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Separation conditions for conformal iterated function systems

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Abstract We extend both the weak separation condition and the finite type condition to include finite iterated function systems (IFSs) of injective C^1 conformal contractions on compact subsets of $\mathbb{R}^d$. For conformal IFSs satisfying the bounded distortion property, we prove that the finite type condition implies the weak separation condition. By assuming the weak separation condition, we prove that the Hausdorff and box dimensions of the attractor are equal and, if the dimension of the attractor is α, then its α-dimensional Hausdorff measure is positive and finite. We obtain a necessary and sufficient condition for the associated self-conformal measure μ to be singular. By using these we give a first example of a singular invariant measure μ that is associated with a *non-linear* IFS with overlaps.

Keywords Conformal iterated function system · Self-conformal measure · Weak separation condition · Finite type condition · Singularity · Absolute continuity · Hausdorff measure

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1 Introduction

Let X be a nonempty compact subset of $\mathbb{R}^d$ and $S_i : X \to X$, $i = 1, \ldots, N$, be mappings. We call $\{S_i\}_{i=1}^N$ an *iterated function system* (*IFS*) on X. It is well known that if the S_i are *contractions*, then there exists a unique non-empty compact subset K of X such that

$$K = \bigcup_{i=1}^{N} S_i(K) \tag{1.1}$$

(see [5,9]). We call K the *invariant set* or *attractor* of the IFS. If we associate the IFS with a set of probability weights $\{p_i\}_{i=1}^N$, then there is a unique probability measure μ with $\mathrm{supp}(\mu) = K$ satisfying

$$\mu(A) = \sum_{i=1}^{N} p_i \mu \circ S_i^{-1}(A), \tag{1.2}$$

for all Borel sets $A \subseteq X$. We call μ the *invariant measure* of the IFS associated with the weights $\{p_i\}_{i=1}^N$. It is well known that the invariant measure is either absolutely continuous or singular continuous with respect to Lebesgue measure.

Recall that an IFS $\{S_i\}_{i=1}^N$ satisfies the well-known *open set condition* (*OSC*) if there exists a nonempty bounded open set $U \subseteq \mathbb{R}^d$, called an *OSC set*, such that $\bigcup_{i=1}^N S_i(U) \subseteq U$ and $S_i(U) \cap S_j(U) = \emptyset$ for all $i \neq j$. In this case, we can lift the measure μ in (1.2) to a symbolic space, and many properties can be derived from there. Conformal finite and infinite IFSs and conformal graph directed Markov systems that satisfy the OSC have been studied extensively by Mauldin and Urbański (see [17–19] and the references therein). Conformal IFSs consisting of finitely many mappings and satisfying the OSC have been studied by Patzschke [21], Fan and Lau [6], Lau et al. [14], Peres et al. [22], and Ye [24,25]. We remark that in this paper, we only study IFSs that consist of finitely many mappings.

IFSs that do not satisfy the OSC are said to have overlaps. In this case, it is in general much harder to understand the structure of the corresponding invariant set K and the invariant measures μ. A well-known family of examples is provided by

$$S_1(x) = \rho x, \quad S_2(x) = \rho x + (1 - \rho), \tag{1.3}$$

with $1/2 < \rho < 1$ and $\mu_\rho = \frac{1}{2}(\mu_\rho \circ S_1^{-1} + \mu_\rho \circ S_2^{-1})$. This class of μ_ρ is known as the infinite Bernoulli convolutions, because it is the distribution of the random variable $(1 - \rho) \sum_{n=0}^{\infty} \rho^n X_n$, where the X_n are i.i.d. Bernoulli random variables taking the values 0 and 1 with probability $1/2$ on each. We refer the reader to the excellent survey article by Peres, Schlag and Solomyak [23] and the references therein.

One of the most important cases in (1.3) is when ρ^{-1} is the golden ratio (or more generally a Pisot number), in which μ_ρ is singular. This and the other examples motivated subsequent studies of two classes of IFSs with overlaps that are governed by two separation conditions: the *weak separation condition* (*WSC*) and the *finite type condition* (*FTC*). The WSC was introduced by Lau and Ngai [11] to extend the open set condition while allowing overlaps on the iteration. It has been studied extensively by Zerner [26], Lau et al. [13], and Lau and Wang [15]. The FTC strengthens the weak separation condition (in the generic cases) to allow tractable calculations. It was formulated by Ngai and Wang in [20] to yield graph directed systems to calculate the Hausdorff dimension of self-similar sets with overlaps. If ρ^{-1} is a Pisot number, the IFS in (1.3) satisfies the WSC [11] and is in fact of finite type [20]. In the same vein, the FTC was also used by Feng [7,8] and Deng et al. [3] to study self-similar measures. Recently it has been generalized by Jin and Yau [10] and Ngai and Lau [12] to include a larger family of contractive similitudes on $\mathbb{R}^d$.

So far in both considerations, the IFSs are either similitudes or affine maps. We note that for the Bernoulli convolution associated with the golden ratio and other examples that have been studied, the singularity of the invariant measures is very much dependent on the number theoretic properties and the affinity of the IFS. It is not clear that there exists a *non-linear* IFS with overlaps such that the associated invariant measure μ is singular and is supported by a set with a non-void interior. Our original goal is to construct such an example. In the process, we extend the WSC and the FTC to include conformal IFSs and give a necessary and sufficient condition for the absolute continuity of the invariant measures associated with such IFSs. This is the main theme of the paper.

Unless otherwise stated we assume that each S_i in an IFS $\{S_i\}_{i=1}^N$ extends to a C^1 contraction $S_i : V \to V$, where V is a fixed open connected neighborhood of X. We use the following sets of indices

$$\Sigma_k := \{1, \ldots, N\}^k, \quad \Sigma_* := \bigcup_{k\geq 0} \Sigma_k, \quad \text{and} \quad \Sigma = \Sigma_1^{\mathbb{N}}$$

(with $\Sigma_0 := \{\emptyset\}$). For $I = (i_1, \ldots, i_k) \in \Sigma_k$, we denote by $|I| = k$ the *length* of I and let $S_I := S_{i_1} \circ \cdots \circ S_{i_k}$. Also let

$$r_I = \inf_{x\in V} |\det S_I'(x)|^{1/d}, \quad r = \min_{1\leq i\leq N} r_i, \ \ R_I = \sup_{x\in V} |\det S_I'(x)|^{1/d}, \quad R = \max_{1\leq i\leq N} R_i. \tag{1.4}$$

If $S = S_I$ for some $I \in \Sigma_*$, we let $R_S := R_I$. Let $\mathcal{L}$ denote the Lebesgue measure on $\mathbb{R}^d$. For any $E \subseteq \mathbb{R}^d$, we let $\dim_{\mathrm{H}} E$, $\dim_{\mathrm{B}} E$, and $\dim_{\mathrm{P}} E$ denote the Hausdorff, box, and packing dimensions of E, respectively; moreover, we let $\mathcal{H}^s(E)$ and $\mathcal{P}^s(E)$ denote the s-dimensional Hausdorff and packing measures of E, respectively.

For $0 < b \leq 1$, we let

$$\mathcal{I}_b = \left\{I = (i_1, \ldots, i_n) : R_I \leq b < R_{i_1\cdots i_{n-1}}\right\} \quad \text{and} \quad \mathcal{A}_b = \{S_I : I \in \mathcal{I}_b\}.$$

Note that it is possible that $S_I = S_{I'}$ for distinct $I, I' \in \mathcal{I}_b$, and we identify such S_I and $S_{I'}$ in $\mathcal{A}_b$. Note also that each $\mathcal{I}_b$ is an antichain in Σ_*, with the partial order $I \preccurlyeq J$ if I is an initial segment of J or $I = J$.

Definition 1.1 We say that $\{S_i\}_{i=1}^N$ satisfies the weak separation condition (WSC) if there exists a constant $\gamma \in \mathbb{N}$ and a subset $D \subseteq X$, with $D^\circ \neq \emptyset$, such that for any $0 < b \leq 1$ and $x \in X$,

$$\#\{S \in \mathcal{A}_b : x \in S(D)\} \leq \gamma.$$

We remark that if the OSC holds, we can take D to be an OSC set and let $\gamma = 1$ to show that the WSC also holds.

Recall that a map $S : V \to V$ is *conformal* on V if for each $x \in V$, $S'(x)$ is a similarity matrix, i.e., a scalar multiple of an orthogonal matrix. Under this assumption, we have

$$|\det S'(x)| = \|S'(x)\|^d, \tag{1.5}$$

where $\|S'(x)\| := \sup\left\{|S'(x)\mathbf{y}| : |\mathbf{y}| = 1\right\}$ is the operator norm of the matrix $S'(x)$. For the purposes of this paper, we will restrict the conformal maps in our IFS to be injective C^1 contractions, as stated in the following definition.

Definition 1.2 We say that $\{S_i\}_{i=1}^N$ is an IFS of injective C^1 conformal contractions on a compact subset $X \subseteq \mathbb{R}^d$ if the S_i can be extended to C^1 injective conformal contractions on some open connected neighborhood V of X and furthermore,

$$0 < \inf_{x \in V} \|S_i'(x)\| \leq \sup_{x \in V} \|S_i'(x)\| < 1, \quad \text{for} \quad 1 \leq i \leq N. \tag{1.6}$$

For such an IFS, we call the associated invariant set in (1.1) a *self-conformal set*, and a measure μ in (1.2) a *self-conformal measure*.

Recall that an IFS $\{S_i\}_{i=1}^N$ has the *bounded distortion property* (*BDP*) if there exists a constant $c_1 > 0$ such that for any index $I \in \Sigma_*$,

$$\frac{|\det S_I'(x)|}{|\det S_I'(y)|} \leq c_1^d \quad \text{for all } x, y \in V. \tag{1.7}$$

It is easy to see that if each $\log|\det S_i'|$ is Hölder continuous, then $\{S_i\}_{i=1}^N$ has the BDP. Moreover, by adopting a proof in [6, Lemma 2.3], it can be shown that the Dini condition on $\log|\det S_i'|$ implies the BDP.

Theorem 1.1 *Let K be the attractor of an IFS $\{S_i\}_{i=1}^N$ of injective C^1 conformal contractions on a compact subset $X \subseteq \mathbb{R}^d$ that has the BDP and satisfies the WSC. Then $\alpha := \dim_{\mathrm{H}} K = \dim_{\mathrm{B}} K = \dim_P K$ and $0 < \mathcal{H}^\alpha(K) \leq \mathcal{P}^\alpha(K) < \infty$.*

Let $\{p_i\}_{i=1}^N$ be probability weights associated with $\{S_i\}_{i=1}^N$ and let μ be the invariant measure. For $S \in \mathcal{A}_b$, let $p_S = \sum\{p_I : S_I = S, I \in \mathcal{I}_b\}$. The following are two useful theorems concerning the singularity and absolute continuity of a self-conformal measure.

Theorem 1.2 *With the same hypotheses and notation as Theorem 1.1, an associated self-conformal measure μ is singular with respect to $\mathcal{H}^\alpha|_K$ if and only if there exist $0 < b \le 1$ and $S \in \mathcal{A}_b$ such that $p_S > R_S^\alpha$.*

Theorem 1.3 *Let $\{S_i\}_{i=1}^N$ be as in Theorem 1.1. If the invariant measure μ is absolutely continuous with respect to $\mathcal{H}^\alpha|_K$, then the Radon-Nikodym derivative of μ is bounded.*

Theorems 1.2 and 1.3 generalize the results in [13, Theorems 1.1 and 1.2] and [15, Theorems 1.1 and 1.2], where the IFS $\{S_i\}_{i=1}^N$ consists of contractive similitudes.

The finite type condition was first introduced in [20] for IFSs of similitudes that have exponentially commensurable contraction ratios. In [10] and [12], this artificial condition on the contraction ratios is removed. The more general definition in this paper is formally the same as that in [12] for similitudes. It is given in Definition 5.3.

Theorem 1.4 *Assume that $\{S_i\}_{i=1}^N$ is an IFS of injective C^1 conformal contractions on a compact subset $X \subseteq \mathbb{R}^d$ that has the BDP. Then the finite type condition implies the weak separation condition.*

Theorem 1.4 allows us to construct a family of conformal IFSs that do not satisfy the OSC but are of finite type and thus satisfy the WSC. For example we can take

$$S_1(x) = ax, \quad S_2(x) = bx^2 + cx + ad, \quad S_3(x) = abx^2 + cx + d, \tag{1.8}$$

where $0 < a, b, c, d < 1$, $ab + c + d = 1$, $b + c + ad \le d$ and $2b + c < 1$. By using Theorem 1.2 and by choosing the parameters and the probability weights suitably, we can control the singularity and absolute continuity of the invariant measure. In Sect. 6 we give a detailed study of such an example and provide additional examples obtained by suitably choosing the four parameter values. To the best of our knowledge these are the first known examples of singular invariant measures which are associated with *non-linear* IFSs and are supported on some interval.

