Available online at www.sciencedirect.com
SCIENCE @ DIRECT®

ACADEMIC PRESS

Journal of Functional Analysis 202 (2003) 67–97

JOURNAL OF Functional Analysis

http://www.elsevier.com/locate/jfa

Cauchy transforms of self-similar measures: the Laurent coefficients☆

Xin-Han Dong*,[1] and Ka-Sing Lau

Department of Mathematics, The Chinese University of Hong Kong, Hong Kong

Received 25 April 2002; accepted 22 July 2002

Communicated by L. Gross

Abstract

The Cauchy transform of a measure has been used to study the analytic capacity and uniform rectifiability of subsets in $\mathbb{C}$. Recently, Lund et al. (Experiment. Math. 7 (1998) 177) have initiated the study of such transform F of self-similar measure. In this and the forecoming papers (Starlikeness and the Cauchy transform of some self-similar measures, in preparation; The Cauchy transform on the Sierpinski gasket, in preparation), we study the analytic and geometric behavior as well as the fractal behavior of the transform F. The main concentration here is on the Laurent coefficients $\{a_n\}_{n=0}^{\infty}$ of F. We give asymptotic formulas for $\{a_n\}_{n=0}^{\infty}$ and for $F^{(k)}(z)$ near the support of μ, hence the precise growth rates on $|a_n|$ and $|F^{(k)}|$ are determined. These formulas are connected with some multiplicative periodic functions, which reflect the self-similarity of μ and K. As a by-product, we also discover new identities of certain infinite products and series.

MSC: primary 28A 80; secondary 30C 55; 30E 20

Keywords: Cauchy transform; Self-similar measure; Laurent coefficient

1. Introduction

Harmonic analysis plays a central role in the study of fractal measures. The aspects of Fourier transform of such measures have been investigated in detail by

☆The research is supported in part by the HKRGC Grant. Xin-Han Dong is also partially supported by the National Natural Science Foundation of China (No. 19871026).
*Corresponding author.
E-mail address: xhdong@hunnu.edu.cn (X.-H. Dong).
[1]Present address: Department of Mathematics, Hunan Normal University, ChangSha, 410081 China.

0022-1236/03/$ - see front matter © 2002 Elsevier Inc. All rights reserved.
PII: S 0 0 2 2 - 1 2 3 6 (0 2) 0 0 0 6 9 - 1

Strichartz [St1,St2,St3] and one of the authors [L1,L2,LW]. For the complex case a natural consideration is on the Cauchy transform defined by

$$F(z) = \int_K \frac{d\mu(w)}{z-w},$$

where μ is a bounded regular Borel measure with support K. The study of such transform can be traced back to that of Cauchy-type integral, which is fundamental in the study of the boundary-value problems for analytic functions. The Cauchy transform is also a useful tool in geometric measure theory [Ma,G]. Two typical examples are the Painlevé Theorem and the Vitushkin conjecture [Ma,G, pp. 265–273, pp. 1–3]. The Painlevé Theorem says that a compact set K is removable (or vanishing analytic capacity) if its one-dimensional Hausdorff measure $\mathscr{H}^1(K)$ is 0, its proof depends on the Cauchy transform, and a direct consequence of the theorem is that the Cauchy transform F of μ with support K cannot be bounded on $\mathbb{C}\backslash K$. The Vitushkin conjecture is that if K is a compact set with $\mathscr{H}^1(K) < \infty$ then K is removable if and only if K is purely 1-unrectifiable, it is equivalent to some Cauchy transforms that cannot be bounded on $\mathbb{C}\backslash K$ [Ma, p. 272]. This conjecture has been solved by David recently [D]. In a new direction, Strichartz et al. [LSV] have initiated an investigation of the self-similar measures through the Cauchy transform. They proved some basic analytic and geometric properties of such F; they also raised some interesting questions based on the computational observations of F on the Sierpinski triangle.

