An Integral Related To The Cauchy Transform On The Sierpinski Gasket

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We estimate an integral on the Sierpinski gasket and justify a theorem in the paper [Lund et al. 98]. The integral relates to the Laplace transform of the Hausdorff measure. It is fundamental and useful in some other contexts [Dong and Lau xx].

1. INTRODUCTION

Let \( K \) be the Sierpinski gasket in the complex plane \( \mathbb{C} \) with three vertices at \( \varepsilon_k = e^{2k\pi i/3}, k = 0, 1, 2 \). It is well known that \( K \) is the attractor of the iterated function system \( \{ S_k \}_{k=0}^{2} \) with \( S_k z = \varepsilon_k + (z - \varepsilon_k)/2 \) and the Hausdorff dimension of \( K \) is \( \alpha = \log 3 / \log 2 \). Let \( \mu \) be the Hausdorff measure \( \mathcal{H}^\alpha \) normalized on \( K \). We define the Cauchy transform of \( \mu \) by

\[
F(z) = \int_K \frac{d\mu(w)}{z - w}.
\]

In [Lund et al. 98], Strichartz et al. initiated the study of the analytic and geometric behavior of the function \( F \). One of the most interesting observations concerns the image of \( K \) under \( F \). Let \( \Delta_0 \) denote the unbounded connected region outside the Sierpinski gasket. The following result was claimed in [Lund et al. 98].

**Theorem 1.1.** \( F(-\frac{1}{2}) \) lies in the interior of \( F(\Delta_0) \).

Note that the point \(-1/2\) is on the boundary curve \( \partial \Delta_0 \) of \( \Delta_0 \). The theorem implies that the image curve \( F(\partial \Delta_0) \) forms a loop near \( F(-\frac{1}{2}) \). By self-similarity, the loops appear everywhere on the image point of each dyadic rational point on \( \partial \Delta_0 \) (see Figure 1). This leads to the conjecture in [Lund et al. 98] that the boundary of \( F(\Delta_0) \) is a simple closed curve and is the image of a Cantor set in \( \partial \Delta_0 \). The reader can also refer to [Dong and Lau 04] for more detail.

As \( F \) is continuous and bounded on \( \mathbb{C} \), \( F(x) < 0 \) for \( x \in (-\infty, -1/2) \) and \( F(-\infty) = 0 \), it follows that...
$F([-\infty, -1/2]) = [a, 0]$ for some $a < 0$. Their proof of the theorem is to conclude $a < F(-1/2) < 0$ by showing that $F(x)$ is increasing for $x < -1/2$ and near $-1/2$. It is equivalent to show that

$$g(x) := F'(-(x + 1/2)) = -\int_K \frac{d\mu(w)}{(1/2 + x + w)^2}$$

$$= \int_K \frac{v^2 - (1/2 + x + u)^2}{(v^2 + (1/2 + x + u)^2)^2} d\mu(w) > 0$$

($w = u + iv$) for small $x > 0$. The difficulty is that it is awkward to handle the integral over the fractal set $K$. In addition, the integrand takes both positive and negative values on $K$. They tried to get around this by using a clever method to show that $g(0) = \infty$, and claimed that a similar argument would imply $g(x) > 0$ for small $x > 0$. However the claim is not so direct, as it is not clear that $\lim_{x \to 0^+} g(x) = g(0)$ (in fact it is not even clear that $g(x) \neq 0$). The main purpose of this note is to justify this step. The integrals in the following are useful and appear in other contexts [Dong and Lau 04].

