supp}x\cap E) \geq \mu(\text{supp}x_n\cap E) > \alpha - n^{-1}.
\]
Suppose now that $\mu(E_0) > 0$ where $E_0 = E \setminus \text{supp} x$. If 

$$ \mu(\text{supp} z \cap E_0) = 0 \quad \text{for all } z \in F, $$

then for $w = (\mu(E_0))^{-1} E_0$, we would have 

$$ \|\lambda z + (1 - \lambda)w\| = 1 \quad \text{for } \lambda \in [0, 1], \ z \in F, $$

hence $\text{conv}(\{w\} \cup F)$ would be a convex subset of $\{x \in B : \|x\| = 1\}$ and thus contained in a maximal face. This contradiction shows that there necessarily exists $z$ in $F$ with 

$$ \mu(\text{supp} z \cap E_0) > 0. $$

By Lemma 1.4, we may assume that $\text{supp} z \subseteq E_0$. Consequently, the function $z_1 = \frac{1}{2} (z + x)$ is in $F$ and 

$$ \text{supp} z_1 = \text{supp} z \cup \text{supp} x. $$

It follows that 

$$ \mu(\text{supp} z_1 \cap E) = \mu(\text{supp} z \cap E_0) + \mu(\text{supp} x \cap E) > \alpha, $$

an impossibility which proves that $E_0 = E \setminus \text{supp} x$ has measure zero. Thus $E \subseteq \text{supp} x$, and the proof for the case $\mu(E) < \infty$ is complete.

If $\mu(E) = \infty$, we let $E = \bigcup_{i=1}^{\infty} E_i$ where $E_i$ are disjoint measurable sets of finite positive measure. For each $E_i$, there exists $y_i \in F$ such that $\text{supp} y_i = E_i$ a.e. Let $y = \sum_{i=1}^{\infty} 2^{-i} y_i$. By Proposition 1.2, $y \in F$ and 

$$ \text{supp} y = \bigcup_{i=1}^{\infty} \text{supp} y_i = \bigcup_{i=1}^{\infty} E_i = E \text{ a.e.} $$

From the above lemma, we see that for any $\sigma$-finite measurable set $E$, there exists $y$ in $F$, $\text{supp} y^+ = E_1$ a.e., $\text{supp} y^- = E_2$ a.e. where $E_1 \cup E_2 = E$, $E_1 \cap E_2 = \emptyset$ and by Lemma 1.3, this decomposition is unique within a set of measure zero.

**Theorem 1.6.** Any two maximal faces of the unit ball $B$ are affinely isometric.

**Proof.** Let $F$ be a maximal face and let 

$$ F_1 = \{x \in B : x \equiv 0, \ |x| = 1\}. $$

We need only show that $F$ and $F_1$ are affinely isometric.

Define $\varphi : F \to F_1$ by $\varphi(x) = |x|$. It is easily seen by Lemma 1.3 that $\varphi$ is affine and isometric. To show that it is onto, let $x \in F_1$ and let $E = \text{supp} x$. Then there exists a decomposition $E = E_1 \cup E_2$ where 

$$ E_1 = \text{supp} y^+, \ E_2 = \text{supp} y^- $$
for some $y$ in $F$. Let $x' = x \cdot (\chi_{E_1} - \chi_{E_2})$. By Lemma 1.4, we have $x' \in F$ and $\varphi(x') = x$.

**Theorem 1.7.** Let $F$ be a maximal face of $B$ then

$$\text{Aff } F = \{ \sum_{i=1}^{n} \lambda_i x_i : \sum_{i=1}^{n} \lambda_i = 1, x_i \in F, i = 1, \ldots, n, n \in \mathbb{N} \}$$

is a hyperplane in $L^1(X, \mu)$.

**Proof.** We need only show that the linear subspace spanned by $F$ is $L^1(X, \mu)$. Let $x \in L^1(X, \mu)$ and let

$$E_1 = \text{supp } x^+, \quad E_2 = \text{supp } x^-.$$

By Lemma 1.5, we can find measurable sets $\{E_{ij}\}_{i,j=1,2}$ such that

$$E_1 = E_{11} \cup E_{12}, \quad E_2 = E_{21} \cup E_{22}$$

and $y_1, y_2 \in F$ with

$$\text{supp } y_1^+ = E_{11}, \quad \text{supp } y_1^- = E_{12}, \quad \text{supp } y_2^+ = E_{21}, \quad \text{supp } y_2^- = E_{22}.$$

Let for $i,j = 1,2$

$$x_{ij} = \frac{x \cdot \chi_{E_{ij}}}{\|x \cdot \chi_{E_{ij}}\|} \quad \text{if } \|x \cdot \chi_{E_{ij}}\| \neq 0, \quad 0 \quad \text{otherwise}.$$

Then

$$x = \|x \cdot \chi_{E_{11}}\| \cdot x_{11} - \|x \cdot \chi_{E_{12}}\| \cdot (-x_{12}) + \|x \cdot \chi_{E_{21}}\| \cdot x_{21} - \|x \cdot \chi_{E_{22}}\| \cdot (-x_{22})$$

where $x_{11}, -x_{12}, x_{21}, -x_{22}$ are in $\text{lin } F$.

**Remark.** By the above theorem, the map $\varphi$ in Theorem 1.6 can be extended to an isometry $\tilde{\varphi}$ of $L^1(X, \mu)$ onto itself. If we let $C_1$, and $C_2$ be the cones generated by the maximal faces $F$ and $F'$ in Theorem 1.6, then $\tilde{\varphi}$ is an order isomorphism with the orderings induced by $C_1$ and $C_2$.

**Theorem 1.8.** Let $(X, \mu)$ be a $\sigma$-finite measure space. Then every maximal face $F$ of the unit ball $B$ of $L^1(X, \mu)$ is of the form

$$F_Y = \{x \in B : \text{supp } x^+ \subseteq Y, \text{supp } x^- \subseteq X \setminus Y \text{ and } \|x\| = 1\},$$

for some measurable set $Y$. Conversely, every set of the form $F_Y$ is a maximal proper face of $B$.

**Proof.** It is easy to check that $F_Y$ is a maximal face. On the other hand, if $F$ is a maximal face, by Lemma 1.5, there exists an $x$ in $F$ such that $\text{supp } x = X$. Let $Y = \text{supp } x^+$, we claim that $F_Y \supseteq F$. Indeed, if $y \in F$, then from Lemma 1.3, we see that $\text{supp } y^+ \subseteq Y$ a.e. and $\text{supp } y^- \cap Y$
has zero measure, hence \(supp y^- \subseteq X \setminus Y\), so \(y \in F_Y\). Since \(F\) is a maximal face, it follows that \(F_Y = F\).