It is not clear to us whether some of the self-conformal measures in the above family are absolutely continuous. Nevertheless, we show in Sect. 7 that by allowing the probability weights to be place-dependent, we can construct absolutely continuous self-conformal measures defined by non-linear IFSs with overlaps.

This paper is organized as follows. Section 2 develops some basic properties of conformal IFSs with the BDP. In Sect. 3, we prove several equivalent conditions for the WSC and use them to prove Theorem 1.1. In Sect. 4 we study the absolute continuity of self-conformal measures and prove Theorems 1.2 and 1.3. In Sect. 5, we study the finite type condition and prove that, for conformal IFSs with the BDP, it implies the WSC (i.e., Theorem 1.4). In Sect. 6, we give a class of examples of conformal IFSs that satisfy the finite type condition, and use this to illustrate the main theorems. Finally, in Sect. 7, we construct another class of self-conformal IFSs with the finite type conditions. Using these IFSs we construct a class of absolutely continuous self-conformal measures associated with place-dependent probabilities.

2 Conformal iterated function systems

In this section we establish some basic properties of conformal IFSs with the BDP, which we will need later in the paper. Some of them are known and we include simple proofs here for completeness.

Lemma 2.1 *Let $\{S_i\}_{i=1}^N$ be an IFS of injective C^1 conformal contractions on a compact subset $X \subseteq \mathbb{R}^d$, let r, R be defined as in (1.4), and let c_1 be the constant in the definition of the BDP (see (1.7)). Suppose $\{S_i\}_{i=1}^N$ has the BDP. Then the following hold.*

(a) *For any $I \in \mathcal{I}_b$, $0 < b \le 1$, and any measurable set $A \subseteq V$,*

$$\left(\frac{br}{c_1}\right)^d \mathcal{L}(A) \le \mathcal{L}(S_I(A)) \le b^d \mathcal{L}(A).$$

(b) *For any $I, J \in \mathcal{I}_b$, $0 < b \le 1$, and any measurable set $A \subseteq V$,*

$$\left(\frac{r}{c_1}\right)^d \mathcal{L}(S_J(A)) \le \mathcal{L}(S_I(A)) \le \left(\frac{c_1}{r}\right)^d \mathcal{L}(S_J(A)).$$

Proof (a) Let $I = (i_1, \dots, i_n) \in \mathcal{I}_b$ and V be as in Definition 1.2. Then by the definition of $\mathcal{I}_b$,

$$\sup_{x \in V} |\det S_I'(x)| \le b^d < \sup_{x \in V} |\det S_{i_1 \cdots i_{n-1}}'(x)|.$$

These inequalities together with the BDP imply that for all $x \in V$,

$$\begin{aligned} b^d &\ge |\det S_I'(x)| = |\det S_{i_1 \cdots i_{n-1}}'(S_{i_n}(x))| \cdot |\det S_{i_n}'(x)| \\ &\ge r^d \inf_{x \in V} |\det S_{i_1 \cdots i_{n-1}}'(x)| \ge \left(\frac{r}{c_1}\right)^d \sup_{x \in V} |\det S_{i_1 \cdots i_{n-1}}'(x)| \ge \left(\frac{br}{c_1}\right)^d. \end{aligned}$$

The result in (a) now follows by integrating over A.

(b) Using part (a) and the hypotheses, we have

$$\mathcal{L}(S_I(A)) \le b^d \mathcal{L}(A) \le \left(\frac{c_1}{r}\right)^d \mathcal{L}(S_J(A)).$$

The other inequality can be proved similarly. □

For all $i \in \Sigma_1 = \{1, \dots, N\}$ and $x \in S_i(V)$, by using $|\det(S_i^{-1})'(x)| = |\det S_i'(S_i^{-1}(x))|^{-1}$, we have

$$R^{-d} \le |\det(S_i^{-1})'(x)| \le r^{-d}. \tag{2.1}$$

Let

$$\mathcal{F} := \{S_J S_I^{-1} : I, J \in \Sigma_*\}.$$

For $I, J \in \Sigma_*$, it is possible that $\tau = S_J S_I^{-1}$ can be simplified to $S_{J'} S_{I'}^{-1}$ and hence the domain of τ is $S_{I'}(V)$ (containing $S_I(V)$). Let $\mathrm{Dom}(\tau)$ denote the domain of τ.

We have an analog of the BDP for the maps in $\mathcal{F}$.

Lemma 2.2 *Assume the same hypotheses of Lemma 2.1. Then for any $I, J \in \Sigma_*$ and $x, y \in \mathrm{Dom}(S_J S_I^{-1})$, we have*

$$\frac{|\det(S_J S_I^{-1})'(x)|}{|\det(S_J S_I^{-1})'(y)|} \le c_1^{2d}.$$

Proof Let $\tau = S_J S_I^{-1} = S_{J'} S_{I'}^{-1}$ with $\mathrm{Dom}(\tau) = S_{I'}(V)$. Then

$$\frac{|\det(S_J S_I^{-1})'(x)|}{|\det(S_J S_I^{-1})'(y)|} = \frac{|\det(S_{J'} S_{I'}^{-1})'(x)|}{|\det(S_{J'} S_{I'}^{-1})'(y)|} = \frac{|\det S'_{J'}(S_{I'}^{-1}(x)) \det(S_{I'}^{-1})'(x)|}{|\det S'_{J'}(S_{I'}^{-1}(y)) \det(S_{I'}^{-1})'(y)|} \le c_1^{2d}.$$

□

We now establish an analog of Lemma 2.1 for the maps in $\mathcal{F}$.

Lemma 2.3 *Assume the same hypotheses of Lemma 2.1. Let $\tau = S_J S_I^{-1} = S_{J'} S_{I'}^{-1} \in \mathcal{F}$ with $\mathrm{Dom}(\tau) = S_{I'}(V)$.*

(a) *For any measurable subset $A \subseteq \mathrm{Dom}(\tau)$,*

$$\left(\frac{r_{J'}}{R_{I'}}\right)^d \mathcal{L}(A) \le \mathcal{L}(\tau(A)) \le \left(\frac{R_{J'}}{r_{I'}}\right)^d \mathcal{L}(A).$$

(b) *Suppose $C > 0$ is a constant such that*

$$C^{-1}\mathcal{L}(B) \le \mathcal{L}(A) \le C\mathcal{L}(B)$$

for all A, B belonging to some collection $\mathcal{C}$ of measurable subsets of V. Then for any $A, B \in \mathcal{C}$ such that $A, B \subseteq \mathrm{Dom}(\tau)$,

$$C^{-1}{c_1}^{-2d}\mathcal{L}(\tau(B)) \le \mathcal{L}(\tau(A)) \le C c_1^{2d}\mathcal{L}(\tau(B)).$$

Proof (a) For $x \in \mathrm{Dom}(\tau)$, let $w = S_{I'}^{-1}(x) \in V$. Then

$$|\det \tau'(x)| = |\det S'_{J'}(S_{I'}^{-1}(x))| \cdot |\det(S_{I'}^{-1})'(x)| = \frac{|\det S'_{J'}(w)|}{|\det S'_{I'}(w)|}.$$

Thus,

$$\left(\frac{r_{J'}}{R_{I'}}\right)^d \leq |\det \tau'(x)| \leq \left(\frac{R_{J'}}{r_{I'}}\right)^d.$$

The result follows by integrating over A.

(b) Using part (a), the given hypotheses, and the BDP, we have

$$\begin{aligned}\mathcal{L}(\tau(A)) &\leq \left(\frac{R_{J'}}{r_{I'}}\right)^d \mathcal{L}(A) \leq \left(\frac{R_{J'}}{r_{I'}}\right)^d C\mathcal{L}(B)\\ &\leq \left(\frac{R_{J'}}{r_{I'}}\right)^d C \left(\frac{R_{I'}}{r_{J'}}\right)^d \mathcal{L}(\tau(B)) \leq Cc_1^{2d}\mathcal{L}(\tau(B)).\end{aligned}$$

The lower bound can be obtained similarly. □

3 The weak separation condition

In this section we first establish several equivalent conditions for the WSC and then use them to study the properties of the attractor K. Let X be a compact subset of $\mathbb{R}^d$ and let $S : X \to \mathbb{R}^d$ be any injective C^1 conformal contraction that can be extended to an open connected neighborhood V of X. Then there exists some constant $c_2 > 0$ such that for all $x, y \in V$,

$$|S(x) - S(y)| \leq c_2 R_S |x - y|, \tag{3.1}$$

where $R_S := \sup_{x\in V} |\det S'(x)|^{1/d}$ (see, e.g., the proof in [21]).

For any $a > 0$ and any bounded subsets $D \subseteq X$ and $U \subseteq \mathbb{R}^d$, we let

$$\mathcal{A}_{a,U,D} = \{S \in \mathcal{A}_{a|U|} :\ S(D) \cap U \neq \emptyset\}, \quad \gamma_{a,D} = \sup_U \#\mathcal{A}_{a,U,D},$$

where $|U|$ denotes the diameter of U. We denote by $B_r(x)$ the closed ball with radius r and center x.

Proposition 3.1 *Let $\{S_i\}_{i=1}^N$ be an IFS of injective C^1 conformal contractions on a compact subset $X \subseteq \mathbb{R}^d$ with $X^\circ \neq \emptyset$. The following are equivalent:*

(a) *$\{S_i\}_{i=1}^N$ satisfies the WSC;*
(b) *there exist $a > 0$ and a nonempty subset $D \subseteq X$ such that $\gamma_{a,D} < \infty$;*
(c) *for any $a > 0$ and any nonempty subset $D \subseteq X$, $\gamma_{a,D} < \infty$;*
(d) *for any subset $D \subseteq X$, there exists $\gamma = \gamma(D)$ (depending only on D) such that for any $0 < b \leq 1$ and $x \in X$, $\#\{S \in \mathcal{A}_b :\ x \in S(D)\} \leq \gamma$.*

Proof (a) ⇒ (b): It suffices to prove that there exists $\gamma' \in \mathbb{N}$ and $D \subseteq X$, with $D^\circ \neq \emptyset$, such that for any $x \in X$ and $0 < b \leq 1$,

$$\#\{S \in \mathcal{A}_b :\ S(D) \cap B_b(x) \neq \emptyset\} \leq \gamma'.$$

To obtain this inequality, let D be as in the definition of the WSC, let $S \in \mathcal{A}_b$ such that $S(D) \cap B_b(x) \neq \emptyset$, and let $x' \in D$ such that $S(x') \in B_b(x)$. Then for any $y \in D$, by (3.1), we have

$$|S(y) - x| \leq |S(y) - S(x')| + |S(x') - x| \leq (1 + c_2|D|)b.$$

Let $\eta := (1 + c_2|D|)b$. We can rewrite the above as $S(D) \subseteq B_\eta(x)$. By (a), each point in X is covered by at most γ of the $S(D)$, where $S \in \mathcal{A}_b$. It follows that

$$\sum \{\mathcal{L}(S(D)) : \ S \in \mathcal{A}_b, \ S(D) \cap B_b(x) \neq \emptyset\} \ \leq \ \gamma \mathcal{L}\left(B_\eta(x)\right).$$

Making use of Lemma 2.1(a), we have

$$(br/c_1)^d \mathcal{L}(D) \, \# \{S \in \mathcal{A}_b : \ S(D) \cap B_b(x) \neq \emptyset\} \leq \gamma C b^d$$

for some constant $C > 0$. The desired inequality follows.

(b) $\Rightarrow$ (c): We prove the contrapositive. Assume (c) is false. Then there exist $a_0 > 0$ and a nonempty subset $D_0 \subseteq X$ such that $\gamma_{a_0, D_0} = \infty$. Hence there exists a sequence $\{U_n\}_{n=1}^\infty$ of nonempty bounded subsets of $\mathbb{R}^d$ such that

$$\# \left\{S \in \mathcal{A}_{a_0|U_n|} : S(D_0) \cap U_n \neq \emptyset\right\} \geq n. \tag{3.2}$$

To prove that (b) must fail, we fix an arbitrary $a > 0$ and an arbitrary nonempty subset $D \subseteq X$. We will show that $\gamma_{a,D} = \infty$. Let

$$\sigma := \sup \{|x - y| : x \in D_0, y \in D\} < \infty.$$

We first claim that for any $S \in \mathcal{A}_{a_0|U_n|}$ and $\delta_n := c_2 a_0 \sigma |U_n|$,

$$S(D_0) \cap U_n \neq \emptyset \quad \Rightarrow \quad S(D) \cap (U_n)_{\delta_n} \neq \emptyset,$$

where $(U_n)_{\delta_n} = \{x \in \mathbb{R}^d : \operatorname{dist}(x, U_n) \leq \delta_n\}$ is the closed δ_n-neighborhood of U_n. To prove the claim, we let $y \in S(D_0) \cap U_n$. Then there exists $x \in D_0$ such that $y = S(x) \in S(D_0)$. Now let $\tilde{x} \in D$ and $\tilde{y} := S(\tilde{x}) \in S(D)$. Then

$$|\tilde{y} - y| = |S(\tilde{x}) - S(x)| \ \leq \ c_2 R_S |\tilde{x} - x| \ \leq \ c_2 R_S \sigma \ \leq \ c_2 a_0 \sigma |U_n| = \delta_n.$$

(The first inequality follows from (3.1) and the third inequality is because $S \in \mathcal{A}_{a_0|U_n|}$). This proves the claim.