Recently, we have carried out a detail study of the Cauchy transform of self-similar measures in the spirit of [LSV]. We consider the following questions: (i) the asymptotic behavior of the Laurent coefficients of F; (ii) the growth rate and the chaotic behavior of F near the support of μ; (iii) the region of starlikeness of F. Since the proofs require certain fine estimations and are quite long, we will present the results separately. The present paper will concentrate on the Laurent coefficients $\{a_n\}_{n=0}^{\infty}$ of F; our first goal is to estimate the order of the growth of $|a_n|$, i.e., to find the maximum α such that $\{n^\alpha |a_n|\}_{n=1}^{\infty}$ is a bounded sequence; then we discuss the limit behavior of $n^\alpha |a_n|$. A related discussion of such limit behavior is Hayman's regularity theorem which asserts that the modulus of the nth coefficient of the *areally mean p-valent function*, $p > \frac{1}{4}$, is asymptotic to a constant multiple of n^{2p-1} (see [H1,H2,Du, pp. 150–161; pp. 163–166]).

The results in the topics (ii) and (iii) will appear elsewhere [DL1,DL2,Do].

We assume that the iterated function system (IFS) $\{S_j\}_{j=1}^m$ is of the form

$$S_j z = z_j + r_j(z - z_j), \tag{1.1}$$

where $0 < r_j < 1$, $|z_j| \leqslant 1$ with at least one $|z_j| = 1$, and the $\{S_j\}_{j=1}^m$ satisfies the open set condition (OSC), a basic condition to ensure separation in the iteration [F]. Let K be the attractor and let μ be the self-similar measure associated with a set of positive probability weights $\{p_j\}_{j=1}^m$:

$$\mu = \sum_{j=1}^{m} p_i \mu \circ S_j^{-1}. \tag{1.2}$$

Let $F(z)$ be the Cauchy transform of μ. Obviously, F is analytic in $\mathbb{C}\backslash K$, hence it has a Laurent expansion

$$F(z) = \sum_{n=1}^{\infty} a_n z^{-n}, \quad |z| > 1 \text{ with } a_{n+1} = \int_K w^n \, d\mu(w). \tag{1.3}$$

This coefficients have been studied by Strichartz et al. [LSV]. For $1 \leqslant j \leqslant m$, let

$$\alpha_j = \log p_j / \log r_j \quad \text{and} \quad \alpha^* := \min\{\alpha_j : 1 \leqslant j \leqslant m\}. \tag{1.4}$$

They gave a crude estimate:

Theorem A. *If $\{S_j\}_{j=1}^m$ satisfies the OSC and if $p_j < r_j$, $1 \leqslant j \leqslant m$, then for any $\beta < \alpha^*$, there exists C such that*

$$n^\beta |a_n| \leqslant C \quad \textit{for all } n > 0. \tag{1.5}$$

In this paper, one of our main efforts is to give the precise growth rate of the Laurent coefficients $\{a_n\}_{n=1}^\infty$. We let

$$\mathscr{J}_1 = \{j : |z_j| = 1\} \quad \text{and} \quad \alpha = \min\{\alpha_j : j \in \mathscr{J}_1\}. \tag{1.6}$$

Theorem 1.1. *Let $\{S_j\}_{j=1}^m$ satisfy the OSC and let μ be the self-similar measure defined by* (1.2). *Then the Laurent coefficients $\{a_n\}_{n=1}^\infty$ defined by* (1.3) *satisfies*

$$a_{n+1} = \sum_{j \in \mathscr{J}_1} n^{-\alpha_j} z_j^n \Phi_j(n) + O\left(\frac{\log^2 n}{n^{\alpha+1}}\right),$$

where the Φ_j are analytic multiplicative periodic functions on $\mathbb{R}^+$ with period r_j, i.e., $\Phi_j(t) = \Phi_j(r_j t)$ (hence Φ_j is bounded).