We conclude this section with some properties of the faces of the unit ball \(B\). These properties will be used later on.

**Lemma 1.9.** (Decomposition lemma.) Suppose that \(V\) is a vector lattice. If \(\{x_i : i \in I\}\) and \(\{y_j : j \in J\}\) are finite sequence of nonnegative elements of \(V\) and if

\[\sum_{i \in I} x_i = \sum_{j \in J} y_j,\]

then there exist \(z_{ij} \geq 0, (i, j) \in I \times J\), such that

\[x_i = \sum_{j \in J} x_{ij} (i \in I) \quad \text{and} \quad y_j = \sum_{i \in I} z_{ij} (j \in J).\]

**Proof.** Cf. [9, p. 61].

**Proposition 1.10.** Let \(F, G\) be proper faces of the unit ball \(B\) such that \(F \cap -G = \emptyset\). Then \(conv (F \cup G)\) is a proper face of \(B\).

**Proof.** We claim that if \(x \in conv (F \cup G)\), then \(\|x\| = 1\). We can write

\[x = \lambda x_1 + (1-\lambda)x',\]

where \(x_1 \in F, x' \in G\) and \(\lambda \in (0, 1)\). By the remark following Theorem 1.7, we see that all the orderings generated by maximal faces are isomorphic, hence we may assume that \(F\) is contained in the maximal face

\[F_1 = \{x \in B : x \geq 0, \|x\| = 1\}.\]

The cone generated by \(F_1\) defines a lattice. Write

\[x' = \alpha x_2 - (1-\alpha)x_3, \quad x_2, x_3, \in F_1, \alpha \in (0, 1) .\]

(If \(\alpha = 1\), the claim follows trivially, for \(\alpha = 0\), the proof is same as below.) Since \(G\) is a face of \(K\), we have \(x_2, x_3 \in G\). Let \(x = x^+ - x^-\). We then have

\[x^+ - x^- = \lambda x_1 + \alpha(1-\lambda)x_2 - (1-\alpha)(1-\lambda)x_3,\]

that is,

\[x^+ + (1-\alpha)(1-\lambda)x_3 = x^- + \lambda x_1 + \alpha(1-\lambda)x_2 .\]

By the decomposition lemma, there exist \(\mu_{ij} \geq 0, z_{ij} \in F_1, i = 1, 2, j = 1, 2, 3,\) such that

\[x^+ = \sum_{j=1}^3 \mu_{1j} z_{1j}, \quad (1-\alpha)(1-\lambda)x_3 = \sum_{j=1}^3 \mu_{2j} z_{2j},\]

\[x^- = \sum_{i=1}^2 \mu_{i1} z_{i1}, \quad \lambda x_1 = \sum_{i=1}^2 \mu_{i2} z_{i2},\]

\[\alpha(1-\lambda)x_2 = \sum_{i=1}^2 \mu_{i3} z_{i3} .\]
Since $x_3 \in -G$, $x_1 \in F$, $x_2 \in G$, by the above equations, we have $z_{22}$, $z_{23} \in -G$, $z_{22} \in F$ and $z_{23} \in G$. But since $F \cap G$ and $G \cap -G$ are void sets, we have $\mu_{21} = 0$, $\mu_{23} = 0$. The second, forth and fifth of the above equations become

$$(1-\lambda)(1-\alpha)x_3 = \mu_{21}z_{21}, \quad \lambda x_1 = \mu_{12}z_{12}, \quad \alpha(1-\lambda)x_2 = \mu_{13}z_{13}.$$ 

Substituting into the first and third equation, we have

$$x^+ = \mu_{11}z_{11} + \lambda x_1 + \alpha(1-\lambda)x_2,$$
$$x^- = \mu_{11}z_{11} + (1-\lambda)(1-\alpha)x_3,$$
so

$$1 \geq ||x|| = ||x^+|| + ||x^-|| = 2\mu_{11} + 1 \geq 1$$

and hence $||x|| = 1$ as asserted. We may therefore assume that both $F$ and $G$ are contained in the maximal face $F_1$. To show that $\text{conv}(F \cup G)$ is a face let

$$\lambda x_1 + (1-\lambda)x_2 \in \text{conv}(F \cup G),$$
where $x_1, x_2 \in B$, $0 < \lambda < 1$. Then

$$\lambda x_1 + (1-\lambda)x_2 = \alpha y_1 + (1-\alpha)y_2,$$
where $y_1 \in F$, $y_2 \in G$, and $0 \leq \alpha \leq 1$. If $\alpha = 0$ (or $\alpha = 1$) then $x_1, x_2$ are in $G$ (or $F$) and hence in $\text{conv}(F \cup G)$. If $0 < \alpha < 1$, by the decomposition lemma, there exist $\mu_{ij} \geq 0$, $z_{ij} \in F_1$, $i, j = 1, 2$ such that

$$\lambda x_1 = \mu_{11}z_{11} + \mu_{12}z_{12}, \quad (1-\lambda)x_2 = \mu_{21}z_{21} + \mu_{22}z_{22},$$
$$\alpha y_1 = \mu_{11}z_{11} + \mu_{21}z_{21}, \quad (1-\alpha)y_2 = \mu_{12}z_{12} + \mu_{22}z_{22}.$$ 

The third and the forth equations imply $z_{11}, z_{21} \in F$ and $z_{12}, z_{22} \in G$. Hence by the first and second equations, we have $x_1, x_2 \in \text{conv}(F \cup G)$.

**Proposition 1.11.** Suppose $F$ is a finite dimensional face in a maximal face $F_1$ of $B$. Then there exists a face $F'$ in $F_1$ such that $F \cap F' = \emptyset$ and $\text{conv}(F \cup F') = F_1$.

Moreover, if $x_1 \in F$ and $x_2 \in F'$, then $||\lambda x_1 + \beta x_2|| = |\lambda| + |\beta|$ for any real $\lambda, \beta$.