Note that $(U_n)_{\delta_n}$ is a set of diameter $2\delta_n + |U_n| = (2c_2 a_0 \sigma + 1)|U_n|$. Let κ be the minimum number of sets of diameter a_0/a required to cover any set of diameter $2c_2 a_0 \sigma + 1$. We can cover $(U_n)_{\delta_n}$ by no more than κ sets of diameter $(a_0/a)|U_n|$. Notice that $\mathcal{A}_{a_0|U_n|} = \mathcal{A}_{a|(a_0/a)U_n|}$. Hence, by (3.2) and the claim, there exists $U_n^* \subseteq \mathbb{R}^d$ with

$|U_n^*| = (a_0/a)|U_n|$ such that

$$\#\left\{S \in \mathcal{A}_{a|U_n^*|} :\ S(D) \cap U_n^* \neq \emptyset\right\} \geq \frac{n}{\kappa}.$$

Since κ is independent of n, we conclude that $\gamma_{a,D} = \infty$.

(c) $\Rightarrow$ (d): Let $D \subseteq X$. Then for any $x \in X$ and $0 < b \leq 1$,

$$\begin{aligned}\#\{S \in \mathcal{A}_b :\ x \in S(D)\} &\leq \#\left\{S \in \mathcal{A}_b : S(D) \cap B_{b/2}(x) \neq \emptyset\right\} \\ &= \#\mathcal{A}_{1, B_{b/2}(x), D} \leq \gamma_{1,D} < \infty.\end{aligned}$$

Since (d) $\Rightarrow$ (a) is trivial, the proof is complete. □

We remark that in [11], the original weak separation condition is defined pointwise: there exist $x_0 \in X$ and $\gamma \in \mathbb{N}$ such that for any $J \in \Sigma_*$, any ball of radius b contains at most γ points of $\{S(S_J(x_0)) : S \in \mathcal{A}_b\}$. It is equivalent to Definition 1.1 if the IFS mappings are contractive similitudes and the attractor does not lie in a hyperplane (see [26]). For the conformal case, Definition1.1 implies the pointwise definition. To see this, we let $x_0 \in X$ and $D := \{S_J(x_0) : J \in \Sigma_*\}$. Then $D \subseteq X$ is a nonempty bounded subset, and for any $J \in \Sigma_*$ and any ball B_b of radius b,

$$\begin{aligned}\#\left(\{S(S_J(x_0)) : S \in \mathcal{A}_b\} \cap B_b\right) &\leq \#\{S \in \mathcal{A}_b : S(S_J(x_0)) \cap B_b \neq \emptyset\} \\ &\leq \#\left\{S \in \mathcal{A}_{\frac{1}{2}|B_b|} : S(D) \cap B_b \neq \emptyset\right\} \\ &\leq \gamma_{1/2,D} < \infty.\end{aligned}$$

which yields the pointwise statement. However, we do not know whether the converse is true for conformal IFSs. We state this as an open question: For a conformal IFS whose attractor does not lie in a hyperplane, does the pointwise weak separation condition, stated above, imply the WSC in Definition 1.1?

Throughout the rest of this section we assume that $\{S_i\}_{i=1}^N$ is an injective conformal IFS on X that has the BDP. It is known (see, for e.g., [21,22,24]) that there exists some positive constant, which will also be denoted by c_2, such that for all $J \in \Sigma_*$ and $x, y \in X$,

$$c_2^{-1} R_J |x - y| \leq |S_J(x) - S_J(y)| \leq c_2 R_J |x - y|. \tag{3.3}$$

Note that for $I, J \in \Sigma_*$, we have

$$\|S'_{IJ}(x)\| = \|S'_I(S_J(x)) S'_J(x)\| = \|S'_I(S_J(x))\| \cdot \|S'_J(x)\|.$$

It follows that $R_{IJ} \leq R_I R_J$ and $r_{IJ} \geq r_I r_J$. In particular, for $S = S_I \in \mathcal{A}_b$, $I = (i_1, \ldots, i_n) \in \mathcal{I}_b$, by the BDP, we have $br < R_{i_1\cdots i_{n-1}} r_{i_n} \leq c_1 r_{i_1\cdots i_{n-1}} r_{i_n} \leq c_1 r_I \leq c_1 R_I$, i.e.,

$$b < \frac{c_1}{r} R_S, \quad \text{for all } S \in \mathcal{A}_b. \tag{3.4}$$

Let π be the projection of Σ to X defined by

$$\pi(I) = \lim_{n\to\infty} S_{i_1\cdots i_n}(x), \quad I = (i_1, i_2, \ldots). \tag{3.5}$$

The above limit is independent of $x \in X$.

Theorem 3.2 *Let $\{S_i\}_{i=1}^N$ be an IFS of injective conformal contractions on X that has the BDP and satisfies the WSC. Let K be the associated attractor. Then $\alpha := \dim_{\mathrm{H}} K = \dim_{\mathrm{B}} K = \dim_{\mathrm{P}} K$ and $0 < \mathcal{H}^\alpha(K) \le \mathcal{P}^\alpha(K) < \infty$.*

Proof To prove $0 < \mathcal{H}^\alpha(K) < \infty$, we make use of [4, Theorems 2 and 4]. We first prove $\mathcal{H}^\alpha(K) > 0$. Note that for any $E \subseteq K$, we have

$$E \subseteq \bigcup \left\{S(K) : \; S \in \mathcal{A}_{|E|}, \; S(K) \cap E \ne \emptyset\right\}.$$

By Proposition 3.1(c), we have $\#\{S \in \mathcal{A}_{|E|} \; : \; S(K) \cap E \ne \emptyset\} \le \gamma_{1,K} < \infty$. For each $S \in \mathcal{A}_{|E|}$ with $S(K) \cap E \ne \emptyset$, consider the map $S^{-1} \; : \; S(K) \to K$. For any $x, y \in S(K)$, let $x', y' \in K$ such that $x = S(x')$ and $y = S(y')$. By (3.3) and by noting that $S \in \mathcal{A}_{|E|}$ implies that $R_S \le |E|$, we have

$$|x - y| = \left|S(x') - S(y')\right| \le c_2 R_S |x' - y'| \le c_2 |E| \cdot \left|S^{-1}(x) - S^{-1}(y)\right|.$$

Hence [4, Theorem 2] implies that $\mathcal{H}^\alpha(K) > 0$.

To prove $\dim_{\mathrm{B}}(K) = \alpha$ and $\mathcal{H}^\alpha(K) \le \mathcal{P}^\alpha(K) < \infty$, let $B_\delta(x)$, $x \in K$, be a ball such that $K \not\subseteq B_\delta(x)$. Then there exists $(i_1, i_2, \ldots) \in \Sigma$ such that $\pi(i_1, i_2, \ldots) = x$. There exists $n \in \mathbb{N}$ such that

$$S_{i_1\cdots i_n}(K) \subseteq B_\delta(x) \quad \text{and} \quad S_{i_1\cdots i_{n-1}}(K) \not\subseteq B_\delta(x).$$

Hence there exists $y \in K$ such that

$$\begin{aligned}\delta &\le \left|S_{i_1\cdots i_{n-1}}(y) - x\right| = \left|S_{i_1\cdots i_{n-1}}(y) - S_{i_1\cdots i_{n-1}}\left(\pi(i_n, i_{n+1}, \ldots)\right)\right| \\ &\le c_2 R_{i_1\cdots i_{n-1}} \left|y - \pi(i_n, i_{n+1}, \ldots)\right| \le c_2 |K| R_{i_1\cdots i_{n-1}}.\end{aligned}$$

Making use of the BDP, we have

$$R_{i_1\cdots i_{n-1}} \le c_1 r_{i_1\cdots i_{n-1}} \le \frac{c_1}{r} r_{i_1\cdots i_{n-1}} r_n \le \frac{c_1}{r} r_{i_1\cdots i_n} \le \frac{c_1}{r} R_{i_1\cdots i_n}.$$

Combining the above estimations yields $R_{i_1\cdots i_n} \ge r\delta/(c_2 c_1 |K|)$. Consider the map $\psi = S_{i_1\cdots i_n} \; : \; K \to B_\delta(x)$. For any $x_1, x_2 \in K$, using (3.3), we have

$$|\psi(x_1) - \psi(x_2)| = \left|S_{i_1\cdots i_n}(x_1) - S_{i_1\cdots i_n}(x_2)\right| \ge c_2^{-1} R_{i_1\cdots i_n} |x_1 - x_2| \ge \frac{r\delta}{c_2^2 c_1 |K|} |x_1 - x_2|.$$

Now, [4, Theorem 4] and [5, Exercise 3.2] imply that $\dim_{\mathrm{B}}(K) = \alpha$ and $\mathcal{P}^\alpha(K) < \infty$. It is well known that $\mathcal{H}^\alpha(K) \le \mathcal{P}^\alpha(K)$. This completes the proof. □

As an important consequence of Theorem 3.2, we obtain the following estimate for $\#\mathcal{A}$ and a formula for the Hausdorff dimension of K.

Corollary 3.3 *Let $\{S_i\}_{i=1}^N$ and α be as in Theorem 3.2. Then there exists a constant $c_3 > 0$ such that for any $0 < b \le 1$,*

$$c_3^{-1} b^{-\alpha} \le \#\mathcal{A}_b \le c_3 b^{-\alpha}. \tag{3.6}$$

Consequently,

$$\alpha = \dim_{\mathrm{H}} K = \dim_{\mathrm{B}} K = \dim_{\mathrm{P}} K = -\lim_{b \to 0^+} \frac{\log \#\mathcal{A}_b}{\log b}.$$

Proof Note that for any $0 < b \le 1$, $K = \bigcup\{S(K) : S \in \mathcal{A}_b\}$. By Proposition 3.1(d), each $x \in K$ is covered by at most γ of the $S(K)$ with $S \in \mathcal{A}_b$. Hence

$$\mathcal{H}^\alpha(K) \le \sum_{S \in \mathcal{A}_b} \mathcal{H}^\alpha(S(K)) \le \gamma \mathcal{H}^\alpha(K). \tag{3.7}$$

For each $S \in \mathcal{A}_b$, by making use of (3.3) and (3.4), we get

$$\mathcal{H}^\alpha(S(K)) \ge c_2^{-\alpha} R_S^\alpha \mathcal{H}^\alpha(K) \ge \left(\frac{r}{c_1 c_2}\right)^\alpha b^\alpha \mathcal{H}^\alpha(K),$$

and

$$\mathcal{H}^\alpha(S(K)) \le c_2^\alpha R_S^\alpha \mathcal{H}^\alpha(K) \le c_2^\alpha b^\alpha \mathcal{H}^\alpha(K).$$

It follows by summing each inequality over $S \in \mathcal{A}_b$ and using (3.7) that

$$\left(\frac{r}{c_1 c_2}\right)^\alpha b^\alpha \mathcal{H}^\alpha(K) \#\mathcal{A}_b \le \gamma \mathcal{H}^\alpha(K),$$

and

$$\mathcal{H}^\alpha(K) \le c_2^\alpha b^\alpha \mathcal{H}^\alpha(K) \#\mathcal{A}_b.$$

Since $0 < \mathcal{H}^\alpha(K) < \infty$, (3.6) and the dimension formula follows immediately. □

4 Absolute continuity of self-conformal measures

In this section we give a necessary and sufficient condition for the absolute continuity (equivalently, singularity) of self-conformal measures by assuming the WSC and the BDP.

Proposition 4.1 *Suppose $\{S_i\}_{i=1}^N$ satisfies the WSC. Then for any finite subset $\Lambda \subseteq \Sigma_*$, the family $\{S_J : J \in \Lambda\}$ also satisfies the WSC.*

Proof We will first prove the proposition for the family $\{S_{12}, S_1, S_2, \ldots, S_N\}$. For convenience, we write $S_0 = S_{12}$. For $0 < b \leq 1$ let $\mathcal{I}'_b$ and $\mathcal{A}'_b$ be the analogs of $\mathcal{I}_b$ and $\mathcal{A}_b$, respectively, that are defined with respect to the index set $\{0, 1, \ldots, N\}$.

