A Note on Lower Semicontinuous Set-valued Maps*)

by

KA-SING LAU

Presented by K. KURATOWSKI on April 10, 1975

**Summary.** Let \( X \) be a topological space, \( K \) a compact subset of a locally convex space \( Y \) and \( c(K) \) the family of closed convex subsets of \( K \). It is shown that a map \( \Phi : X \to c(K) \) is lower semicontinuous if the set \( \{ x : \Phi(x) \cap H \neq \emptyset \} \) is open for any \( H = \{ x : f(x) > r \} \) where \( f \) is a continuous functional on \( Y \). A simple example in \( R^2 \) shows that the compactness assumption on \( K \) is essential.

1. **Introduction.** Let \( X, Y \) be topological spaces. We use \( 2^Y \) to denote the family of closed subsets of \( Y \). A set-valued map \( \Phi : X \to 2^Y \) is called lower semicontinuous if for any open set \( U \) in \( Y \), the set \( \{ x \in X : \Phi(x) \cap U \neq \emptyset \} \) is open in \( X \). It is well known that the lower semicontinuous set-valued maps play a significant role in the continuous selection theories (cf. [3, 4]). If \( Y \) is a locally convex (Hausdorff) linear topological space and \( K \) a closed subset of \( Y \), we let \( c(K) \) denote the family of closed convex subsets of \( K \). A set-valued map \( \Phi : X \to 2^Y \) is called weakly lower semicontinuous if the set \( \{ x \in X : \Phi(x) \cap H \neq \emptyset \} \) is open for any open half space \( H = \{ y \in Y : f(y) > r \} \), where \( f \in Y^*, \ r \in R \). Our main purpose is to prove

**Theorem 1.** Suppose \( X \) is a topological space, \( Y \) a locally convex space and \( K \) a compact subset of \( Y \). Let \( \Phi : X \to c(K) \) be a set-valued map. Then \( \Phi \) is lower semicontinuous if and only if it is weakly lower semicontinuous.

Although the given topology and the \( w(Y, Y^*) \) topology coincide in \( K \), it is not immediate that the sets \( \{ x \in X : \Phi(x) \cap H_i \neq \emptyset \}, \ i = 1, \ldots, n \) are open, will imply that \( \{ x \in X : \Phi(x) \cap \bigcap_{t=1}^n H_t \neq \emptyset \} \) is open. Hence one side of the theorem is non-trivial.

2. **Proof of the theorem.** Theorem 1 will result from the following lemmas. In the proofs, we will make use of the Vietoris topology [1]. Let \( X \) be a topological space. A subbase for the Vietoris topology on \( 2^X \) consists of all sets having one of the following forms:

\[ \{ F \in 2^X : F \cap U \neq \emptyset \}, \quad \{ F \in 2^X : F \subseteq U \}, \]

where \( U \) is an arbitrary open set in \( X \).

*) Revised version received on August 5, 1975.

[271]
Lemma 2. Let $X$ be a topological space and let $2^X$ be given the Vietoris topology, then

(i) if $X$ is compact, so is $2^X$.

(ii) if $X$ is regular, let $\{F_a\}_{a \in I}$ be a net in $2^X$ converging to $F_0$, then for each $x \in X$, we have an equivalence: $x \in F_0$ if and only if there exists a net $\{x_\alpha\}_{\alpha \in I}$, $x_\alpha \in F_a \forall \alpha \in I$, converging to $x$ in $X$.

Proof. (i) follows from Theorem 15 in [1]. Suppose the necessity part of (ii) were not true, there exist a subnet $\{F_\beta\}_{\beta \in J}$ of $\{F_a\}_{a \in I}$ and an open neighborhood $U$ of $x$ such that $F_\beta \cap U = \emptyset$ for each $\beta$. Note that $\{F_\beta\}_{\beta \in J}$ converges to $F_0$. The family

$$\mathcal{J} = \{F \in 2^X : F \cap U = \emptyset\}$$

is a closed subset in $2^X$ and $F_\beta \in \mathcal{J}$ for all $\beta \in J$. But $F_0 \notin \mathcal{J}$ (for $x \in F_0 \cap U$), a contradiction. The sufficiency follows immediately from the definition of the Vietoris topology and the regularity of the space $X$.

Lemma 3. Let $X$ be a subset of a locally convex space and let $c(X)$ be the family of closed convex subsets of $X$, then $c(X)$ is closed in $2^X$.

Proof. Let $\{F_a\}_{a \in I}$ be a net in $c(X)$ converging to $F_0$. We only need to show that $F_0$ is convex. Consider $\lambda x + (1 - \lambda) y, x, y \in F_0, 0 < \lambda < 1$. By Lemma 2 (ii), there exist two nets $\{x_\alpha\}_{\alpha \in I}, \{y_\alpha\}_{\alpha \in I}, x_\alpha, y_\alpha \in F_a \forall \alpha \in I$, converging to $x, y$, respectively. Since $F_a$ is convex, $\lambda x_\alpha + (1 - \lambda) y_\alpha$ is in $F_a$ for each $\alpha \in I$. That $\lambda x + (1 - \lambda) y$ and Lemma 2 (ii) imply that $\lambda x + (1 - \lambda) y$ is in $F_0$. Hence $F_0$ is convex.