**Proof.** If $F_1 = F$, then take $F_1$ to be the empty set. Hence, assuming that $F_1 + F$, we will first show that there exists a face $G$ in $F_1$ disjoint from $F$. In fact, let $x \in F_1 \setminus F$ and let $K = F_1 \cap \text{Aff}(F \cup \{x\})$. Then $K$ is a finite dimensional compact convex subset of $F_1$. Since $K + F$, there exists an extreme point $x_0$ of $K$ which is not in $F$. Consider

$$G = \{z : \lambda z + (1-\lambda)z' = x_0, \quad 0 < \lambda < 1, \quad z, z' \in F_1\}.$$
Then $G$ is a face in $F_1$ disjoint from $F$. Let $F'$ be the union of all faces in $F_1$ disjoint from $F$; it too is a face, in fact, let $x_1, x_2 \in F$, then there exists two faces $G_1, G_2$ in $F_1$ disjoint from $F$ such that $x_1 \in G_1, x_2 \in G_2$. By Proposition 1.10, the set $\text{conv}(G_1 \cup G_2)$ is a face contained in $F_1$ disjoint from $F$, thus

$$\lambda x_1 + (1-\lambda)x_2 \in F \quad \text{for some} \quad 0 < \lambda < 1.$$ 

To show that it is a face, let

$$\lambda x_1 + (1-\lambda)x_2 \in F', \quad 0 < \lambda < 1.$$ 

Then $\lambda x_1 + (1-\lambda)x_2 \in H$ for some face $H$ in $F'$, hence $x_1, x_2 \in H \subseteq F'$. We claim that $\text{conv}(F \cup F') = F_1$, for if this were not true, then there exists $x_1 \in F_1 \setminus \text{conv}(F \cup F')$ and arguing as above, we can find a face containing $x_1$ disjoint from $F$ and not contained in $F'$. This is a contradiction.

To show the last assertion, we see that if $\alpha$ and $\beta$ have the same sign, then it is clear that equality holds since the norm is additive on the cone generated by each maximal face. If $\alpha > 0, \beta < 0$, say, let $F'' = \text{conv}(F \cup -F')$, it suffices to show that this, too, is a maximal face. As $F_1$ is of codimension 1 in $B$, also $F''$ (which is a face by Proposition 1.10) has codimension 1 in $B$, hence it is maximal. By the remark following theorem 1.7, we have $B = \text{conv}(F'' \cup -F'')$, and the norm is additive on the cone generated by $F''$, thus

$$\|\alpha x_1 + (-\beta)(-x_1)\| = |\alpha| + |\beta|.$$

2. Facial properties of $L$-polytopes.

Let $K$ be a convex subset of a linear space and let $H$ be a convex subset in $K$. We say that $H$ is of codimension $n$ in $K$ if there exists an affinely independent set $\{x_1, \ldots, x_n\}$ in $K \setminus \text{Aff}H$ such that

$$\text{Aff}(H \cup \{x_1, \ldots, x_n\}) = \text{Aff}K.$$ 

Suppose that $h_1, \ldots, h_n$ are affine functions on $K$ and that

$$M_K = \{x \in K : h_i(x) = 0, \ i=1,\ldots,n\}.$$ 

Then $M_K$ is called a finite codimensional slice of $K$. If $K$ is a compact convex set and if $M_K$ is closed in $K$, we call $M_K$ a closed finite codimensional slice of $K$.

**Definition 2.1.** A compact convex set $H$ is called an $L$-polytope if $H$ is affinely homeomorphic to some $M_K$ where $K$ is an $L$-ball. (The unit ball of the dual of a Lindenstrauss space with the $w^*$-topology cf. [7].)
Note that in the definition of $L$-polytopes, we do not assume $M_K$ to be of finite codimension in $K$. We will show, however, that an $L$-polytope is affinely homeomorphic to some $M_K$ such that $M_K$ is of finite codimension in $K$. We first give two lemmas which will be useful in what follows.

**Lemma 2.2.** Let $F$ be a convex subset of a linear space $E$, and let

\[
M = \{ x \in F : h_i(x) = 0, \ i = 1, \ldots, n \}, \\
M_j = \{ x \in F : h_i(x) = 0, \ i \neq j, i \in \{1, \ldots, n\} \}, \quad j = 1, \ldots, n,
\]

where $h_i, i = 1, \ldots, n$, are affine functions on $F$. Suppose that for each $j$, there exist $x_j, y_j \in M_j$ such that $h_j(x_j) < 0, h_j(y_j) > 0$; then we have:

(i) For each $z \in F$, there exists $\lambda > 0, \lambda_i, \beta_i \geq 0, i = 1, \ldots, n$, such that

\[
\lambda + \sum_{i=1}^n \lambda_i + \sum_{i=1}^n \beta_i = 1
\]

and

\[
\lambda z + \sum_{i=1}^n \lambda_i x_i + \sum_{i=1}^n \beta_i y_i \in M,
\]

(ii) $M$ is of finite codimension $n$ in $F$.

Furthermore, if $F$ is a compact convex set and if $h_i, i = 1, \ldots, n$ are continuous affine functions on $F$, then there exists $k \in \mathbb{R}^+$ such that the $\lambda$ in (i) may be chosen greater than $k$ for all $z \in F$.

**Proof.** Without loss of generality, we assume that $0 \in M$. Define the map

\[
T : \mathbb{R}^n \to \text{lin} F / \text{lin} M
\]

such that $T(e_i) = \overline{x}_i, i = 1, \ldots, n$, where $\overline{x}_i$ is the equivalence class of $x_i$. For any $z \in F$, we have

\[
z - \sum_{i=1}^n h_i(z)x_i \in \text{lin} M,
\]

thus $\bar{z} = \sum_{i=1}^n h_i(z)x_i$. If we let $\alpha = (\alpha_1, \ldots, \alpha_n) = (h_1(z), \ldots, h_n(z))$, then $T(\alpha) = \bar{z}$. Let $I$ be the subset of $\{1, \ldots, n\}$ such that $\alpha_i < 0$ and let $J = \{1, \ldots, n\} \setminus I$. Then

\[
0 = \bar{z} + \sum_I (-\alpha_i) T(e_i) - \sum_J (-\alpha_i) T(e_i) = \bar{z} + \sum_I (-\alpha_i) \bar{x}_i + \sum_J \alpha_i (-h_i(x_i)/h_i(y_i)) \bar{y}_i.
\]

(Here we use the fact that $\bar{x}_i = (h_i(x_i)/h_i(y_i)) \bar{y}_i$.) We let

\[
(\ast) \quad \lambda = \left(1 + \sum_I (-\alpha_i) + \sum_J \alpha_i (-h_i(x_i)/h_i(y_i))\right)^{-1}
\]

Further for $i = 1, \ldots, n$ we let

\[
\lambda_i = 0 \quad \text{if} \quad \alpha \notin I,
\]

\[
= \alpha_i \lambda \quad \text{if} \quad \alpha \in I,
\]
\[ \beta_i = 0 \quad \text{if } \alpha \notin J, \]
\[ = \alpha_i(-h_i(x_i)/h_i(y_i))\lambda \quad \text{if } \alpha \in J. \]