Let $T = 1 - K$ be the relocation of the Sierpinski gasket with the new vertices at $0, \sqrt{3}e^{\pi i/6}, \sqrt{3}e^{-\pi i/6}$, and let $T_j = 1 - K_j$ where $K_j = S_j K$, $j = 0, 1, 2$. Let

$$A_0 = \bigcup_{n=-\infty}^{\infty} 2^n (T_1 \cup T_2)$$

be the “Sierpinski cone” generated by $T$ (see Figure 2). It is easy to see that $T = A_0 \cap T = A_0 \cap \{z = x + yi : x \leq 3/2\}$ and $A_0 = \lim_{r \to +\infty} A_0 \cap \{z = x + yi : x \leq r\}$. We still use $\mu$ to denote the normalized Hausdorff measure (i.e., $\mu(T) = 1$) on $C$. We define

$$H(x) := \int_{e^{\pi i/3} A_0 \cup e^{-\pi i/3} A_0} \frac{d\mu(w)}{(x + w)^2}$$

(see Figure 3 for the domain of integration of $H$, the union of the rotations of $A_0$ by $e^{\pi i/3}$ and $e^{-\pi i/3}$). We can reduce the consideration of $F'$ to $H$ as follows:

**Proposition 1.2.** $F'(-(x + 1/2)) = -H(x) + \psi(x), \ x > 0$ for some real function $\psi(x)$, bounded and continuous for $x \geq 0$.

Our main result is the following:

**Proposition 1.3.** $H(x)$ is continuous and is $< 0$ for $x > 0$. 

![Figure 2. The Sierpinski cone $A_0$.](image2)

![Figure 3. The region $e^{\pi i/3} A_0 \cup e^{-\pi i/3} A_0$.](image3)
By using $\mu(2E) = 3\mu(E)$, it is easy to show that $H(2x) = (3/4)H(x)$. Combining this with Proposition 1.3, we have the following:

**Corollary 1.4.** $\lim_{x \to 0^+} H(x) = -\infty$.

It follows immediately from Proposition 1.2 and Corollary 1.4 that $F'(-(x + 1/2)) > 0$ for small $x > 0$, hence Theorem 1.1 holds.

The major part of the proof is to show that $H(x) < 0$ in Proposition 1.3. We overcome the difficulty in [Lund et al. 98] by considering the Laplace transform $\Phi(t)$ of $\mu$ on $A_0$, which is given by an infinite product of simple functions [Dong and Lau 03]. We use Mathematica and MATLAB to help prove the following interesting fact: $0.4715 < t^n\Phi(t) < 0.4795$ for all $t > 0$. (It is known that $t^n\Phi(t)$ is not a constant [Dong and Lau 03, Theorem 5.6].) This small variation in the values of $t^n\Phi(t)$ allows us to prove Proposition 1.3.

## 2. THE PROOFS

By using the scaling property $\mu(2E) = 3\mu(E)$ and the rotational invariance of $\mu$, we have

$$H(x) = 2\text{Re} \int_{A_0} \frac{d\mu(w)}{(x + \omega e^{i\pi/3})^2}$$

$$= 2\text{Re} \sum_{n=-\infty}^{\infty} \frac{1}{3^n} \int_{T_1 \cup T_2} \frac{d\mu(w)}{(x + 2^{-n} e^{i\pi/3})^2}$$

$$= 2\text{Re} \left( \sum_{n=0}^{\infty} \frac{1}{3^n} \int_{T_1 \cup T_2} \frac{d\mu(w)}{(x + 2^{-n} e^{i\pi/3})^2} + \sum_{n=1}^{\infty} \frac{3}{4} \int_{T_1 \cup T_2} \frac{d\mu(w)}{(2^{-n} x + e^{i\pi/3})^2} \right).$$

It follows that the above series converges absolutely and uniformly on each compact subset of $\mathbb{R}^+$, therefore $H(x)$ is well defined for $x > 0$ and is continuous.