Theorem 1.2. (i) *If $\{S_j\}_{j=1}^m$ satisfies the OSC, then there exists $C > 0$ such that*

$$n^\alpha |a_n| \leqslant C, \quad n \geqslant 1.$$

(ii) *If $\{S_j\}_{j=1}^m$ satisfies the separated OSC, then*

$$\varlimsup_{n \to \infty} n^\alpha |a_n| > 0.$$

If we only assume the OSC in (ii), we have the following lower bound estimate:

Proposition 1.3. *If* $\{S_j\}_{j=1}^m$ *satisfies the OSC and if* $0<\alpha<\pi/\theta_0$, *then*

$$\overline{\lim_{n\to\infty}}\; n^{\alpha}|a_n| > 0, \tag{1.7}$$

where $\theta_0 = \inf_{j\in\mathscr{I}_1,\alpha_j=\alpha}(\sup\{\arg(w-z_j)-\arg(\xi-z_j): w,\xi\in K\backslash\{z_j\}\})$.

We remark that the supremum in the above expression is the angle subtended by K at the vertex z_j and is less than π (Lemma 2.3(i)). In particular, when $m=2$ we have $\theta_0=0$, hence (1.7) always holds for this case. The case for Sierpinski triangle $(m=3)$ also holds as $\pi/\theta_0 = 3 > \alpha = \log 3/\log 2 \approx 1.5849$.

As a remark of Theorem A and our theorems, we note that if $\{S_j\}_{j=1}^m$ satisfies the OSC, then

$$\sum_{j=1}^m r_j^s = 1 = \sum_{j=1}^m r_j^{\alpha_j} \leqslant \sum_{j=1}^m r_j^{\alpha^*},$$

where $s=\dim_{\mathscr{H}} K$ is the Hausdorff dimension of K. Hence $0<\alpha^*\leqslant s$. In addition the condition $p_j<r_j$ implies $\alpha^*>1$. It follows that

$$1<\alpha^*\leqslant s\leqslant 2$$

for the case in Theorem A. On the other hand, the α in Theorems 1.1 and 1.2 lies in $(0,+\infty)$ depends on the weights.

Our proof is different from [LSV], it is based on some accurate estimations of $a_{n+1}=\int_K w^n\,d\mu(w)$, making use of the self-similar properties of $\{S_j\}_{j=1}^m$ and μ as well as a special decomposition of the integral $\int_K w^n\,d\mu(w)$ (see (3.1), (3.2)). The extra condition on α in Proposition 1.3 is used to justify an auxiliary function $H_k(z)\not\equiv 0$ (see (4.2)), a seemingly trivial statement but technically difficult to prove. We believe that the statement $\overline{\lim}_{n\to\infty}\, n^{\alpha}|a_n|>0$ is true without such condition.

For the special case when $z_j=e^{2\pi ij/m}, r_j=r$ and $p_j=1/m$, the attractor K and the self-similar measure μ are m-fold symmetric, so is the Cauchy transform F. The Laurent series can be expressed as

$$F(z) = z^{-1} + \sum_{n=1}^{\infty} a_{nm+1} z^{-(nm+1)}, \quad |z|>1.$$

In this case, we find constants ρ_m (see (5.5)) such that for $0<r\leqslant\rho_m$, $\{S_j\}_{j=1}^m$ satisfies the OSC; the functions Φ_j in Theorem 1.1 are all equal (denoted by Φ_0) and can be expressed more explicitly by an infinite product (Corollary 3.5 or (5.7)). The growth rate of the $\{a_{nm+1}\}_{n=1}^{\infty}$ can be described more precisely as follows.

Theorem 1.4. *For* $0 < r \leqslant \rho_m$, $\dim_{\mathscr{H}} K = \alpha = \log m / |\log r|$, *there exists* $C > 0$ *such that the sequence of Laurent coefficients* $\{a_{nm+1}\}_{n=1}^{\infty}$ *satisfies*

$$C^{-1} \leqslant (nm+1)^{\alpha} a_{mn+1} \leqslant C, \quad \forall n \geqslant 0.$$