Let $I = (i_1, \ldots, i_n) \in \mathcal{I}'_b$ and let $S = S_I$. Then $R_I \leq b < R_{i_1 \cdots i_{n-1}}$. Note that $I = (j_1, \ldots, j_m) \in \mathcal{I}_b$ for some $m \geq n$. If $i_n \neq 0$, then it is clear that $S \in \mathcal{A}_b$. If $i_n = 0$, then

$$\text{either} \quad R_I \leq b < R_{i_1 \cdots i_{n-1} 1} \quad \text{or} \quad R_{i_1 \cdots i_{n-1} 1} \leq b < R_{i_1 \cdots i_{n-1}},$$

i.e., either $S \in \mathcal{A}_b$ or $S = \tilde{S} S_2$ with $\tilde{S} \in \mathcal{A}_b$. Thus

$$\mathcal{A}'_b \subseteq \mathcal{A}_b \bigcup (\mathcal{A}_b S_2).$$

Let D be given as in the definition of the WSC (Definition 1.1). It follows from above and Proposition 3.1(d) that for any $x \in X$,

$$\begin{aligned} \# \left\{ S \in \mathcal{A}'_b : x \in S(D) \right\} &\leq \# \{ S \in \mathcal{A}_b : x \in S(D) \} + \# \{ S \in \mathcal{A}_b : x \in S S_2(D) \} \\ &\leq \gamma(D) + \gamma \left(S_2(D) \right). \end{aligned}$$

Hence the proposition is true for $\Lambda = \{12, 1, 2, \ldots, N\}$. By repeating this argument, we see that it is also true for $\Lambda = \bigcup_{k=1}^{n} \{1, \ldots, N\}^k$ for all $n \geq 1$. The general statement follows by observing the trivial fact that if an IFS satisfies the WSC, then so does any of its subfamilies. $\square$

We need to introduce more notations. For $I \in \Sigma_*$, let $[I] = \{I' \in \Sigma_* : S_I = S_{I'}\}$ and let C_I be the cylinder set in Σ with initial segment I. For $\Lambda \subseteq \Sigma_*$, let $C_\Lambda = \bigcup \{C_I : I \in \Lambda\}$. Let P be the product probability measure on Σ induced by the probability weights $\{p_i\}_{i=1}^N$. For $\Lambda \subseteq \Sigma_*$, we will use the abbreviated notation $P(\Lambda)$ to denote $P(C_\Lambda)$. Recall that π is the projection of Σ to X defined by (3.5). Note that $\pi(C_I) = S_I(K)$ for all $I \in \Sigma_*$, and $\mu = P\pi^{-1}$. Hence if $S = S_I$, then

$$\mu(S(K)) = P\left(\pi^{-1} S(K)\right) \geq P\left(\{C_{I'} : S_{I'} = S\}\right) = \sum_{I' \in [I]} p_{I'}.$$

We also recall that $p_S = \sum \{p_I : S_I = S, I \in \Sigma_*\}$.

Lemma 4.2 *Let $\{S_i\}_{i=1}^N$ be an IFS of injective C^1 conformal contractions on a compact subset $X \subseteq \mathbb{R}^d$ and assume that it has the BDP and satisfies the WSC. Let $\{p_i\}_{i=1}^N$ be the associated probability weights, and let K be the attractor with $\dim_{\mathrm{H}}(K) = \alpha$. For $0 < b \leq 1$ and $\Lambda \subseteq \mathcal{I}_b$, let*

$$\tilde{\Lambda} = \left\{ I \in \Lambda : \sum_{I' \in [I] \cap \Lambda} p_{I'} > \frac{b^\alpha}{4c_3} \right\},$$

where c_3 is as in Corollary 3.3. Then $P(\Lambda) > \frac{1}{2}$ implies that $P(\tilde{\Lambda}) > \frac{1}{4}$.

Proof By Corollary 3.3, we have $\#\mathcal{A}_b \leq c_3 b^{-\alpha}$. It follows that

$$P(\Lambda\backslash\tilde{\Lambda}) = \sum\{p_I : I \in \Lambda\backslash\tilde{\Lambda}\} = \sum_{[I]}\sum\{p_{I'} : I' \in [I] \cap (\Lambda\backslash\tilde{\Lambda})\} \leq \#\mathcal{A}_b \cdot \frac{b^\alpha}{4c_3} \leq \frac{1}{4}$$

and $P(\tilde{\Lambda}) = P(\Lambda) - P(\Lambda\backslash\tilde{\Lambda}) > 1/2 - 1/4 = 1/4$. □

We now prove Theorems 1.2 and 1.3.

Theorem 4.3 *Assume the same hypotheses as in Lemma 4.2. Then a self-conformal measure μ is singular with respect to $\mathcal{H}^\alpha|_K$ if and only if there exist $0 < b \leq 1$ and $S \in \mathcal{A}_b$ such that $p_S > R_S^\alpha$.*

Proof To prove the necessity, we suppose that μ is singular with respect to $\mathcal{H}^\alpha|_K$. Then there exists $K_0 \subseteq K$ such that $\mu(K_0) = 1$ but $\mathcal{H}^\alpha(K_0) = 0$. Thus for any $\varepsilon > 0$, there exists a δ-cover $\{U_i\}$ of K_0 such that $\sum_i |U_i|^\alpha < \varepsilon$. Let $b_i = |U_i|$. Then $U_i \subseteq B_{b_i}(x_i) =: B_i$, where x_i is any fixed element in U_i. Note that

$$\begin{aligned}\mu(B_i) &= \sum\left\{p_S\mu \circ S^{-1}(B_i) : \ S \in \mathcal{A}_{b_i}, \ S(K) \cap B_i \neq \emptyset\right\}\\ &\leq \sum\left\{p_S : \ S \in \mathcal{A}_{b_i}, \ S(K) \cap B_i \neq \emptyset\right\}.\end{aligned}$$

If the necessity is not true, then $p_S \leq R_S^\alpha \leq b^\alpha$ for all $S \in \mathcal{A}_b$ and all $0 < b \leq 1$. Hence the above inequality and Proposition 3.1(c) imply that $\mu(B_i) \leq \gamma_{1/2,K} b_i^\alpha$ and thus

$$1 = \mu(K_0) \leq \sum_i \mu(B_i) \leq \gamma_{1/2,K} \sum_i b_i^\alpha < \varepsilon\gamma_{1/2,K}.$$

This is a contradiction because $\varepsilon > 0$ is arbitrary.

The proof of the sufficiency follows by using Lemma 4.2 and modifying the technique in the proof of [13, Theorem 3.1] and [15, Theorem 1.1]; we omit the details. □

Theorem 4.4 *Let $\{S_i\}_{i=1}^N$ be as in Theorem 4.3. If the self-conformal measure μ is absolutely continuous with respect to $\mathcal{H}^\alpha|_K$, then the Radon-Nikodym derivative of μ is bounded.*

Proof Let $\nu = \mathcal{H}^\alpha|_K$ and let f be the Radon-Nikodym derivative of μ with respect to ν. Suppose f is unbounded. Then a density theorem [16, Sect. 2.14] implies that for any $M, \delta > 0$, there exist $x \in K$ and $b > 0$ such that

$$\nu\left(\{t \in K : f(t) > M\} \cap B_{b\delta}(x)\right) > \frac{1}{2}\,\nu\left(B_{b\delta}(x)\right).$$

Hence

$$\begin{aligned}\mu\left(B_{b\delta}(x)\right) &= \int\limits_{B_{b\delta}(x)} f(t)\mathrm{d}\nu(t)\\ &\geq M\nu\left(\{t\in K : f(t) > M\}\cap B_{b\delta}(x)\right)\\ &> \frac{1}{2}M\nu\left(B_{b\delta}(x)\right).\end{aligned}$$

Let $\delta = c_2|K|$ in the above inequality (where c_2 is as in (3.3)). Note that $x \in K = \bigcup\{S(K) : S \in \mathcal{A}_b\}$, and hence there exists $S \in \mathcal{A}_b$ such that $x \in S(K)$. Making use of (3.3), we have $|S(K)| \leq c_2 R_S|K| \leq b\delta$, which implies that $S(K) \subseteq B_{b\delta}(x)$. (3.3) and (3.4) also imply that

$$\nu\left(S(K)\right) \geq c_2^{-\alpha} R_S^{\alpha}\nu(K) > c_2^{-\alpha}\left(\frac{rb}{c_1}\right)^{\alpha}\nu(K) = \left(\frac{r}{c_1c_2}\right)^{\alpha} b^{\alpha}\nu(K).$$

Hence

$$\mu\left(B_{b\delta}(x)\right) > \frac{1}{2}M\nu\left(S(K)\right) \geq \frac{1}{2}M\left(\frac{r}{c_1c_2}\right)^{\alpha} b^{\alpha}\nu(K). \tag{4.1}$$

On the other hand,

$$\begin{aligned}\mu\left(B_{b\delta}(x)\right) &= \sum\left\{p_S\mu\circ S^{-1}\left(B_{b\delta}(x)\right) : S\in\mathcal{A}_b,\ S(K)\cap B_{b\delta}(x)\neq\emptyset\right\}\\ &\leq \sum\left\{p_S : S\in\mathcal{A}_b,\ S(K)\cap B_{b\delta}(x)\neq\emptyset\right\}.\end{aligned}$$

By Proposition 3.1(c), there are at most $\gamma_{1/(2\delta),K}$ terms in the last summation. We choose M such that $(r/c_1c_2)^{\alpha}M\nu(K)/2 > \gamma_{1/(2\delta),K}$. Then (4.1) and the above inequality imply that there exists $S \in \mathcal{A}_b$ such that $p_S > b^{\alpha} \geq R_S^{\alpha}$. It follows from Theorem 1.2 that μ is singular with respect to ν, a contradiction. □

5 The finite type condition

In this section we generalize the finite type condition in [12] and use it to study conformal IFSs with the BDP and investigate its relationship with the weak separation condition. Since contractivity is not required, we will formulate the finite type condition for IFSs of injections. The setup is the same as for similitudes (see [12]); we summarize it here and refer the reader to [12] for details.

The definition of the finite type condition consists of two parts. The first part is the definition of a sequence of nested index sets (Definition 5.1), which generalizes the notion of "level of iteration". It is defined in [12]; we include it here for completeness. The second part is the concept of neighborhood types (Definition 5.2), which is slightly modified from that for IFSs of contractive similitudes. The finite type condition is formulated to generalize the open set condition to compute the Hausdorff dimension of self-similar sets defined by certain IFSs with overlaps.

Fix an IFS $\{S_i\}_{i=1}^N$ of injections (not assuming the C^1 property) on a subset $X \subseteq \mathbb{R}^d$ (not necessarily compact). Recall the partial order $\preccurlyeq$ on Σ_* defined for $I, J \in \Sigma_*$ by $I \preccurlyeq J$ if I is an initial segment of J or $I = J$. We denote by $I \not\preccurlyeq J$ if $I \preccurlyeq J$ does not hold. Consider a sequence of index sets $\{\mathcal{M}_k\}_{k=0}^\infty$, where for all $k \geq 0$, $\mathcal{M}_k$ is a finite subset of Σ_*. Let

$$\underline{m}_k = \underline{m}_k(\mathcal{M}_k) := \min\{|I| : I \in \mathcal{M}_k\} \quad \text{and} \quad \overline{m}_k = \overline{m}_k(\mathcal{M}_k) := \max\{|I| : I \in \mathcal{M}_k\}.$$

Definition 5.1 We say that $\{\mathcal{M}_k\}_{k=0}^\infty$ is a sequence of nested index sets if it satisfies the following conditions:

(1) both $\{\underline{m}_k\}$ and $\{\overline{m}_k\}$ are nondecreasing, and $\lim_{k\to\infty} \underline{m}_k = \lim_{k\to\infty} \overline{m}_k = \infty$;
(2) for each $k \geq 0$, $\mathcal{M}_k$ is an antichain in Σ_*;
(3) for each $J \in \Sigma_*$ with $|J| > \overline{m}_k$, there exists $I \in \mathcal{M}_k$ such that $I \preccurlyeq J$;
(4) for each $J \in \Sigma_*$ with $|J| < \underline{m}_k$, there exists $I \in \mathcal{M}_k$ such that $J \preccurlyeq I$;
(5) there exists a positive integer L, independent of k, such that for all $I \in \mathcal{M}_k$ and $J \in \mathcal{M}_{k+1}$ with $I \preccurlyeq J$, we have $|J| - |I| \leq L$.

(We allow $\mathcal{M}_k \cap \mathcal{M}_{k+1} \neq \emptyset$.)

Condition (2) means that the indices in $\mathcal{M}_k$ are incomparable, and (3) means that $\mathcal{M}_k$ covers Σ. We also remark that (4) actually follows from (3).