**Proof of Proposition 1.2:** For $T = 1 - K$, we let $T_j = 1 - K_j$ where $K_j = S_j K$, $j = 0, 1, 2$. It is easy to see that $T_0 = \bigcup_{n \leq -1} 2^n (T_1 \cup T_2) \cup \{0\}$, as the “cap” of the Sierpinski cone $A_0$. Note that

$$F'(-(x + 1/2)) = -\int_{K+1/2} \frac{d\mu(w)}{(x + w)^2}$$

and $K + 1/2 = (e^{i\pi/3} T_0) \cup (e^{-i\pi/3} T_0) \cup (K_0 + 1/2)$. We have

$$F(-(x + 1/2)) = -H(x) + \left( \int_{\hat{A}} - \int_{K_0 + 1/2} \right) \frac{d\mu(w)}{(x + w)^2},$$

where $\hat{A} = (e^{i\pi/3} (A_0 \setminus T_0)) \cup (e^{-i\pi/3} (A_0 \setminus T_0))$. Let $\psi(x)$ be the above integral. Since $\hat{A}$ is bounded away from 0 and the integrand is integrable on $\hat{A}$ for each $x \geq 0$ (use the same argument as in the above series expression of $H(x)$), it is easy to see that $\psi(x)$ is bounded and continuous for $x \geq 0$.

The remaining task is to prove $H(x) < 0$ in Proposition 1.3. We need to establish a few lemmas. Let $\Phi(t) = \int_{A_0} e^{-t\omega} d\mu(w)$, $t > 0$ be the Laplace transform of $\mu$ on $A_0$ [Dong and Lau 03, page 78]. Similar to $H(x)$, it is easy to see that

$$\Phi(t) = \sum_{n=0}^{\infty} \frac{1}{3^n} \int_{T_1 \cup T_2} e^{-t2^{-n} w} d\mu(w)$$

and $\Phi(t)$ is continuous. In [Dong and Lau 03, Example 2], we proved that

$$\Phi(t) = \prod_{k=1}^{\infty} q(2^k t) \prod_{k=0}^{\infty} \frac{q(2^{k+1} t)}{3}, \quad t > 0,$$  \hspace{2cm} \tag{2.1}

where

$$q(t) = 1 + 2e^{-3t/4} \cos\left(\frac{\sqrt{3}t}{4}\right).$$  \hspace{2cm} \tag{2.2}

Let $\Phi_0(t) = t^n \Phi(t)$. Since $\Phi_0(2t) = \Phi_0(t)$ and $\Phi_0$ is continuous, $\Phi_0$ is bounded on $\mathbb{R}^+$. Let

$$M = \max_{1/2 \leq t \leq 1} \Phi_0(t),$$

$$m = \min_{1/2 \leq t \leq 1} \Phi_0(t).$$

**Lemma 2.1.** $0.4715 < m \leq M < 0.4795$.

**Proof:** We approximate $\Phi_0(t)$ by the finite product

$$f(t) = t^n \prod_{k=1}^{4} q(2^k t) \prod_{k=0}^{5} \frac{q(2^{k+1} t)}{3}.$$

For this elementary function $f$, we can use “fminbnd” of MATLAB to obtain the maximum and minimum estimation on $[1/2, 1]$:

$$0.4790 < f(t) < 0.4832.$$  \hspace{2cm} \tag{2.4}

Our main estimation is on the two truncated parts of $\Phi_0(t)$. From (2.2), we have

$$1 - 2e^{-3t-3} \leq q(2^k t) \leq 1 + 2e^{-3t-3}, \quad 1/2 \leq t \leq 1.$$  \hspace{2cm} \tag{2.3}

Consider $1 - 2e^{-3x}$; we look for a $d_x$ such that

$$-d_x x^{-7} \leq \log(1 - 2e^{-3x}), \quad x \geq 4.$$  \hspace{2cm} \tag{2.4}
By a direct differentiation of \( g(x) = \log(1 - 2e^{-3x}) + d_1 x^{-7} \), we have

\[
g'(x) = \frac{6}{e^{3x} - 2} \left( 1 - \frac{7d_1 (e^{3x} - 2)}{6x^8} \right), \quad x \geq 4.
\]

If we take \( d_1 = (6 \cdot 4^8)/(7(e^{12} - 2)) \), then \( g'(x) < 0 \); from \( g(\infty) = 0 \), we conclude that \( g(x) > 0 \) for \( x \geq 4 \) as needed.