Then we have \( \lambda > 0, \lambda_i, \beta_i \geq 0 \) for \( i = 1, \ldots, n, \)
\[ \lambda + \sum_{i=1}^{n} \lambda_i + \sum_{i=1}^{n} \beta_i = 1 \]
and
\[ \lambda \bar{z} + \sum_{i=1}^{n} \lambda_i \bar{x}_i + \sum_{i=1}^{n} \beta_i \bar{y}_i = 0, \]
so
\[ \lambda z + \sum_{i=1}^{n} \lambda_i x_i + \sum_{i=1}^{n} \beta_i y_i \in F \cap \text{lin } M = M. \]

Hence (i) is proved and (ii) follows from this directly. To verify the last assertion, we notice that when \( F \) is compact and each \( h_i, i = 1, \ldots, n \) is continuous, the set
\[ \{ h_i(z) : i = 1, \ldots, n, z \in F \} \]
is a bounded set in \( \mathbb{R} \), hence the equation (*) is uniformly bounded away from 0. That is there exists \( k > 0 \) such that \( \lambda > k > 0 \) for all \( z \in F \).

**Lemma 2.3.** Let \( K \) be a convex set and let \( M \) be a finite codimensional slice of \( K \). Suppose that \( M_0 \) is a face of \( M \) and that \( F \) is the smallest face of \( K \) containing \( M_0 \); then \( M_0 \) is of finite codimension in \( F \).

Suppose \( K \) is compact. If \( M \) is a closed finite codimensional slice of \( K \) and \( M_0 \) is closed, then \( F \) is compact.

**Proof.** Let
\[ M = \{ x \in K : h_i(x) = 0, \quad i = 1, \ldots, n \}, \]
where \( h_i, i = 1, \ldots, n \) are affine functions on \( K \). Since \( F \) contains \( M_0 \), we have
\[ M_0 = \{ x \in F : h_i(x) = 0, \quad i = 1, \ldots, n \}. \]

We may assume that \( n \) is the smallest integer such that the above equality holds. Let
\[ M_j = \{ x \in F : h_i(x) = 0, \quad i \neq j, i = 1, \ldots, n \}. \]

Then \( M_0 \not\subseteq M_j \not\subseteq F \). We claim that for each \( j \in \{ 1, \ldots, n \} \), there exist \( x_j, y_j \) such that \( h_j(x_j) > 0, h_j(y_j) < 0 \). Indeed, let
\[ F' = \{ x \in F : \lambda x + (1-\lambda)y \in M \text{ where } y \in F \text{ and } 0 < \lambda < 1 \}. \]

Then \( F' \) is a face of \( F \) containing \( M_0 \) and thus \( F' = F \). Since \( M_j \not\subseteq F \),
there exists $x_j \in F \setminus M_j$ and we have $h_j(x_j) > 0$ (or $< 0$). There also exist $y_j \in F$ and $0 < \lambda < 1$ such that

$$\lambda x_j + (1 - \lambda) y_j \in M_0.$$ 

It follows that $h(y_j) < 0$. Hence we have found $\{x_j\}_{j=1}^n, \{y_j\}_{j=1}^n$ which satisfy the conditions of Lemma 2.2, and therefore $M_0$ is of finite codimension in $F$.

To show the last part, let $\{x_\alpha\}_{\alpha \in I}$ be a net in $F$. By the above lemma, we have $k > 0, \lambda_\alpha > k > 0, \lambda_{i_\alpha}, \beta_{i_\alpha} \geq 0, \alpha \in I, i = 1, \ldots, n$, such that

$$\lambda_\alpha + \sum_{i=1}^n \lambda_{i_\alpha} + \sum_{i=1}^n \beta_{i_\alpha} = 1$$

and

$$\lambda_\alpha x_\alpha + \sum_{i=1}^n \lambda_{i_\alpha} x_i + \sum_{i=1}^n \beta_{i_\alpha} y_i \in M_0.$$ 

By compactness, we may assume that $\{\lambda_\alpha\}$ converges to $\lambda > 0$, $\{\lambda_{i_\alpha}\}$ converges to $\lambda_i, \{\beta_{i_\alpha}\}$ converges to $\beta_i, i = 1, \ldots, n$ and $\{z_\alpha\}$ converges to $z$. Hence

$$\lambda z + \sum_{i=1}^n \lambda_i x_i + \sum_{i=1}^n \beta_i y_i \in M_0 \subseteq F$$

and

$$\lambda + \sum_{i=1}^n \lambda_i + \sum_{i=1}^n \beta_i = 1.$$

Since $F$ is a face and $\lambda > 0$, we have $z \in F$ which shows that $F$ is compact.

**Corollary 2.4.** (Lazar). Suppose $K$ is a compact convex set and that $M$ is a closed finite codimensional slice in $K$. If $F$ is a closed face of $M$, then there exists a closed face $F'$ of $K$ such that $F = M \cap F'$. If $F'$ is a $G_\delta$ set in $H_1 \cap K$, then $F$ is a $G_\delta$ set in $K$.

**Proof.** The first part follows directly from Lemma 2.3. The second part follows from the last part of [10, Lemma 3.4].

**Proposition 2.5.** Let $H$ be an $L$-polytope. Then there exists a closed finite codimensional slice $M_K$ of an $L$-ball $K$ such that $H$ is affinely homeomorphic to $M_K$ and $M_K$ is of finite codimension in $K$.

**Proof.** Without loss of generality, we assume that

$$H = \{x \in K' : k_i(x) = 0, i = 1, \ldots, n\}$$

where $k_1, \ldots, k_n \in A(K')$ and $K'$ is an $L$-ball. Let $F$ be the smallest face of $K'$ containing $H$. If $F = K'$, then by Lemma 2.3, the proof is complete. If $F \neq K'$, then $F$ is a proper face of $K'$. Again, by the same lemma, it
is compact, thus it is a Choquet simplex. Let \( K = \text{conv} F \cup -F \). Then \( K \) is an \( L \)-ball and

\[
H = \{ x \in K : h_i(x) = 0, \ i = 1, \ldots, n \},
\]

where \( h_i \) is the restriction of \( k_i \) to \( K \), \( i = 1, \ldots, n \). Lemma 2.3 shows that \( H \) is of finite codimension in \( F \), hence \( H \) is of finite codimension in \( K \).

Let \( H \) be an \( L \)-polytope. We call an \( L \)-ball \( K \) with the property of Proposition 2.5 to be an envelope of \( H \). Our next three propositions are concerned with maximal faces of \( L \)-polytopes. We make the convention that maximal face shall mean maximal proper face.