Furthermore, except for the case $m = 2, 4$ *with* $r = \rho_m = \frac{1}{2}$, *the sequence* $\{(nm+1)^{\alpha} a_{nm+1}\}_{n=0}^{\infty}$ *is dense in the non-degenerated line segment* $m\Phi_0([r, 1])$. *In this case*

$$\varlimsup_{n \to \infty} (nm+1)^{\alpha} a_{nm+1} = \max_{r \leqslant t \leqslant 1} m\Phi_0(t);$$
$$\varliminf_{n \to \infty} (nm+1)^{\alpha} a_{nm+1} = \min_{r \leqslant t \leqslant 1} m\Phi_0(t).$$

When $m = 2, 4$ with $r = \rho_m = \frac{1}{2}$, the sequence $\{(nm+1)^{\alpha} a_{nm+1}\}_{n=1}^{\infty}$ is very simple and the limits exist (see (5.15), (5.18)). This and Theorem 1.1 allow us to obtain a few identities of certain infinite products and series (Section 5; see also Section 3).

From the first part of Theorem 1.4, we can also conclude the behavior of the kth derivative $F^{(k)}$ near 1 (and hence for the vertices $z_j = e^{2\pi ij/m}$).

Corollary 1.5. *For* $0 < r \leqslant \rho_m$ (*except for* $m = 2, 3, 4$ *with* $r = \rho_m = \frac{1}{2}$), *if* $k + 1 - \alpha > 0$, *then*

$$0 < \varliminf_{t \to 1^+} (t-1)^{k+1-\alpha} |F^{(k)}(t)| < \varlimsup_{t \to 1^+} (t-1)^{k+1-\alpha} |F^{(k)}(t)| < \infty.$$

The corollary also holds for $m = 3$, $r = \rho_m = \frac{1}{2}$, but it needs a different proof which is given in [DL2]. While for the case $m = 2$, 4 and $r = \rho_m = \frac{1}{2}$, the $\varliminf$ and the $\varlimsup$ are actually equal (see Section 5).

For the organization of the paper, we give some preliminary results in Section 2. We introduce the multiperiodic functions Φ_j and prove some basic estimations in Section 3. Theorems 1.1, 1.2 and Proposition 1.3 are proved in Section 4. In Section 5 we consider the special cases of the self-similar m-gasket; Theorems 1.4 and Corollary 1.5 are proved there.

2. Preliminaries

The following proposition is probably known and we include it here for completeness.

Proposition 2.1. *Let* μ *be a positive, bounded regular Borel measure on* $\mathbb{C}$ *and has a compact support* K. *Then* $F(z) = \int_K (z-w)^{-1}\, d\mu(w)$ *is analytic on* $\mathbb{C} \backslash K$ *but is not analytic on* K.

Proof. The analyticity of F on $\mathbb{C}\backslash K$ is clear and the main concern is on the *non-analyticity* of F on K. We first note that for any $R>0$,

$$\int_K \int_{|z-z_0|\leqslant R} |w-z|^{-1}\, d\mathscr{L}^2(z)\, d\mu(w) < \infty,$$

hence the Fubini theorem implies that $F(z)$ exists $\mathscr{L}^2$-a.e. on K. Let $z_0 \in K$ and let $B_r(z_0)$ denote the ball with center at z_0 and radius r. We claim that

$$\mu(B_r(z_0)) = \frac{1}{2\pi i}\int_{|z-z_0|=r} F(z)\, dz, \quad r\in\mathbb{R}^+, \ \mathscr{L}\text{-a.e.} \tag{2.1}$$

Indeed we apply the Fubini theorem and the polar coordinate on the above double integral to conclude that

$$\int_{|z-z_0|=r}\int_K \frac{d\mu(w)}{|z-w|}\,|dz| < \infty, \quad r\in\mathbb{R}^+, \ \mathscr{L}\text{-a.e.}, \tag{2.2}$$

we also exclude the countably many r such that $\mu(\{|z-z_0|=r\})>0$ in (2.2). For the remaining r, we apply the Fubini theorem again to the right-hand side of (2.1) to obtain

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} F(z)\, dz = \int_K \left(\frac{1}{2\pi i}\int_{|z-z_0|=r}\frac{dz}{z-w}\right) d\mu(w).$$

The integral inside the parentheses equals 1 if $|w-z_0|<r$, and equals 0 if $|w-z_0|>r$ (by the Cauchy formula). This together with $\mu(\{|z-z_0|=r\})=0$ implies that the integral equals $\mu(B_r(z_0))$ and the claim follows.