Clearly, by letting $\mathcal{M}_k = \Sigma_k$ for all $k \geq 0$, we obtain an example of a sequence of nested index sets.

To define neighborhood types, we fix a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^\infty$. For each integer $k \geq 0$, let $\mathcal{V}_k$ be the set of *kth level vertices* (with respect to $\{\mathcal{M}_k\}$) defined as

$$\mathcal{V}_0 := \{(I, 0)\} \quad \text{and} \quad \mathcal{V}_k := \{(S_I, k) : I \in \mathcal{M}_k\} \quad \text{for all } k \geq 1.$$

We call $(I, 0)$ the *root vertex* and denote it by $\boldsymbol{v}_{root}$. Let $\mathcal{V} := \bigcup_{k\geq 0} \mathcal{V}_k$ be the set of all vertices. For $\boldsymbol{v} = (S_I, k) \in \mathcal{V}_k$, we use the convenient notation $S_{\boldsymbol{v}} := S_I$.

Assume that there exists a nonempty open set $\Omega \subseteq X$ which is *invariant* under $\{S_i\}_{i=1}^N$, i.e., $\bigcup_{i=1}^N S_i(\Omega) \subseteq \Omega$. Such an Ω exists if the S_i are contractions on $\mathbb{R}^d$. Two kth level vertices $\boldsymbol{v}, \boldsymbol{v}' \in \mathcal{V}_k$ (allowing $\boldsymbol{v} = \boldsymbol{v}'$) are said to be *neighbors* (with respect to Ω and $\{\mathcal{M}_k\}$) if $S_{\boldsymbol{v}}(\Omega) \cap S_{\boldsymbol{v}'}(\Omega) \neq \emptyset$. The set of vertices

$$\Omega(\boldsymbol{v}) := \{\boldsymbol{v}' : \boldsymbol{v}' \in \mathcal{V}_k \text{ is a neighbor of } \boldsymbol{v}\}$$

is called the *neighborhood* of $\boldsymbol{v}$ (with respect to Ω and $\{\mathcal{M}_k\}$). Note that $\boldsymbol{v} \in \Omega(\boldsymbol{v})$ by definition. Recall that $\mathcal{F} := \{S_J S_I^{-1} : I, J \in \Sigma_*\}$.

We define an equivalence relation on $\mathcal{V}$.

Definition 5.2 Under the above assumptions, two vertices $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{v}' \in \mathcal{V}_{k'}$ are equivalent, denoted by $\boldsymbol{v} \sim_\Omega \boldsymbol{v}'$ (or simply $\boldsymbol{v} \sim \boldsymbol{v}'$) if, for $\tau := S_{\boldsymbol{v}'} S_{\boldsymbol{v}}^{-1} (\in \mathcal{F}) : \bigcup_{\boldsymbol{u}\in\Omega(\boldsymbol{v})} S_{\boldsymbol{u}}(X) \to X$, the following conditions hold:

(1) $\{S_{\boldsymbol{u}'} : \boldsymbol{u}' \in \Omega(\boldsymbol{v}')\} = \{\tau S_{\boldsymbol{u}} : \boldsymbol{u} \in \Omega(\boldsymbol{v})\}$; in particular, $\tau S_{\boldsymbol{u}}$ is defined for all $\boldsymbol{u} \in \Omega(\boldsymbol{v})$.
(2) For $\boldsymbol{u} \in \Omega(\boldsymbol{v})$ and $\boldsymbol{u}' \in \Omega(\boldsymbol{v}')$ such that $S_{\boldsymbol{u}'} = \tau S_{\boldsymbol{u}}$, and for any positive integer $\ell \geq 1$, an index $I \in \Sigma_*$ satisfies $(S_{\boldsymbol{u}} S_I, k + \ell) \in \mathcal{V}_{k+\ell}$ if and only if it satisfies $(S_{\boldsymbol{u}'} S_I, k' + \ell) \in \mathcal{V}_{k'+\ell}$.

It can be verified directly that $\sim$ is an equivalence relation. We denote the equivalence class containing $\boldsymbol{v}$ by $[\boldsymbol{v}]$ and call it the *neighborhood type* of $\boldsymbol{v}$ (with respect to Ω and $\{\mathcal{M}_k\}$). Condition (2) is essential in establishing the important property that equivalent vertices generate the same number of offspring of each neighborhood type (see Proposition 5.3).

Definition 5.3 Let $\{S_i\}_{i=1}^N$ be an IFS of injections on a subset $X \subseteq \mathbb{R}^d$. We say that $\{S_i\}_{i=1}^N$ is of finite type (or that it satisfies the finite type condition) if there exists a sequence of nested index sets $\{\mathcal{M}_k\}_{k=0}^\infty$ and a nonempty invariant open set $\Omega \subseteq X$ such that, with respect to Ω and $\{\mathcal{M}_k\}$, the set of equivalence classes $\mathcal{V}/_\sim := \{[\boldsymbol{v}] : \boldsymbol{v} \in \mathcal{V}\}$ is finite. We call such an Ω a finite type condition set (or FTC set).

Remark 5.1 Suppose $\{S_i\}_{i=1}^N$ satisfies the OSC, i.e., there exists a nonempty bounded open set $\Omega \subseteq \mathbb{R}^d$ such that $\bigcup_{i=1}^N S_i(\Omega) \subseteq \Omega$ and $S_i(\Omega) \cap S_j(\Omega) = \emptyset$ for all $i \neq j$. Let $\mathcal{M}_k = \Sigma_k$ for all $k \geq 0$. Then $\mathcal{V}/_\sim$ consists of just one element. In fact, for each vertex $\boldsymbol{v}$, $\Omega(\boldsymbol{v}) = \{\boldsymbol{v}\}$ and thus each vertex is equivalent to the root vertex $\boldsymbol{v}_{\text{root}}$. Hence $\{S_i\}_{i=1}^N$ is of finite type.

Remark 5.2 We remark that the generalized finite type condition for IFSs of similitudes defined in [12], and extended above, is essentially equivalent to the general finite type condition defined by Jin and Yau [10] using the notions of sections, flags, and recurrentable flags. Each section is equal to some $\mathcal{M}_k$ in Definition 5.1. A recurrentable flag corresponds to a nested index set that satisfies the conditions in Definition 5.2.

Let $\{S_i\}_{i=1}^N$ be an IFS of injections on X as defined above, $\Omega \subseteq X$ be an invariant open set, $\{\mathcal{M}_k\}_{k=0}^\infty$ be a fixed sequence of nested index sets, and $\sim$ be the equivalence relation on the set of vertices as defined above. We need two infinite graphs $\mathcal{G}$ and $\mathcal{G}_R$. The graph $\mathcal{G}$ has vertex set $\mathcal{V}$ and directed edges defined as follows. Let $\boldsymbol{v} \in \mathcal{V}_k$ and $\boldsymbol{u} \in \mathcal{V}_{k+1}$. Suppose there exists $I \in \mathcal{M}_k$, $J \in \mathcal{M}_{k+1}$ and $L \in \Sigma_*$ such that

$$\boldsymbol{v} = (S_I, k), \quad \boldsymbol{u} = (S_J, k+1), \quad \text{and} \quad J = (I, L).$$

Then we connect a directed edge $L : \boldsymbol{v} \to \boldsymbol{u}$. We call $\boldsymbol{v}$ a *parent* of $\boldsymbol{u}$ and $\boldsymbol{u}$ an *offspring* of $\boldsymbol{v}$. We write $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ where $\mathcal{E}$ is the set of all directed edges defined above.

The reduced graph $\mathcal{G}_R$ is constructed from $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ as follows. We first remove all but the smallest (in the lexicographic order) directed edges going to a vertex, and denote the resulting graph by $\mathcal{G}'_R$. It is possible that a vertex in $\mathcal{V}$ does not have any offspring in $\mathcal{G}'_R$ (an example is given in [12]). To finish the construction, we remove all vertices that do not have any offspring in $\mathcal{G}_{R'}$, together with all vertices and edges leading only to them. The resulting graph is the reduced graph, denoted by $\mathcal{G}_R = (\mathcal{V}_R, \mathcal{E}_R)$, where $\mathcal{V}_R$ is the set of all vertices and $\mathcal{E}_R$ is the set of all edges.

We will illustrate the construction of the reduced graph in Example 6.2; other examples can be found in [12] and [20].

Proposition 5.3 *Let $\boldsymbol{v}$ and $\boldsymbol{v}'$ be two vertices in $\mathcal{V}$ with offspring $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ and $\boldsymbol{u}'_1, \ldots, \boldsymbol{u}'_\ell$ in $\mathcal{G}_R$, respectively. Suppose $[\boldsymbol{v}] = [\boldsymbol{v}']$. Then*

$$\{[\boldsymbol{u}_i] : 1 \le i \le m\} = \left\{[\boldsymbol{u}'_i] : 1 \le i \le \ell\right\} \tag{5.1}$$

counting multiplicity. In particular, $m = \ell$.

The proof of this proposition is the same as that of [12, Proposition 2.4]; we omit the details.

We remark that the definition of the finite type condition and the properties proved above only require that the S_i are injective. For the rest of this section we assume that $\{S_i\}_{i=1}^N$ is conformal.

Lemma 5.4 *Let $\{S_i\}_{i=1}^N$ be an IFS of injective C^1 conformal contractions on a compact subset $X \subseteq \mathbb{R}^d$ as defined in Definition 1.2. Assume that $\{S_i\}_{i=1}^N$ has the BDP and is of finite type with respect to some $\Omega \subseteq X$. Then there exists a constant $c_4 > 0$ such that for any two neighboring vertices $\boldsymbol{u}_1, \boldsymbol{u}_2$, we have*

$$c_4^{-1} \le \frac{\mathcal{L}(S_{\boldsymbol{u}_1}(\Omega))}{\mathcal{L}(S_{\boldsymbol{u}_2}(\Omega))} \le c_4.$$

Proof Fix a neighborhood type $\mathcal{T}$ and a vertex $\boldsymbol{v}$ such that $[\boldsymbol{v}] = \mathcal{T}$. Let

$$\Omega(\boldsymbol{v}) = \{\boldsymbol{v}_0 = \boldsymbol{v}, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\}.$$

Then there exists a constant $C > 0$, depending only on $\boldsymbol{v}_0, \boldsymbol{v}_1, \ldots, \boldsymbol{v}_m$ and Ω, such that

$$C^{-1}\mathcal{L}(S_{\boldsymbol{v}_0}(\Omega)) \le \mathcal{L}(S_{\boldsymbol{v}_i}(\Omega)) \le C\mathcal{L}(S_{\boldsymbol{v}_0}(\Omega)), \quad i = 0, 1, \ldots, m. \tag{5.2}$$

Now let $\boldsymbol{v} \sim_\Omega \boldsymbol{v}'$, $\tau = S_{\boldsymbol{v}'}S_{\boldsymbol{v}}^{-1} \in \mathcal{F}$, and

$$\Omega(\boldsymbol{v}') = \{\boldsymbol{v}'_0 = \boldsymbol{v}', \boldsymbol{v}'_1, \ldots, \boldsymbol{v}'_m\}.$$

Then, upon rearranging the $\boldsymbol{v}'_i$ if necessary, we can assume that

$$S_{\boldsymbol{v}'_i} = \tau S_{\boldsymbol{v}_i}, \quad i = 0, 1, \ldots, m.$$

Note that condition (1) in Definition 5.2 implies that $S_{\boldsymbol{v}_i}(\Omega) \subseteq \mathrm{Dom}(\tau)$ for all $i = 0, 1, \ldots, m$. Hence, using (5.2) and substituting $S_{\boldsymbol{v}_i}(\Omega) = A$, $S_{\boldsymbol{v}_0}(\Omega) = B$ and $\tau = S_{\boldsymbol{v}'}S_{\boldsymbol{v}}^{-1}$ in Lemma 2.3(b), we have, for all $i \in \{0, 1, \ldots, m\}$,

$$\mathcal{L}(S_{\boldsymbol{v}'_i}(\Omega)) = \mathcal{L}(\tau S_{\boldsymbol{v}_i}(\Omega)) \le Cc_1^{2d}\mathcal{L}(\tau S_{\boldsymbol{v}_0}(\Omega)) = Cc_1^{2d}\mathcal{L}(S_{\boldsymbol{v}'_0}(\Omega)). \tag{5.3}$$

Similarly,

$$\mathcal{L}(S_{\boldsymbol{v}_i'}(\Omega)) = \mathcal{L}(\tau S_{\boldsymbol{v}_i}(\Omega)) \geq C^{-1}{c_1}^{-2d}\mathcal{L}(\tau S_{\boldsymbol{v}_0}(\Omega)) = C^{-1}{c_1}^{-2d}\mathcal{L}(S_{\boldsymbol{v}_0'}(\Omega)). \quad (5.4)$$

Combining (5.3) and (5.4) yields the conclusion of the lemma for any two neighboring vertices $\boldsymbol{u}_1, \boldsymbol{u}_2$ with one of them being of type $\mathcal{T}$. Since there are only finitely many distinct neighborhood types, the result follows. □

We now prove Theorem 1.4.