**Lemma 2.6.** Let \( X \) be a Banach space isometric to an \( L^1(\mu) \) space. Suppose that \( F \) is a maximal face of the unit ball \( B(X) \). Then every maximal face of \( F \) is of codimension 1 in \( F \).

**Proof.** Let \( F_1 \) be a maximal face of \( F \). If \( F_1 \) is not of codimension 1 in \( F \), there exist \( x, y \) in \( F \) such that \( F_1, x \) and \( y \) are affinely independent. Since \( F \) is a linearly closed and linearly bounded, we may assume that \( x, y \) are such that \( \text{Aff} \{x, y\} \cap F = [x, y] \). Let

\[
F_2 = \{ z \in F : \lambda z + (1 - \lambda)z' = x \text{ for some } z' \in F \text{ and } 0 < \lambda < 1 \}.
\]

Then \( F_2 \) is a face which is not equal to \( F_1 \) and does not contain the point \( y \). By Proposition 1.10, \( \text{conv} (F_1 \cup F_2) \) is a proper face of \( F \). This contradicts the fact that \( F_1 \) is a maximal face of \( F \).

**Proposition 2.7.** Suppose that \( H_1 \) is a face of an \( L \)-polytope \( H \). Then \( H_1 \) is a maximal face of \( H \) if and only if \( H_1 \) is of codimension 1 in \( H \). (We assume that \( H \) is not a single point.)

**Proof.** We need only prove the necessity. Let \( K \) be an envelope of \( H \). Let \( K_0 \) be the smallest face of \( K \) containing \( H \). Then \( K_0 \) is either an \( L \)-ball or a Choquet simplex (Lemma 2.3). In the latter case it is a maximal face of an \( L \)-ball. Suppose now that \( H \) is of codimension \( n \) in \( K_0 \). Let \( K_1 \) be a maximal proper face of \( K_0 \) containing \( H_1 \). By Lemma 2.6, it is of codimension 1 in \( K_0 \). We can find \( y_1, \ldots, y_n \in K_1 \) such that \( H, y_1, \ldots, y_n \) are affinely independent and

\[
\text{Aff} (H \cup \{y_1, \ldots, y_n\}) = \text{Aff} K_0.
\]

If we can show that

\[
\text{Aff} (H_1 \cup \{y_1, \ldots, y_n\}) = \text{Aff} K_1,
\]
then $H_1$ is of codimension $(n+1)$ in $K_0$ hence of codimension 1 in $H$. In fact, for $x \in K_1$, we have

$$x = \lambda y + \sum_{i=1}^{n} \lambda_i y_i,$$

where $\lambda + \sum_{i=1}^{n} \lambda_i = 1$ and $y \in H$.

If $\lambda = 0$, then $x$ is in $\text{Aff}(H_1 \cup \{y_1, \ldots, y_n\})$. If $\lambda \neq 0$, then

$$\lambda^{-1} - \lambda^{-1} \sum_{i=1}^{n} \lambda_i = 1$$

and

$$\lambda^{-1} - \lambda^{-1} \sum_{i=1}^{n} \lambda_i y_i \in \text{Aff} K_1,$$

which implies that $y$ is in $H \cap \text{Aff} K_1$. But $H \cap \text{Aff} K_1$ is a proper face of $H$ containing $H_1$, hence $H \cap \text{Aff} K_1 = H_1$. This shows that

$$y \in H_1 \quad \text{and} \quad x \in \text{Aff}(H \cup \{y_1, \ldots, y_n\}).$$

The reverse inclusion is obvious, so we have

$$\text{Aff}(H_1 \cup \{y_1, \ldots, y_n\}) = \text{Aff} K_1$$

and the proof is complete.

**Lemma 2.8.** (Dubins [3]) Let $K$ be a linearly closed, linearly bounded convex set and let $M$ be a finite codimensional slice in $K$. Let $x$ be an extreme point in $M$, then $x$ is a finite convex combination of extreme points of $K$.

**Proof.** (This proof differs from that of Dubins.) Let $F$ be the smallest face of $K$ containing $M$. Then $M$ is of finite codimension in $F$. Let

$$F' = \{y \in F : \lambda y + (1-\lambda)z = x \text{ where } z \in F \text{ and } 0 < \lambda < 1\},$$

Then $F'$ is a face of $F$. Since $x$ is an extreme point of $M$, we have

$$\text{Aff} F' \cap \text{Aff} M = \{x\}$$

and since $M$ is of finite codimension in $F$, this implies that $\text{Aff} F'$ is finite dimensional. Since $F'$ is linearly closed and linearly bounded, it is compact. Hence $x$ is a convex combination of finitely many extreme points of $F'$ and these are also extreme points of $K$.

**Proposition 2.9.** If $H$ is an infinite dimensional $L$-polytope, then every maximal face $H_1$ of $H$ contains infinitely many extreme points.

**Proof.** Let $K_0, K_1$ be the faces containing $H, H_1$ respectively defined as in Proposition 2.7. We see that $K_0$ is an infinite dimensional $L$-ball or a Choquet simplex and $K_1$ is a maximal face of $K_0$. Hence it contains infinitely many extreme points. We will let $\partial_C C$ denote the set of extreme
points on a convex set \( C \). Suppose \( \partial_e H_1 \) were finite. By Dubins' lemma, there exists a finite set \( A \) contained in \( \partial_e K_1 \) such that each point of \( \partial_e H_1 \) is a convex combination of points in \( A \). Let \( B \subseteq \partial_e K_1 \setminus A \) such that \( B \) is a finite set and \( \text{conv} \ B \) has dimension greater than \( n \) where \( n \) is the codimension of \( H \) in \( K \). It is obvious that \( \text{conv} \ B \cap H \) is nonempty, compact and is a face of \( H \) hence contains an extreme point of \( H \). Furthermore,

\[
\text{conv} \ B \cap H \subseteq K_1 \cap H = H_1,
\]

so we can find an extreme point in \( H_1 \) which is not in \( \partial_e H_1 \) which is a contradiction.