Our key step is to prove the following proposition.

Proposition 4. Let $Y$ be a locally convex space, $K$ a compact subset of $Y$, $y_0 \in K$ and $U$ an open neighborhood of $y_0$ in $Y$. Then there are open half spaces $H_1, ..., H_n$ in $Y$ containing $y_0$ such that every closed convex subset $S \subseteq K$ which intersects $H_1, ..., H_n$ must intersect $U$.

Proof. Let $D = K \setminus U$. Then $D$ is compact, and so is $2^D$ with the Vietoris topology. By Lemma 3, $c(D)$ is closed and hence compact in $2^D$. Let $\mathcal{H}$ be the collection of open half spaces in $Y$ containing $y_0$. Since $Y$ is locally convex, by the separation theorem, each $F$ in $c(D)$ is a subset of $Y \setminus \mathcal{H}$ for some $H \in \mathcal{H}$. Hence the sets

$$\mathcal{U}_H = \{F \in 2^D : F \subseteq Y \setminus H\}, \quad H \in \mathcal{H}$$

form an open cover of $c(D)$. There exists a finite subcover $\mathcal{U}_{H_1}, ..., \mathcal{U}_{H_n}$. These $H_1, ..., H_n$ satisfy the requirement. Indeed, if $S$ is a closed convex subset in $K$ such that $S \cap U = \emptyset$, then $S \subseteq D$ and $S \subseteq \mathcal{U}_{H_i}$ for some $i = 1, ..., n$. This implies that $S \cap H_i = \emptyset$ for some $i = 1, ..., n$.

Proof of Theorem 1. The necessity is clear. To prove the sufficiency, it is enough to prove that for any open set $U$ in $Y$ such that $\Phi(x_0) \cap U = \emptyset$, the set $\{x \in X : \Phi(x) \cap U \neq \emptyset\}$ is a neighborhood of $x_0$. Let $y_0 \in \Phi(x_0) \cap U$ and let
$H_1, \ldots, H_n$ be the open half spaces constructed in the above proposition. Since each $\Phi(x)$ is closed and convex, it follows that

$$\bigcap_{i=1}^{n} \{x \in X : \Phi(x) \cap H_i \neq \emptyset\} \subseteq \{x \in X : \Phi(x) \cap U \neq \emptyset\}.$$  

By assumption each set on the left side is an open neighborhood of $x_0$, hence so is $\{x \in X : \Phi(x) \cap U \neq \emptyset\}$.

3. Remarks. Combining Theorem 1 and the Michael selection theorem, we have

COROLLARY 5. Suppose $X$ is a compact Hausdorff space, $Y$ a metrizable locally convex space and $K$ a compact subset of $Y$. Let $\Phi : X \rightarrow c(K)$ be a weakly lower semicontinuous map and let $f$ be a continuous function defined on a closed subset $F$ in $X$ with values in $Y$ and such that $f(x) \in \Phi(x)$ for each $x \in F$. Then $f$ can be extended to a continuous function $\overline{f}$ on $X$ such that $\overline{f}(x) \in \Phi(x)$ for each $x \in X$.

An application of this corollary is shown in [2]. We finally remark that Theorem 1 will not be true if we do not assume that each of the $\Phi(x)$ is contained in a compact subset of $Y$. Consider the map $\Phi$ from $X = [0, 1]$ into $c(R^2)$ with $\Phi(0) = \{(1, y_2) : y_2 \in R\}$ and $\Phi(x) = \{(y_1, y_2) : x \cdot y_2 = y_1, y_1, y_2 \in R\}$ for $x \neq 0$. Then $\Phi$ is not lower semicontinuous. But for any open half space $H$ in $R^2$, the set $\{x \in X : \Phi(x) \cap H \neq \emptyset\}$ is either $[0, 1]$ or $[0, 1] \setminus \{x'\}$ for some $x'$ in $X$, hence $\Phi$ is weakly lower semicontinuous. If $Y = R$, the two conditions will be equivalent even without the compactness condition. For in this case we have

$$\{x \in X : \Phi(x) \cap (a, b) \neq \emptyset\} = \{x \in X : \Phi(x) \cap (-\infty, b) \neq \emptyset\} \cap \{x : \Phi(x) \cap (a, \infty) \neq \emptyset\}$$

for any $a, b$ in $R$ with $a < b$.

UNIVERSITY OF PITTSBURGH, PITTSBURGH, PA. 15260 (U.S.A.)

REFERENCES


Ка-Синг Лоу, Заметка о полунепрерывных снизу многовалентных преобразованиях

Содержание. Пусть $X$ будет топологическим пространством, $K$ — компактным множеством локально выпуклого пространства $Y$, $c(K)$ — семейством замкнутых выпуклых подмножеств. Докажем, что преобразование $\Phi : X \rightarrow c(K)$ есть полунепрерывное снизу если множество $\{x : \Phi(x) \cap H \neq \emptyset\}$ открыто для любого $H = \{x : f(x) > r\}$, где $f$ является непрерывным функционалом на $Y$. Простейший пример в $R^2$ показывает, что предположение компактности $K$ существенно.