Theorem 5.5 *Let $\{S_i\}_{i=1}^N$ be an IFS of injective C^1 conformal contractions on a compact subset $X \subseteq \mathbb{R}^d$ and assume that it has the BDP. Then the finite type condition implies the weak separation condition.*

Proof Assume that $\{S_i\}_{i=1}^N$ is a finite type conformal IFS on X and let $\{\mathcal{M}_k\}_{k=0}^\infty$ and Ω be as in the definition of the finite type condition. We will show that there exists an integer $\gamma > 0$ such that for all $0 < b \leq 1$ and $x \in X$,

$$\#\{S \in \mathcal{A}_b : x \in S(\Omega)\} \leq \gamma.$$

Let $\mathcal{S} = \{S \in \mathcal{A}_b : x \in S(\Omega)\}$. List all elements of $\mathcal{S}$ as $S_{I_1}, \ldots, S_{I_m}$. (The choice of I_j does not affect the following proof.) For each $j \in \{1, \ldots, m\}$, let $I_j = (\tilde{I}_j, \widetilde{J}_j)$, where $\tilde{I}_j \in \mathcal{M}_{k_j}$ is the longest initial segment of I_j that belongs to some $\mathcal{M}_k$. We assume without loss of generality that

$$k_1 = \min\{k_j : 1 \leq j \leq m\} = k \quad \text{and} \quad \tilde{I}_1 \in \mathcal{M}_k.$$

For each $j \in \{1, \ldots, m\}$, let I_j' be the initial segment of I_j such that $I_j' \in \mathcal{M}_k$. In particular, $I_1' = \tilde{I}_1$. Since $x \in S(\Omega)$ for all $S \in \mathcal{S}$, it follows that

$$\boldsymbol{v}_2 = (S_{I_2'}, k), \ldots, \boldsymbol{v}_m = (S_{I_m'}, k),$$

not necessarily distinct, are neighbors of $\boldsymbol{v}_1 = (S_{I_1'}, k)$. The finite type condition implies that the number of vertices in each neighborhood is uniformly bounded by some constant M independent of x, b, and the choice of I_j. That is,

$$\#\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_m\} \leq M.$$

By Lemma 5.4, for $j = 2, \ldots, m$,

$$c_4^{-1} \leq \frac{\mathcal{L}(S_{I_1'}(\Omega))}{\mathcal{L}(S_{I_j'}(\Omega))} \leq c_4,$$

while from Lemma 2.1(b) we get

$$(r/c_1)^d \leq \frac{\mathcal{L}(S_{I_j}(\Omega))}{\mathcal{L}(S_{I_1}(\Omega))} \leq (c_1/r)^d.$$

Combining these inequalities gives

$$c_4^{-1}(r/c_1)^d \le \frac{\mathcal{L}(S_{I_j}(\Omega))}{\mathcal{L}(S_{I_j'}(\Omega))} \cdot \frac{\mathcal{L}(S_{I_1'}(\Omega))}{\mathcal{L}(S_{I_1}(\Omega))} \le c_4(c_1/r)^d. \tag{5.5}$$

For $j \in \{1, \dots, m\}$, we can write uniquely $I_j = I_j' J_j$. By putting $S_I^{-1} = S_{I'}^{-1}$ equal to the identity and $S_J = S_{J'} = S_{I_j'}$ in Lemma 2.3(a), and then using the BDP, we have

$$\frac{\mathcal{L}(S_{I_j}(\Omega))}{\mathcal{L}(S_{I_j'}(\Omega))} = \frac{\mathcal{L}(S_{I_j' J_j}(\Omega))}{\mathcal{L}(S_{I_j'}(\Omega))} \le \frac{R_{I_j'}^d \mathcal{L}(S_{J_j}(\Omega))}{r_{I_j'}^d \mathcal{L}(\Omega)} \le c_1^d \frac{\mathcal{L}(S_{J_j}(\Omega))}{\mathcal{L}(\Omega)}. \tag{5.6}$$

Since each S_i is contractive, i.e., there exists some $0 < \rho < 1$ such that for all $i \in \{1, \dots, N\}$ and $x, y \in X$, $|S_i(x) - S_i(y)| \le \rho|x - y|$, it follows that

$$\mathcal{L}(S_{J_j}(\Omega)) \le \rho^{|J_j|d} \mathcal{L}(\Omega). \tag{5.7}$$

Combining (5.5), (5.6) and (5.7) yields

$$\frac{\mathcal{L}(S_{I_1}(\Omega))}{\mathcal{L}(S_{I_1'}(\Omega))} \le (c_1^2/r)^d c_4 \rho^{|J_j|d}, \quad \text{for all } j = 1, \dots, m. \tag{5.8}$$

A similar argument as that in (5.6) shows that

$$\frac{\mathcal{L}(S_{I_1}(\Omega))}{\mathcal{L}(S_{I_1'}(\Omega))} \ge c_1^{-d} \frac{\mathcal{L}(S_{J_1}(\Omega))}{\mathcal{L}(\Omega)} \ge c_1^{-d} r_{J_1}^d \ge c_1^{-d} r^{|J_1|d}. \tag{5.9}$$

Recall that I_1' is the longest initial segment of $I_1 = I_1' J_1$ that belongs to some $\mathcal{M}_k$. In view of Definition 5.1(3), there exists some $J_1' \in \Sigma_*$ such that $I_1 J_1' = I_1' J_1 J_1' \in \mathcal{M}_{k+1}$, and condition (5) of the same definition says $|J_1| \le |J_1| + |J_1'| \le L$. It follows by combining this with (5.8) and (5.9) that there exists some constant $\bar{c} > 0$ (which can be taken to be, say, $r^{L+1}/(c_4^{1/d} c_2^3)$) such that

$$\bar{c} \le \rho^{|J_j|} \quad \text{for all } j = 1, \dots, m.$$

If we let $\ell := \lfloor \log \bar{c} \big/ \log \rho \rfloor + 1$, then $|J_j| \le \ell$ (where $\lfloor x \rfloor$ is the integer part of x). It follows that

$$\#\{S \in \mathcal{A}_b : x \in S(\Omega)\} \le M N^{\ell},$$

which completes the proof of the theorem. □

6 Examples

In this section, we first illustrate the main results in this paper by an example. Then we provide a family of examples with similar properties.

Consider the following IFS on [0, 1]:

$$S_1(x) = \frac{1}{2}x, \quad S_2(x) = \frac{1}{16}x^2 + \frac{9}{32}x + \frac{11}{32}, \quad S_3(x) = \frac{1}{32}x^2 + \frac{9}{32}x + \frac{11}{16}. \tag{6.1}$$

We first observe that $S_1(0,1) \cap S_2(0,1) \neq \emptyset$ and $S_2(1) = S_3(0)$ so that the attractor is [0, 1] and the IFS does not satisfy the OSC. Note also the crucial property that $S_{13} = S_{21}$. We establish other properties of the IFS below.

Lemma 6.1 $\{S_i\}_{i=1}^3$ *defined in* (6.1) *is an IFS of* C^1 *injective conformal contractions on* [0, 1] *that has the BDP.*

Proof First it is direct to check that S_2 and S_3 are increasing on some neighborhood of [0, 1], and they are contractive on this neighborhood. $\{S_i\}_{i=1}^3$ also has the BDP, because each S_i is a C^∞ function (see, e.g., [17, Remark 2.3]). This completes the proof. □

Example 6.2 The IFS $\{S_i\}_{i=1}^3$ in (6.1) is of finite type.

Proof By Lemma 6.1, the S_i are injections on [0, 1]. Let $\Omega = (0, 1)$, $\mathcal{M}_k = \Sigma_k$ for all $k \geq 0$, and $\mathcal{T}_0 = [\boldsymbol{v}_{\text{root}}]$. $\boldsymbol{v}_{\text{root}}$ has three offspring $\boldsymbol{v}_i = (S_i, 1)$, $i = 1, 2, 3$. $[\boldsymbol{v}_3] = [\boldsymbol{v}_{\text{root}}] = \mathcal{T}_0$. Let $[\boldsymbol{v}_i] = \mathcal{T}_i$, $i = 1, 2$. Upon one more iteration, we see that no new neighborhood types are generated. In fact, since $S_{13} = S_{21}$ and $S_2(\Omega) \cap S_3(\Omega) = \emptyset$, $\boldsymbol{v}_1$ has two offspring of neighborhood type $\mathcal{T}_1$ and one of neighborhood type $\mathcal{T}_2$, and $\boldsymbol{v}_2$ has three offspring of distinct neighborhood types $\mathcal{T}_i$, $i = 0, 1, 2$ (see Fig. 1). Thus by Proposition 5.3, $\mathcal{V}/_\sim = \{\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2\}$. □

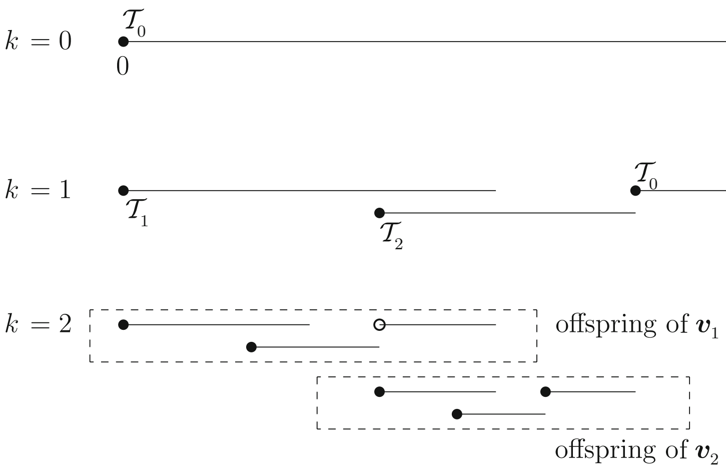

Fig. 1 Vertices in $\mathcal{V}_k$ for $k = 0, 1, 2$ in Example 6.3

Figure 1 shows how all the neighborhood types are generated. Overlapping vertices are separated vertically to show distinction and multiplicity. Iterates of the point 0 under the maps are represented by dots (or circles). For $k = 2$, only offspring of $\boldsymbol{v}_1 = (S_1, 1)$ and $\boldsymbol{v}_2 = (S_2, 1)$ are shown, and the one indicated by a circle is to be removed when constructing $\mathcal{G}_R$.

We illustrate Theorem 1.2 by using the above example.

Example 6.3 Consider the IFS $\{S_i\}_{i=1}^3$ in (6.1). Let $\{p_i\}_{i=1}^3$ be any associated probability weights. Then the corresponding self-conformal measure is singular (with respect to the Lebesgue measure on $\mathbb{R}$).

Proof It is direct to check that the self-conformal set is $K = [0, 1]$. By Example 6.2, $\{S_i\}_{i=1}^3$ is of finite type and thus by Theorem 1.4, it satisfies the WSC. Note that for each $\epsilon > 0$ sufficiently small, each S_i can be extended to a C^1 injective conformal mapping on the interval $V_\epsilon = (-\epsilon, 1 + \epsilon)$. Moreover, with respect to V_ϵ, we have

$$R_{S_1} = \frac{1}{2}, \quad R_{S_2} = \frac{13}{32} + \frac{\epsilon}{8}, \quad R_{S_3} = \frac{11}{32} + \frac{\epsilon}{16}.$$

Let $b = 1$. Then $S_1, S_2, S_3 \in \mathcal{A}_b$. If $p_1 > 1/2 = R_{S_1}$, then the singularity of μ follows from Theorem 1.2. Similarly, if $p_1 < 1 - R_{S_2} - R_{S_3} = 1/4 - 3\epsilon/16$, then at least one of the inequalities $p_2 > R_{S_2}$ or $p_3 > R_{S_3}$ must hold, and again μ is singular.

Now consider the case $p_1 \in [1/4 - 3\epsilon/16, 1/2]$. Note that $S_{13} = S_{21} =: S$ and $R_S = 11/64 + \epsilon/32$. Thus, for all $\epsilon > 0$ sufficiently small,

$$p_S = p_1 p_3 + p_2 p_1 = p_1(1 - p_1) \geq \frac{3}{16} - \frac{3\epsilon}{32} - \left(\frac{3\epsilon}{16}\right)^2 > R_S.$$

Hence μ is also singular in this case. □

The IFS in (6.1) can be generalized as follows.