**Proposition 2.10.** A maximal face of an infinite dimensional \( L \)-polytope cannot be centrally symmetric.

**Proof.** Let \( H, H_1, K_0, K_1 \) be defined as in Proposition 2.7. If \( \bar{H}_1 \subseteq K_1 \) then by a proof similar to Lemma 2.3, the face \( K_1 \) is compact and hence it is a Choquet simplex. Suppose \( \bar{H}_1 \) is symmetric about \( a \). \( \bar{H}_1 \) is also symmetric about \( a \). Let \( x + a \) be an extreme point of \( \bar{H}_1 \), then \( -x + a \) is also an extreme point and both of them are finite convex combinations of extreme points of \( K_1 \), hence the same is true for

\[
a = \frac{1}{2}(x + a) + \frac{1}{2}(-x + a).
\]

Let \( \mu \) be a probability measure representing \( a \) and supported by extreme points \( x_1, \ldots, x_n, y_1, \ldots, y_m \) of \( K_1 \) such that \( x + a \) and \( -x + a \) are convex combinations of \( x_1, \ldots, x_n \) and \( y_1, \ldots, y_m \) respectively. Since \( \bar{H}_1 \) is infinite dimensional, we can find another extreme point \( y \in \bar{H}_1 \) which is not in the affine variety generated by the above extreme points. Hence we can find another boundary probability representation for \( a \). This contradicts the existence of a unique boundary probability measure representing each point of a Choquet simplex.

Next, consider the case where \( \bar{H}_1 \not\subseteq K_1 \). Then there exists \( x \in \bar{H}_1 \setminus K_1 \)

\[
\subseteq K \setminus K_1. \quad \text{If} \ H_1 \ \text{has a symmetric center} \ a, \ \text{then} \ H_1 \ \text{also has} \ a \ \text{as a center of symmetry. Hence} \ x, \ -x + 2a \ \text{are symmetric with respect to} \ a \ \text{and}
\]

\[
\frac{1}{2}x + \frac{1}{2}(-x + 2a) = a \in \bar{H}_1 \subseteq K_1.
\]

If \( -x + 2a \in K \), then since \( K_1 \) is a face, \( x \) will be in \( K_1 \) which is impossible. Hence \( -x + 2a \notin K \), which also contradicts the fact that \( \bar{H}_1 \subseteq K \). We conclude that \( \bar{K}_1 \) cannot be centrally symmetric.

In [10], Phelps showed that the \( \beta \)-polytopes have certain extension properties and he also characterized the \( G_\beta \) face of such polytopes. These results can be generalized to the class of \( L \)-polytopes.
Lemma 2.11. (Phelps [10].) Suppose that $K$ is a compact convex set and $M$ is a closed finite codimensional slice of $K$. If $M$ is contained in no proper face of $K$, then any continuous affine function on $M$ can be extended to a continuous affine function on $K$.

Theorem 2.12. Suppose that $H$ is an $L$-polytope and that $F$ is a closed face of $H$. If $g$ is a continuous affine function on $F$, then $g$ admits an extension to a continuous affine functional $f$ on $H$.

Furthermore, there is a uniform bound on the norm of the extension.

Proof. Let $H = M_K$, where $K$ is an envelope of $H$ and $M$ is a closed finite codimensional slice of $K$. By Corollary 2.4, there exists a closed face $F$ of $K$ such that $F_1 \cap H = F$ and by Lemma 2.11, we can extend $g$ to $g'$ on $F_1$. By [7, Proposition 2.5], we can extend $g'$ to $g''$ on $K$. Let $f$ be the restriction of $g''$ to $H$; $f$ has the required property.

The last assertion follows from Alfsen [2, p. 114].

Theorem 2.13. If $H$ is an $L$-polytope and if $F$ is a closed face of $H$ which is $G_\delta$ in $H$, then there exists a continuous affine function $f \geq 0$ on $H$ such that

$$F = \{x \in H : f(x) = 0\}.$$ 

Proof. Let $H = M_K$, where $K$ is an $L$-ball and $M$ is a closed finite codimensional slice of $K$. By Corollary 2.4, we can find a closed $G_\delta$ face $F_1$ in $H$ such that $F = H \cap F_1$ and by [7, Proposition 2.7] there exists a continuous affine function $g \geq 0$ on $K$ such that

$$F_1 = \{x \in K : g(x) = 0\}.$$ 

Let the restriction of $g$ to $H$ be denoted by $f$; then $f$ is the required function.

3. Polyhedral vertices of $L$-polytopes.

Definition 3.1. Let $K$ be a compact convex subset of a locally convex space and define

$$\text{cone}(x, K) = x + \bigcup_{\lambda \geq 0} \lambda(K - x).$$

A point $x$ in $K$ is called a polyhedral vertex of $K$ if $\text{cone}(x, K)$ is closed and proper.

The definition was introduced by Alfsen and Nordseth [1], who proved that every extreme point of a Choquet simplex is a polyhedral vertex.
Hall-Pedersen [6] proved that this is also true for an $\alpha$-polytope. In what follows, we show that it is the case for an $L$-polytope.

**Lemma 3.2.** Every extreme point of an $L$-ball is a polyhedral vertex.

**Proof.** Let $K$ be the $L$-ball embedded into $A_0(K)^*$. Suppose $a$ is an extreme point of $K$, and suppose that $C = \text{cone}(0, K - a)$. We want to show that $C$ is $w^*$-closed in $A_0(K)^*$. By [4, Theorem 3.2, Theorem 4.1], we see that

$$K = (a - C) \cap (C - a)$$

for each extreme point $a$ of $K$. Hence

$$C \cap K = C \cap (a - C) \cap (C - a)$$

$$= C \cap (a - C) \cap (a - C) \cap (C - a)$$

$$= [(\frac{1}{2}C - \frac{1}{2}a) + \frac{1}{2}a] \cap [(\frac{1}{2}a - \frac{1}{2}C) + \frac{1}{2}a] \cap K$$

$$= \frac{1}{2}[(C - a) \cap (a - C) + a] \cap K$$

$$= \frac{1}{2}(K + a) \cap K$$

which is $w^*$-compact. By the Krein-Smulian theorem, the set $C$ is $w^*$-closed. That the cone is proper follows from the fact that $x$ is an extreme point of $K$. Thus, it is a polyhedral vertex of $K$.

**Lemma 3.3.** Let $C$ be a closed cone in a locally convex space and let $F$ be a finite dimensional subspace. Then $F + C$ is a closed cone.