Similarly we can take \( d_2 = (6 \cdot 4^8)/(7(e^{12} + 2)) \) so that

\[
\log(1 + 2e^{-3x}) \leq d_2 x^{-7}, \quad x \geq 4.
\]

Combining these estimates, we have

\[
e^{-d' t} = e^{-d_1 \sum_{k=5}^{\infty} 2^{-7(k-3)}} = \\
\leq \prod_{k=5}^{\infty} q(2^k t) \\
\leq e^{d_2 \sum_{k=5}^{\infty} 2^{-7(k-3)}} = e^{d_2 t}
\]

(2-5)

for \( 1/2 \leq t \leq 1 \), where \( d_1' = d_1/(2^7 \cdot (2^7 - 1)), \quad i = 1, 2 \).

Next we estimate \( \prod_{k=5}^{\infty} q(2^{-k} t)/3 \). It is easy to check that for \( 0 \leq x \leq 1/64 \),

\[
(3e^{-c x} - q(x))' = \frac{3}{2} e^{-3x/4} \\
\times \left( \frac{2\sqrt{3}}{3} \cos(\frac{\pi}{6} - \frac{\sqrt{3}x}{4}) - 2ce^{(3/4-c)x} \right) \\
\geq \frac{3}{2} e^{-3x/4} (1 - 2ce^{(3/4-c)x}).
\]

If we take \( c = 2^{-1} e^{-3/256} = 0.494175 \ldots \), the above expression is positive, hence

\[
q(x) = 1 + \cos(\sqrt{3} x/4) e^{-3x/4} \leq 3 e^{-c x}, \quad 0 < x \leq 1/64.
\]

Combining this and (5.10) in [Dong and Lau 03], we have

\[
3e^{-1/2(k+1)} \leq q(2^{-k} t) \leq 3e^{-c/2(k+1)}
\]

for \( k \geq 6 \) and \( 1/2 \leq t \leq 1 \); hence

\[
e^{-1/2^6} \leq \prod_{k=5}^{\infty} q(2^{-k} t)/3 \leq e^{-c/2^6}, \quad 1/2 \leq t \leq 1.
\]

(2-6)

By (2-1) and (2-3)–(2-6)

\[
0.4715 < 0.4790 e^{-d'_1 - 1/2^6} < \Phi_0(t) < 0.4832 e^{d_2 - c/2^6} < 0.4795
\]

and Lemma 2.1 follows.

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We remark that the choice of the number of factors in \( f(t) \) and the \( x^{-7} \) are by trial and error so as to get two bounds accurate enough to fit in Lemma 2.3 in the sequel to get a positive value. We also remark that for the \( f(t) \) in (2-3), we can actually show that \( f'(t) > 0 \) for \( 1/2 \leq t \leq 1 \), hence \( f(1/2) \leq f(t) \leq f(1) \) for \( t \in [1/2, 1] \) (see Figure 4). However, the proof is lengthy and does not have much significance, so the above MATLAB approximation is enough for our purpose.

**Lemma 2.2.** There exists a constant \( C > 0 \) such that for \( x > 0 \),

\[
x^{2-\alpha} H(x) = \\
- C \int_0^\infty \Phi_0 \left( \frac{2\pi t}{\sqrt{x}} \right) t^{1-\alpha} e^{-\pi t/\sqrt{x}} \sin(\frac{\pi}{6} - \pi t) \, dt \\
:= -C \phi(x).
\]

**Proof:** Let \( x > 0 \) be fixed. Using integration by parts, we have

\[
\int_{A_0} \int_0^\infty |te^{-t(w+xe^{-\pi i/3})}| dt \, d\mu(w) \\
= \int_{A_0} \frac{1}{(\text{Re} w + x/2)^2} d\mu(w) < +\infty.
\]