Example 6.4 Let $0 < a < 1$ and define

$$S_1(x) = ax, \quad S_2(x) = bx^2 + cx + ad, \quad S_3(x) = abx^2 + cx + d. \tag{6.2}$$

Assume either of the following conditions holds:

(C1) $ab + c + d = 1,\ b + c + ad \leq d$ and either
   (a) $b \geq 0,\ c > 0,\ 2b + c < 1$ or
   (b) $b < 0,\ 2b + c > 0,\ 2ab + c < 1.$
(C2) $d = 1,\ ab + c + 1 > a,\ b + c + a \geq 0$ and either
   (a) $b \geq 0,\ 2b + c < 0,\ 2ab + c > -1$ or
   (b) $b < 0,\ c < 0,\ 2b + c > -1.$

(Note that for each case the set of parameters is nonempty.) Then $\{S_i\}_{i=1}^3$ is a finite type conformal IFS on [0, 1] that has the BDP and hence it satisfies the WSC.

To understand the family, we note that $S_{13} = S_{21}$ and therefore the OSC is not satisfied. Both conditions (C1) and (C2) imply that the convex hull of the self-conformal set K is the interval $[0, 1]$, $S_1[0, 1] \cap S_2[0, 1] \neq \emptyset$, and $S_2(0, 1) \cap S_3(0, 1) = \emptyset$. Conditions (a) and (b) in (C1) guarantee that both S_2 and S_3 are increasing and have derivatives bounded between 0 and 1 on some neighborhood of $[0, 1]$, while conditions (a) and (b) in (C2) guarantee that both S_2 and S_3 are decreasing and have derivatives bounded between -1 and 0 on some neighborhood of $[0, 1]$. We also remark that under either condition (C1) or (C2), three of the parameters a, b, c, d are free, and the special case for $b = 0$ (self-similar) is studied in detail in [15].

Example 6.4 can be proved by modifying the arguments in Lemma 6.1 and Example 6.2 slightly.

By extending the argument in Example 6.3 and using Theorem 1.2, we can show that for values of the parameters belonging to a quite large region that satisfies either of the conditions (C1) or (C2) in Example 6.4, all associated self-conformal measures are singular. However, we do not know whether singularity holds for *all* associated self-conformal measures for all possible parameter values satisfying condition (C1) or (C2). Nevertheless, we show in the next section that if we allow the probability weights to be place-dependent, then absolutely continuous self-conformal measures can be constructed. The method illustrated in Example 6.3 may fail for some of the IFSs in (6.2); a simple example of such an IFS is given by the parameter values $a = 1/2, b = 1/4, c = 1/8, d = 3/4$.

7 Absolutely continuous measures

We will show that if the probability weights are allowed to be place-dependent, then the corresponding self-conformal measure can be absolutely continuous, and can even have a continuous density function. The following family of examples are modified from those in [2] and [25].

Then S be C^1 on an open interval containing $[0, 1]$ and assume that S satisfies

$$\begin{aligned} &S(0) = 0, \quad S(1) = \frac{1}{2}, \quad S'(0) = S'(1), \quad \text{and} \\ &0 < C_1 \le S'(x) \le C_2 < 1 \quad \text{on } [0, 1], \end{aligned} \tag{7.1}$$

where C_1, C_2 are constants. $S(x) = x/2$ and $S(x) = (-2x^3 + 3x^2 + 2x)/6$ are examples of such a function. Define $S_0 : \mathbb{R} \to \mathbb{R}$ by extending S as follows

$$S_0(x) = S(x - k) + k/2, \quad \text{if } x \in [k, k+1) \text{ and } k \in \mathbb{Z}.$$

Equivalently,

$$S_0(x) = S(x - \lfloor x \rfloor) + \lfloor x \rfloor/2. \tag{7.2}$$

(Recall that $\lfloor x \rfloor$ is the integer part of x.) It follows from (7.1) that S_0 is C^1 on $\mathbb{R}$.

Now fix a positive integer $N \geq 2$ and define mappings $S_i : \mathbb{R} \to \mathbb{R}$, $i = 1, \ldots, N$, by

$$S_i(x) = S_0(x) + i/2, \quad x \in \mathbb{R}. \tag{7.3}$$

The following identities are useful and can be derived directly from (7.2):

$$S_i(x+k) = S_i(x) + k/2 \quad \text{and} \quad S_i^{-1}(x+k/2) = S_i^{-1}(x) + k,$$
$$i = 0, 1, \ldots, N, \; k \in \mathbb{Z}. \tag{7.4}$$

By combining (7.3) and (7.4) we also have

$$S_i^{-1}(x) = S_0^{-1}(x - i/2) = S_0^{-1}(x) - i, \quad x \in \mathbb{R}. \tag{7.5}$$

Note that $\{S_i\}_{i=0}^N$ is a conformal IFS on $[0, N]$.

Proposition 7.1 *$\{S_i\}_{i=0}^N$ is a finite type conformal IFS on $[0, N]$.*

Proof We first illustrate the proof by using the case $N = 2$. Let $\Omega = (0, 2)$ and let $\mathcal{T}_0, \mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3$ denote the neighborhood types of $(I, 0)$, $(S_0, 1)$, $(S_1, 1)$, and $(S_2, 1)$, respectively. The following observations can be proved easily:

(a) $S_0S_2 = S_1S_0$ and $S_1S_2 = S_2S_0$.
(b) (i) $S_0S_0(2) = S_0S_2(0)(= S_1S_0(0)) = 1/2$.
(ii) $S_0S_2(2)(= S_1S_0(2)) = S_1S_2(0)(= S_2S_0(0)) = 1$.
(iii) $S_1S_2(2)(= S_2S_0(2)) = S_2S_2(0) = 3/2$.
(iv) $S_0S_1(2) = S_1S_1(0) = S_0(1/2) + 1/2$.
(v) $S_1S_1(2) = S_2S_1(0) = S_0(1/2) + 1$.

It follows from these observations that

$$[(S_0S_0, 2)] = \mathcal{T}_1, \quad [(S_0S_1, 2)] = [(S_1S_1, 2)] = [(S_2S_1, 2)] = \mathcal{T}_2, \quad [(S_2S_2, 2)] = \mathcal{T}_3.$$

It remains to show that

$$[(S_1S_0, 2)] = [(S_1S_2, 2)] = \mathcal{T}_2.$$

Let $\tau = S_1S_0S_1^{-1}$. Then $\tau S_1 = S_1S_0$. To show that $[(S_1S_0, 2)] = [(S_1, 1)]$, it suffices to show that

$$\tau S_0 = S_0S_1 \quad \text{and} \quad \tau S_2 = S_1S_1.$$

We will prove the first equality; the second one can be established similarly. Note that for all $x \in \mathbb{R}$,

$$
\begin{aligned}
\tau S_0(x) &= S_0 S_2 S_1^{-1} S_0(x) \qquad \text{(by (a))} \\
&= S_0 S_2 S_0^{-1}(S_0(x) - 1/2) \qquad \text{(by (7.5))} \\
&= S_0(S_0 S_0^{-1}(S_0(x) - 1/2) + 1) \qquad \text{(by (7.3))} \\
&= S_0(S_0(x) + 1/2) \\
&= S_0 S_1(x).
\end{aligned}
$$

The proof for $[(S_1S_2, 2)]$ is the same. This completes the proof that $\{S_i\}_{i=0}^{2}$ is of finite type.

For $N > 2$, we let $\Omega = (0, N)$ and let $\mathcal{T}_{i,j}$ denote the neighborhood type of a vertex with i left neighbors and j right neighbors, in the reduced graph. Thus, $\mathcal{T}_{0,0}$ denotes the neighborhood type of the root vertex. Upon one iteration, the following $N+1$ new neighborhood types are generated:

$$\mathcal{T}_{0,N-1},\ \mathcal{T}_{1,N-1},\ \mathcal{T}_{N-1,1},\ \mathcal{T}_{N-1,0}, \quad \text{and} \quad \{\mathcal{T}_{i,N-i}\}_{i=2}^{N-2}.$$

Upon one more iteration, the following $2N-5$ new neighborhood types are generated:

$$\{\mathcal{T}_{i,N-1}\}_{i=2}^{N-1} \quad \text{and} \quad \{\mathcal{T}_{N-1,N-i}\}_{i=2}^{N-2}.$$

No more new neighborhood types are generated upon another iteration. Thus, the total number of neighborhood types is $3N-3$. We leave the details for the reader. □

To study self-conformal measures associated to the IFS $\{S_i\}_{i=0}^{N}$, we let $\{p_i(x)\}_{i=0}^{N}$ be strictly positive weight functions so that each p_i is continuous and $\log p_i$ satisfies the Dini condition. It is known [6] that under these assumptions, there exists a unique probability measure μ satisfying

$$\rho\mu = \sum_{i=0}^{N} p_i \mu \circ S_i^{-1}, \tag{7.6}$$

where $\rho \geq 1$ is the spectral radius of the Ruelle operator $T : C[0, N] \to C[0, N]$ (the space of continuous functions on $[0, N]$) defined by

$$Tf(x) = \sum_{i=0}^{N} p_i(S_i(x)) f(S_i(x)).$$

We assume that

$$\sum_{i=0}^{N} \int_0^N p_i(x)\, d\mu(x) = 1 \tag{7.7}$$

so that $\rho = 1$ and (7.6) becomes

$$\mu = \sum_{i=0}^{N} p_i \mu \circ S_i^{-1}, \tag{7.8}$$

which means

$$\mu(A) = \sum_{i=0}^{N} \int_{S_i^{-1}(A)} p_i \, d\mu,$$

for any μ measurable set A, and for any integrable f,

$$\int_A f \, d\mu = \sum_{i=0}^{N} \int_{S_i^{-1}(A)} (f \circ S_i) p_i \, d\mu.$$

As in [25], we can express the measure μ in a vector form. To see this, note that for any Borel subset $D \subseteq \mathbb{R}$ and $i = 1, \ldots, N$, we can use (7.4) and (7.5) repeatedly to write

$$\begin{aligned}\mu(D \cap [0,1] + i - 1) = &\sum_{j=\max\{2i-N-1,1\}}^{\min\{2i-1,N\}} p_{2i-j-1} \mu(S_0^{-1}(D) \cap [0,1] + j - 1) \\ &+ \sum_{j=\max\{2i-N,1\}}^{\min\{2i,N\}} p_{2i-j} \mu(S_1^{-1}(D) \cap [0,1] + j - 1).\end{aligned}$$

Define $N \times N$ matrices $P_0 = P_0(x)$ and $P_1 = P_1(x)$ by

$$(P_0)_{ij} = p_{2i-j-1}, \quad (P_1)_{ij} = p_{2i-j}, \quad 1 \le i, j \le N.$$

Equivalently,

$$P_0 = \begin{bmatrix} p_0 & 0 & 0 & 0 & \ldots & 0 & 0 \\ p_2 & p_1 & p_0 & 0 & \ldots & 0 & 0 \\ \vdots & & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & p_N & p_{N-1} \end{bmatrix},$$

$$P_1 = \begin{bmatrix} p_1 & p_0 & 0 & 0 & \ldots & 0 & 0 \\ p_3 & p_2 & p_1 & p_0 & \ldots & 0 & 0 \\ \vdots & & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \ldots & 0 & p_N \end{bmatrix}.$$

Then

$$\begin{bmatrix} \mu(D\cap[0,1]) \\ \mu(D\cap[0,1]+1) \\ \vdots \\ \mu(D\cap[0,1]+N-1) \end{bmatrix} = P_0 \begin{bmatrix} \mu(S_0^{-1}(D)\cap[0,1]) \\ \mu(S_0^{-1}(D)\cap[0,1]+1) \\ \vdots \\ \mu(S_0^{-1}(D)\cap[0,1]+N-1) \end{bmatrix}$$
$$+P_1 \begin{bmatrix} \mu(S_1^{-1}(D)\cap[0,1]) \\ \mu(S_1^{-1}(D)\cap[0,1]+1) \\ \vdots \\ \mu(S_1^{-1}(D)\cap[0,1]+N-1) \end{bmatrix}.$$

We will show that if the weight functions $\{p_i\}_{i=1}^N$ are suitably chosen, then the corresponding self-conformal measure μ is absolutely continuous and may even have a continuous density function f. We start by considering the dilation equation corresponding to (7.8):

$$f(x) = \sum_{i=0}^{N} (S_i^{-1})'(x)\, p_i(S_i^{-1}(x))\, f(S_i^{-1}(x)). \tag{7.9}$$

Note that μ is absolutely continuous if and only if there exists an L^1-solution f to (7.9) and in this case, $d\mu/dx = f$.