**Proof.** Cf. [5, Proposition 7.5].

**Proposition 3.4.** If $H$ is an $L$-polytope, then every extreme point of $H$ is a polyhedral vertex.

**Proof.** Let $K$ be an envelope of $H$, so that

$$H = K \cap \{ x \in A_0(K)^* : h_i(x) = 0, \ i = 1, \ldots, n \},$$

$h_1, \ldots, h_n \in A(K)$. Let $a$ be an extreme point of $H$. First we claim that

$$\text{cone}(a, H) = M \cap \text{cone}(a, K)$$

where $M$ is the affine variety generated by $H$. Indeed, let

$$x \in M \cap \text{cone}(a, K)$$

and write $x = a + \lambda(y - a)$, where $\lambda \geq 0$ and $y \in K$. If $\lambda = 0$, then

$$x = a \in \text{cone}(a, H).$$

If $\lambda \neq 0$, then since $y \in M \cap K$, we have $y \in H$ and $x \in \text{cone}(a, H)$. 
If $a$ is an extreme point of $H$, then by Dubin's lemma, it is a convex combination of extreme points $\{x_1, \ldots, x_n\}$ of $K$. Let $F$ be the affine variety generated by $\{x_1, \ldots, x_n\}$; then

$$\text{cone}(a, K) = \text{cone}(x_1, K) + F.$$  

Indeed, by translation, we may let $a = 0$, so that $F$ is a linear subspace. For $\lambda k \in \text{cone}(0, K)$, $\lambda \geq 0$ and $k \in K$, we have

$$\lambda k = x_1 + \lambda(k - x_1) - (1 - \lambda)x_1 \in \text{cone}(x_1, K) + F.$$  

Conversely, suppose $z = x_1 + \lambda(k - x_1) + y \in \text{cone}(x_1, k) + F$ where $\lambda \geq 0$, $k \in K$ and $y \in F$. Since $(x_1 + y)$ and $\lambda(k - x_1)$ are in $\text{cone}(0, K)$, $\lambda \geq 0$ and $k \in K$, we have

$$\text{cone}(a, K) = \text{cone}(x_1, K) + F.$$  

By Lemma 3.1 and Lemma 3.3, the set $\text{cone}(a, K)$ is closed and by the first part of the proof, we see that $\text{cone}(a, H)$ is closed. It is also a proper cone since $a$ is an extreme point of $H$. Thus $a$ is a polyhedral vertex of $H$.

The following result was proved for Choquet simplexes by Alfsen and Nordseth [1]. We use a similar technique.

**Proposition 3.5.** If $H$ is an $L$-polytope such that the set $\text{cone}(x, H)$ is closed for each $x \in H$, then $H$ is finite dimensional.

**Proof.** Let $H = M_K$, where $K$ is an envelope of $H$ and $M$ is a closed finite codimensional slice of $K$. We will prove the proposition by the following steps:

1. Let

$$F_x = \{y \in H : \lambda y + (1 - \lambda)z = x \text{ for some } z \in H \text{ and } 0 < \lambda < 1\}.$$  

We claim that

$$F_x = H \cap (2x - \text{cone}(x, H))$$  

which will show that every $F_x$ is closed. Now, $y \in F_x$ if and only if there exist $z \in H$ and $0 < \lambda < 1$ such that

$$\lambda y + (1 - \lambda)z = x.$$  

Equivalently,

$$y = \lambda^{-1}x - (\lambda^{-1} - 1)z$$  

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where $z \in H$ and $0 < \lambda < 1$, that is,

$$y = 2x - ((\lambda^{-1} - 1)(z - x) + x), \quad z \in H, \quad 0 < \lambda < 1,$$

which means $y \in H \cap (2x - \text{cone}(x, H))$.

(ii) If $H_1$ is a maximal face of $H$ which contains infinitely many extreme points, then there exists a point $x$ in $H_1$ such that $F_x \cap \partial_e H_1$ is a countable set. Indeed, let $\{x_i\}_{i=1}^{\infty}$ be a countable set of extreme point in $\partial_e H_1$. Let

$$x = \sum_{i=1}^{\infty} \lambda_i x_i, \quad \text{where } \lambda_i > 0 \text{ and } \sum_{i=1}^{\infty} \lambda_i = 1.$$

Let $B$ be the set of extreme points in the envelope $K$ such that each member of $B$ is a convex component of the $x_i$. By Dubins’ lemma, the set $B$ is countable. Let $G_x$ be the smallest face on $K$ containing $F_x$. By Lemma 2.3 and by (i), we see that $G_x$ is compact and hence it is a Choquet simplex. We claim that

$$G_x \cap \partial_e K = B.$$

It is clear that $G_x \cap \partial_e K \supseteq B$. On the other hand, if

$$y \in (G_x \cap \partial_e K) \setminus B,$$

then $x = \lambda y + (1 - \lambda)z$ for some $z \in K$ and $0 < \lambda < 1$. Let $\mu_z$ be a boundary probability measure representing $z$. Then $\lambda \mu_y + (1 - \lambda)\mu_z$ is a boundary probability measure representing $x$. We thus have two boundary probability measures representing $x$ and supported by $B$. This contradicts the fact that $G_x$ is a Choquet simplex. Hence $G_x \cap \partial_e H = B$. It follows from Dubins’ lemma and $G_x \cap H = F_x$ that $F_x$ contains only countably many extreme points.

(iii) Give $\partial_e H_1$ a topology generated by the family of subbasic closed sets of $I$ consisting of $\partial_e H_1$ and sets of the form $F \cap \partial_e H_1$ where $F$ is closed face of $H_1$. We claim that $\partial_e H_1$ is compact under this topology. Indeed, let $\{A_\alpha\}$ be a family of subbasic closed sets in $I$ with the finite intersection property, then there exists a family of closed faces $\{G_\alpha\}$ of $H_1$ such that $G_\alpha \cap \partial_e H_1 = A_\alpha$ for each $\alpha$. We know that $\bigcap G_\alpha$ is a nonvoid compact face in $H_1$, so it contains extreme points of $\partial_e H_1$. This shows that $\bigcap A_\alpha$ is nonempty and thus $\partial_e H_1$ is compact.

(iv) The set $\partial_e H_1$ is a finite set. Indeed, suppose it is an infinite set, by (ii) we can find a closed face $F_0 \subseteq H_1$ with countably many extreme points, say, $\partial_e F_0 = \{x_i\}_{i=1}^{\infty}$. Let $F_n$ be the face generated by $\{x_i\}_{i=n}^{\infty}$ as in (ii) and let $A_n$ denote the set of extreme points of $F_n$. We see that $\{A_n\}_{n=1}^{\infty}$ is a family of compact sets in $\partial_e H$ and has the finite inter-
section property, so \( \cap_{n=1}^{\infty} A_n \neq \emptyset \) which contradicts the fact that the family \( \{A_n\}_{n=1}^{\infty} \) has void intersection.

(v) By (iv) and Proposition 1.9, we conclude that \( H \) is a finite dimensional polytope.

4. Some examples.

In this section, we are going to give some examples which show that some properties of a finite dimensional polytope do not hold in the class of \( L \)-polytopes.