By Fubini’s theorem,

\[
\int_0^\infty \phi(t)te^{txe^{-\pi i/3}} \, dt = \int_{A_0} \left( \int_0^\infty te^{-t(w+xe^{-\pi i/3})} \, dt \right) d\mu(w) \\
= \int_{A_0} \frac{d\mu(w)}{(w + x e^{-\pi i/3})^2}.
\]
It follows from the definition of $H(x)$ that

$$
H(x) = 2\text{Re} \int_{A_0} \frac{d\mu(w)}{x + w e^{\pi i/3}}
= 2\text{Re} \left(e^{-2\pi i/3} \int_{A_0} \frac{d\mu(w)}{w + x e^{-\pi i/3}}\right)
= -2\text{Re} \int_0^\infty \Phi(t) t^{-x+\pi i/3} \left(\frac{\pi i}{3} + \frac{\sqrt{3}t}{2}\right) dt
= -2 \int_0^\infty \Phi(t) t e^{xt/2} \left(\frac{\pi i}{3} + \frac{\sqrt{3}t}{2}\right) dt
= -C x^{\alpha-2} \int_0^\infty \Phi_0 \left(\frac{2\pi t}{\sqrt{3}x}\right) t^{1-\alpha} e^{-\pi t/\sqrt{3}} \sin\left(\frac{\pi}{6} - \pi t\right) dt.
$$

The last equality follows by a change of variable and by replacing $\Phi$ with $\Phi_0$.

\textbf{Lemma 2.3.} Let $\phi(x)$ be the integral given in Lemma 2.2 and let

$$
a = \int_0^{1/6} t^{1-\alpha} e^{-\pi t/\sqrt{3}} \sin\left(\frac{\pi}{6} - \pi t\right) dt,
$$

$$
b = \int_{1/6}^{7/6} t^{1-\alpha} e^{-\pi t/\sqrt{3}} \sin(\pi t - \pi/6) dt.
$$

Then $\phi(x) > ma - Mb$ for all $x > 0$.

\textbf{Proof:} Let

$$
t_n = t + n + 1/6,
$$

and let

$$
c_n = \int_0^1 t_n^{1-\alpha} e^{-\pi t/\sqrt{3}} \sin(\pi t) dt.
$$

Obviously $c_n > c_{n+1} > 0$. By using the $2\pi$ periodicity of the sine function, we have

$$
\phi(x) = \left(\int_0^{1/6} + \int_{1/6}^{7/6}\right) \Phi_0 \left(\frac{2\pi t}{\sqrt{3}x}\right) t^{1-\alpha} e^{-\pi t/\sqrt{3}} \sin\left(\frac{\pi}{6} - \pi t\right) dt
+ e^{-\pi/(6\sqrt{3})} \sum_{n=1}^\infty (-1)^{n-1} e^{-n\pi/\sqrt{3}} \int_0^1 \Phi_0 \left(\frac{2\pi t}{\sqrt{3}x}\right) t_n^{1-\alpha} e^{-\pi t/\sqrt{3}} \sin(\pi t) dt
d > ma - Mb
+ (m - Me^{-\pi/\sqrt{3}}) e^{-\pi/(6\sqrt{3})} \sum_{k=1}^\infty e^{-(2k-1)\pi/\sqrt{3}} c_{2k-1}.
$$

Lemma 2.1 implies that the last term is positive. Therefore $\phi(x) > ma - Mb$.

\textbf{Proof of Proposition 1.3:} The continuity follows from the remark in the beginning of this section. We use Mathematica to estimate the two constants $a$ and $b$ in Lemma 2.3: $a > 0.3890$, $b < 0.3270$. This together with Lemma 2.1 implies that $\phi(x) > ma - Mb > 0.025$. By Lemma 2.2, $H(x) < 0$ for $x > 0$.

\textbf{ACKNOWLEDGMENTS}

The authors thank Professor R. Strichartz for various comments to improve the paper. They also thank Dr. Xiang-Yang Wang for help in preparing the pictures. The research is supported in part by an HKRGC Grant and a Direct Grant from CUHK. The first author is also partially supported by the NNSFC, Grant No. 19871026.

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Received January 8, 2004; accepted June 7, 2004.