Define the linear operator L on the space of functions g on $[0, N]$ as

$$Lg(x) = \sum_{i=0}^{N} (S_i^{-1})'(x)\, p_i(S_i^{-1}(x))\, g(S_i^{-1}(x)).$$

We can also write (7.9) in a vector form, in a similar fashion as for the corresponding measure. Define

$$\mathbf{g}(x) = \begin{bmatrix} g(x) \\ g(x+1) \\ \vdots \\ g(x+N-1) \end{bmatrix} \quad \text{and} \quad \mathbf{Lg}(x) = \begin{bmatrix} Lg(x) \\ Lg(x+1) \\ \vdots \\ Lg(x+N-1) \end{bmatrix}, \quad x \in [0,1].$$

Let $T_0 = T_0(x)$ and $T_1 = T_1(x)$ be $N \times N$ matrices defined as

$$(T_0(x))_{i,j} = \frac{p_{2i-j-1}(x+j-1)}{S'_{2i-j-1}(x+j-1)}, \quad (T_1(x))_{i,j} = \frac{p_{2i-j}(x+j-1)}{S'_{2i-j}(x+j-1)}, \quad 1 \le i, j \le N.$$

Then it can be derived as above that

$$\mathbf{Lg}(x) = \begin{cases} T_0(S_0^{-1}(x))\mathbf{g}(S_0^{-1}(x)), & \text{if } 0 \le x \le 1/2 \\ T_1(S_1^{-1}(x))\mathbf{g}(S_1^{-1}(x)), & \text{if } 1/2 \le x \le 1. \end{cases}$$

Define

$$\sigma x = \begin{cases} S_0^{-1}(x), & \text{if } 0 \le x \le 1/2 \\ S_1^{-1}(x), & \text{if } 1/2 \le x \le 1 \end{cases} \quad \text{and} \quad d(x) = \begin{cases} 0, & \text{if } 0 \le x \le 1/2 \\ 1, & \text{if } 1/2 \le x \le 1. \end{cases}$$

Then

$$\mathbf{Lg}(x) = \begin{cases} T_0(\sigma x)\mathbf{g}(\sigma x), & \text{if } 0 \le x \le 1/2 \\ T_1(\sigma x)\mathbf{g}(\sigma x), & \text{if } 1/2 \le x \le 1, \end{cases}$$

and for all $k \ge 1$,

$$\mathbf{Lg}(x) = T_{d(x)}(\sigma x)T_{d(\sigma x)}(\sigma^2 x) \cdots T_{d(\sigma^{k-1}x)}(\sigma^k x)\mathbf{g}(\sigma^k x). \tag{7.10}$$

To obtain examples of L^1 and continuous solutions f to the dilation equation (7.9), we choose each weight function p_i, $i = 0, 1, \dots, N$, to be proportional to S_i', i.e.,

$$p_i(x) = c_i S_i'(x), \quad x \in \mathbb{R}, \tag{7.11}$$

for some constant c_i. In this case, (7.9) becomes

$$f(x) = \sum_{i=0}^{N} c_i f(S_i^{-1}(x)), \tag{7.12}$$

and T_0 and T_1 reduce to constant matrices:

$$T_0 = \begin{bmatrix} c_0 & 0 & 0 & 0 & \dots & 0 & 0 \\ c_2 & c_1 & c_0 & 0 & \dots & 0 & 0 \\ \vdots & & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & c_N & c_{N-1} \end{bmatrix},$$

$$T_1 = \begin{bmatrix} c_1 & c_0 & 0 & 0 & \dots & 0 & 0 \\ c_3 & c_2 & c_1 & c_0 & \dots & 0 & 0 \\ \vdots & & & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 & c_N \end{bmatrix}.$$

Let M be the $(N-1) \times (N-1)$ matrix obtained by deleting the first row and first column of T_0, or equivalently, the last row and last column of T_1.

Suppose now that 1 is an eigenvalue of M with an associated eigenvector $\mathbf{a} = (a_1, \ldots, a_{N-1})$ that can be normalized so that $\sum_{i=1}^{N-1} a_i = 1$ (a sufficient condition for this is given in the theorem below). Let

$$f(i) = \begin{cases} a_i, & i = 1, \ldots, N-1 \\ 0, & i \le 0 \text{ or } i \ge N. \end{cases}$$

Let $f_0(x)$ be the function which is linear on every interval $[i, i+1]$ and satisfies $f_0(i) = f(i)$ for all i, and let

$$f_k(x) = L^k f_0(x), \quad k \in \mathbb{N}. \tag{7.13}$$

Let $a_0 = a_N = 0$ and define

$$\mathbf{v}_0 = \begin{bmatrix} a_0(1-x) + a_1 x \\ a_2(1-x) + a_3 x \\ \vdots \\ a_{N-1}(1-x) + a_N x \end{bmatrix} \quad \text{and} \quad \mathbf{v}_k(x) = \begin{bmatrix} f_k(x) \\ f_k(x+1) \\ \vdots \\ f_k(x+N-1) \end{bmatrix}, \quad k \ge 1 \tag{7.14}$$

Then $\mathbf{v}_k(x) = \mathbf{L}^k \mathbf{v}_0(x)$.

The following theorem can be easily proved by modifying [2, Theorem 2.5]; we omit the details.

Theorem 7.2 *Let $\{S_i\}_{i=0}^N$ be defined as in (7.2) and (7.3). Assume that*

$$\sum c_{2i} = \sum c_{2i+1} = 1. \tag{7.15}$$

Let E_1 be the $(N-1)$-dimensional subspace orthogonal to $\mathbf{e}_1 = (1, \ldots, 1)$, the left eigenvector of T_0 and T_1 for the eigenvalue 1. Assume that there exist $\lambda < 1$ and $C > 0$ such that for all $m \in \mathbb{N}$,

$$\max\{\|T_{d_1} \cdots T_{d_m}|_{E_1}\| : d_j = 0 \text{ or } 1, \ j = 1, \ldots, m\} \le C\lambda^m. \tag{7.16}$$

Then the following hold:

(a) 1 *is a simple eigenvalue of M and there exists an associated eigenvector $\mathbf{a} = (a_1, \ldots, a_{N-1})$ that can be normalized so that $\sum_{i=1}^{N-1} a_i = 1$.*
(b) *The functions $\mathbf{v}_k(x)$ defined in (7.14) satisfy $\mathbf{e}_1 \cdot \mathbf{v}_k(x) = 1$ for all $k \in \mathbb{N}$ and all $x \in [0, 1]$.*
(c) *The corresponding functions f_k in (7.13) converge uniformly to a continuous function f and*

$$\|f_k - f\|_\infty \le C 2^{-k|\ln \lambda|/\ln 2}.$$

Moreover, f is an L^1-solution of (7.9) and satisfies $\int_0^N f(x)\,dx = 1$.

It is shown in [2] that the condition in (7.16) can be expressed in more convenient forms. For $m \in \mathbb{N}$, define

$$\lambda_m = \max\{\|T_{d_1} \cdots T_{d_m}|_{E_1}\|^{1/m} : d_j = 0 \text{ or } 1,\ j = 1, \ldots, m\}.$$

Also, given two matrices A_0, A_1, define the *joint spectral radius* of A_0, A_1 by

$$\hat{\rho}(A_0, A_1) = \limsup_{m\to\infty} \max\{\|A_{d_1} \cdots A_{d_m}\|^{1/m} : d_j = 0 \text{ or } 1,\ j = 1, \ldots, m\}.$$

Under the assumptions of Theorem 7.2, it follows from [2] that each of the following conditions is equivalent to (7.16):

$$\lambda_m < 1 \text{ for some } m \in \mathbb{N} \tag{7.17}$$

or

$$\hat{\rho}(T_0|_{E_1}, T|_{E_1}) < 1. \tag{7.18}$$

There are many known examples of positive constants $\{c_i\}_{i=0}^N$ that satisfy these conditions (see, e.g., [1]). They provide us with examples of continuous solutions to (7.9) and thus absolutely continuous self-conformal measures defined by (7.8).

Finally, we remark that if (7.15) and (7.16) are satisfied so that a continuous scaling function f exists, then for $p_i(x)$ satisfying (7.11), the corresponding self-conformal measure satisfies (7.7) automatically. In fact, by Theorem 7.2(c) and (7.12),

$$\begin{aligned} 1 = \int_0^N f(x)\,dx &= \sum_{i=0}^N c_i \int_0^N f(S_i^{-1}(x))\,dx \\ &= \sum_{i=0}^N c_i \int_0^N f(x) S_i'(x)\,dx = \sum_{i=0}^N \int_0^N p_i(x)\,d\mu(x). \end{aligned}$$

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References

1. Colella, D., Heil, C.: The characterization of continuous, four-coefficient scaling functions and wavelets. IEEE Trans. Inform. Theory **38**, 876–881 (1992)
2. Daubechies, I., Lagarias, J.C.: Two-scale difference equations. II. Local regularity, infinite products of matrices and fractals. SIAM J. Math. Anal. **23**, 1031–1079 (1992)

3. Deng, Q.-R., He, X.-G., Lau, K.-S.: Self-affine meausres and vector-valued representations, Studia Math. (2008, in press)
4. Falconer, K.J.: Dimensions and measures of quasi self-similar sets. Proc. Amer. Math. Soc. **106**, 543–554 (1989)
5. Falconer, K.J.: Techniques in fractal geometry. Wiley, Chichester (1997)
6. Fan, A.-H., Lau, K.-S.: Iterated function system and Ruelle operator. J. Math. Anal. Appl. **231**(2), 319–344 (1999)
7. Feng, D.: The smoothness of L^q-spectrum of self-similar measures with overlaps. J. Lond. Math. Soc. **68**, 102–118 (2003)
8. Feng, D.: The limited Rademacher functions and Bernoulli convolutions associated with Pisot numbers. Adv. Math. **195**, 24–101 (2005)
9. Hutchinson, J.E.: Fractals and self-similarity. Indiana Univ. Math. J. **30**, 713–747 (1981)
10. Jin, N., Yau, S.S.T.: General finite type IFS and M-matrix. Comm. Anal. Geom. **13**, 821–843 (2005)
11. Lau, K.-S., Ngai, S.-M.: Multifractal measures and a weak separation condition. Adv. Math. **141**, 45–96 (1999)
12. Lau, K.-S., Ngai, S.-M.: A generalized finite type condition for iterated function systems. Adv. Math **208**, 647–671 (2007)
13. Lau, K.-S., Ngai, S.-M., Rao, H.: Iterated function systems with overlaps and self-similar measures. J. Lond. Math. Soc. (2) **63**(1), 99–116 (2001)
14. Lau, K.-S., Rao, H., Ye, Y.: Corrigendum: "Iterated function system and Ruelle operator" [J. Math. Anal. Appl. 231 (1999), 319–344] by Lau, K.-S., Fan, A. H. J. Math. Anal. Appl. **262**, 446–451 (2001)
15. Lau, K.-S., Wang, X.-Y.: Iterated function systems with a weak separation condition. Studia Math. **161**, 249–268 (2004)
16. Mattila, P.: Geometry of sets and measures in Euclidean spaces. Fractals and rectifiability, Cambridge Studies in Advanced Mathematics, vol. 44. Cambridge University Press, Cambridge (1995)
17. Mauldin, R.D., Urbański, M.: Dimensions and measures in infinite iterated function systems. Proc. Lond. Math. Soc. (3) **73**, 105–154 (1996)
18. Mauldin, R.D., Urbański, M.: Conformal iterated function systems with applications to the geometry of continued fractions. Trans. Amer. Math. Soc. **351**, 4995–5025 (1999)
19. Mauldin, R.D., Urbański, M.: Graph directed Markov systems. Geometry and dynamics of limit sets, Cambridge Tracts in Mathematics, vol. 148. Cambridge University Press, Cambridge (2003)
20. Ngai, S.-M., Wang, Y.: Hausdorff dimension of self-similar sets with overlaps. J. Lond. Math. Soc. (2) **63**(3), 655–672 (2001)
21. Patzschke, N.: Self-conformal multifractal measures. Adv. Appl. Math. **19**(4), 486–513 (1997)
22. Peres, Y., Rams, M., Simon, K., Solomyak, B.: Equivalence of positive Hausdorff measure and the open set condition for self-conformal sets. Proc. Amer. Math. Soc. **129**(9), 2689–2699 (2001)
23. Peres, Y., Schlag, W., Solomyak, B.: Sixty years of Bernoulli convolutions. Fractal geometry and stochastics, II (Greifswald/Koserow, 1998) Progr. Probab., vol. 46. Birkhäuser, Basel, pp. 39–65 (2000)
24. Ye, Y.-L.: Separation properties for self-conformal sets. Studia Math. **152**(1), 33–44 (2002)
25. Ye, Y.-L.: Multifractal of self-conformal measures. Nonlinearity. **18**(5), 2111–2133 (2005)
26. Zerner, M.P.W.: Weak separation properties for self-similar sets. Proc. Amer. Math. Soc. **124**, 3529–3539 (1996)