**Example 4.1.** The product of two \( L \)-polytopes need not be an \( L \)-polytope.

Let \( F \) be an infinite dimensional Choquet simplex, considered as a subset of \( A(F)^* \). Let

\[
K = \text{conv}(F \cup -F),
\]

that is \( K = \) unit ball of \( A(F)^* \). Then \( K \) is an \( L \)-polytope. We show that \( K \times K \) is not an \( L \)-polytope. If \( K \times K \) were an \( L \)-polytope, we would have \( K \times K = M_K \), where \( K' \) is an envelope of \( K \times K \) and \( M_K \), is a closed finite codimensional slice of \( K \). The set \( K \times F \) is a maximal face of \( K \times K \) and is compact. Let \( F_1 \) be a maximal face of \( K' \) containing \( K \times F \). Then since \( K \times F \) is finite codimensional in \( F_1 \), we see that \( F_1 \) is compact and is a Choquet simplex.

Let \( x_1, x_2 \) be two extreme points of \( K \) such that \( x_2 \in F \), so \( (x_1, x_2), (-x_1, x_2) \) are extreme points of \( K \times F \). We can write

\[
(x_1, x_2) = \sum_{i=1}^{n} \lambda_i y_i, \quad (-x_1, x_2) = \sum_{j=1}^{m} \beta_j z_j,
\]

where \( \lambda_i, \beta_j \geq 0 \), \( \sum_{i=1}^{n} \lambda_i = 1 \), \( \sum_{j=1}^{m} \beta_j = 1 \) and \( y_i, z_j \) are extreme points of \( F_1 \), \( i = 1, \ldots, n \), \( j = 1, \ldots, m \), and hence

\[
(0, x_2) = \frac{1}{2}(\sum_{i=1}^{n} \lambda_i y_i + \sum_{j=1}^{m} \beta_j z_j).
\]

Since \( K \) is infinite dimensional, there exists an extreme point \( x_3 \) of \( K \) such that \( (x_3, x_2) \) is not in the linear subspace generated by \( y_i, z_j, i = 1, \ldots, n \), \( j = 1, \ldots, m \). Similarly, we have

\[
(x_3, x_2) = \sum_{i=1}^{k} \lambda'_i y'_i, \quad (-x_3, x_2) = \sum_{j=1}^{k} \beta'_j z'_j
\]

and

\[
(0, x_2) = \frac{1}{2}(\sum_{i=1}^{k} \lambda'_i y'_i + \sum_{j=1}^{m} \beta'_j z'_j),
\]

where \( \sum_{i=1}^{k} \lambda'_i = 1, \sum_{j=1}^{k} \beta'_j = 1, \lambda'_i, \beta'_j \geq 0 \) and \( y'_i, z'_j \) are extreme points of \( F_1 \), thus we can find two boundary measures representing the point.
$(0,x)$ of the Choquet simplex $F_1$. This contradiction shows that $K \times K$ is not an $L$-polytope.

**Example 4.2.** The intersection of two $L$-polytopes is not necessarily an $L$-polytope.

Let $c$ be the set of real sequences of the form $y=(y_n)_{n=1}^\infty$ such that

$$y_1 = \lim_{n \to \infty} y_n,$$

with supremum norm. Then $l_1$ is the dual of $c$ and we let $B$ denote the unit ball of $l_1$. If

$$S = \{x : x_n \geq 0 \text{ for each } n, \sum_{n=1}^\infty x_n = 1\},$$

then $S$ is $w^*$-closed and is a Choquet simplex. Take $x \in S$ to be the sequence

$$x = (\frac{1}{2} + 2^{-2}, 2^{-3}, 2^{-4}, \ldots).$$

Consider the set $x - B$. Each $y \in x - B$ can be written uniquely as $y = z + \alpha x$ where $z \in \text{lin}(x-S)$. We define

$$B_1 = \{z - \alpha x : z + \alpha x \in x - B, \ z \in \text{lin}(x-S)\}$$

and let $B_2 = B - x$. Both $B_1$ and $B_2$ are $L$-polytopes, but we claim that $B_1 \cap B_2$ is not an $L$-polytope. Since $S - x$ and $x - S$ are maximal faces of $B_1, B_2$ respectively, $(S - x) \cap (x - S)$ is a face of $B_1 \cap B_2$. It is a proper face because

$$-x \in (B_1 \cap B_2) \setminus ((S - x) \cap (x - S)).$$

To show that it is maximal, we need only show that it is of codimension 1. We let $\delta_n = (x_i)_{i=1}^\infty$ be the points in $S$ such that $x_i = 0$ for $i \neq n$ and $x_n = 1$. Note that

$$2^{-(n-1)}(\delta_n - \delta_1) \in (S - x) \cap (x - S)$$

for $n > 1$ and $\text{lin}(S - x)$ is the $w^*$-closed subspace generated by $\{\delta_n - x\}_{n=1}^\infty$. Furthermore, $\text{lin}((S - x) \cap (x - S))$ is the $w^*$-closed subspace generated by $\{\delta_n - \delta_1\}_{n=2}^\infty$. Now

$$\delta_1 - x = (2^{-2}, 2^{-3}, 2^{-4}, \ldots) = \sum_{n=2}^\infty 2^{-(n+1)}(\delta_1 - \delta_n).$$

This implies that $(\delta_1 - x)$ is in $\text{lin}((S - x) \cap (x - S))$ and

$$\delta_n - x = (\delta_n - \delta_1) + (\delta_1 - x) \in \text{lin}((S - x) \cap (x - S)) \cap \text{lin}((S - x) \cap (x - S)).$$

Hence

$$\text{lin}((S - x) \cap (x - S)) = \text{lin}(S - x).$$
Thus, we conclude that \((S-x)\cap(x-S)\) is of codimension 1 in \(B_1 \cap B_2\) and hence it is a maximal face of \(B_1 \cap B_2\). Notice that \((S-x)\cap(x-S)\) is a symmetric set with center of symmetry 0. By Proposition 2.10, the set \(B_1 \cap B_2\) cannot be an \(L\)-polytope.

**Example 4.3.** There exists a compact convex set \(K\) such that each extreme point is a polyhedral vertex, but \(K\) is not an \(L\)-polytope.

We first observe that if \(K_1, K_2\) are two compact convex sets such that \(x_1 \in K_1, x_2 \in K_2\) are polyhedral vertexes, then \((x_1, x_2) \in K_1 \times K_2\) is a polyhedral vertex of \(K_1 \times K_2\). Now, we can choose \(K_1, K_2\) to be two \(L\)-polytopes such that \(K_1 \times K_2\) is not an \(L\)-polytope (Example 4.1) but every extreme point of \(K_1 \times K_2\) is a polyhedral vertex.

**BIBLIOGRAPHY**


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