Now if $F(z)$ were analytic for some $z_0\in K$, then there exists $\varepsilon>0$ such that F is analytic on $\{z: |z-z_0|<\varepsilon\}$. By using the Cauchy theorem,

$$\frac{1}{2\pi i}\int_{|z-z_0|=r} F(z)\, dz = 0 \quad \text{for all } 0<r<\varepsilon,$$

so that $\mu(B_r(z_0))=0$. This contradicts that $\mu(B_r(z_0))>0$ since $z_0\in K=\operatorname{supp}\mu$. □

Throughout the paper we assume that the IFS $\{S_j\}_{j=1}^m$ is of the form

$$S_j z = z_j + r_j(z-z_j), \quad j=1,2,\ldots,m, \tag{2.3}$$

where $0<r_j<1$ and $|z_j|\leqslant 1$ with equality holding for some j (we let $z_1=1$ for convenience). Also, we assume $\{S_j\}_{j=1}^m$ satisfies the OSC [Hu]: there exists an open set U such that

$$S_jU\subseteq U \quad \text{and} \quad S_iU\cap S_jU=\emptyset \ \text{for } i\neq j.$$

If in addition $S_i\bar{U}\cap S_j\bar{U}=\emptyset$ for $i\neq j$, we call it *separated OSC*; in this case K is totally disconnected. Let K be the attractor and let μ be the self-similar measure associated with a set of positive probability weights $\{p_j\}_{j=1}^m$. Let $\mathscr{J}=\{1,\dots,m\}$ denote the index sets for the S_j's; for the multi-indices, we write $J=(j_1,\dots,j_n)$, $S_J=S_{j_1}\circ\cdots\circ S_{j_n}$, $K_J=S_JK$ and $p_J=p_{j_1}\cdots p_{j_n}$. Under the OSC, the measure μ has the *measure separation property* [S]:

$$\mu(S_JK\cap S_{J'}K)=0\quad\text{for } J\neq J' \text{ and } |J|=|J'|. \tag{2.4}$$

Since we are only interested on the self-similar measures, we will assume that μ is such a measure if we make no specification. The above proposition is applied to yield a Laurent series expansion on $|z|>1$ as in (1.3). Our goal in this section is to set up some basic tools for the estimation of the coefficients of the series.

Proposition 2.2. *Suppose that $\{S_j\}_{j=1}^m$ satisfies the OSC and μ is a self-similar measure. Then for any $f\in L^1(\mu)$ and for any Borel subset E,*

$$\int_{S_JE} f(w)\,d\mu(w)=p_J\int_E f(S_Jw)\,d\mu(w).$$

Moreover, for any $j\in\mathscr{J}_1$ and $k\geqslant 0$,

$$\int_{S_j^kK} f(w)\,d\mu(w)=\sum_{\ell=k}^{\infty}p_j^{\ell}\int_{K\backslash S_jK} f(S_j^{\ell}(w))\,d\mu(w).$$

Proof. It suffices to prove the first identity for $f=\chi_A$ where A is any Borel subset in S_JE. Let $B=S_J^{-1}A$, then by applying the self-similar identity and the OSC, we have

$$\mu(A)=\mu(S_{j_1\cdots j_n}(B))=p_{j_1}\mu(S_{j_2\cdots j_n}(B))=\cdots=p_J\mu(B)$$

and the identity follows directly from this and $\chi_A(S_Jw)=\chi_B(w)$. For the second identity, it is observed that under the OSC,

$$S_j^kK\backslash\{z_j\}=\bigcup_{\ell=0}^{\infty}S_j^{\ell+k}(K\backslash S_jK).$$

The measure separation property (2.4) implies that the measure of the intersection of any two sets in the union is zero. We can apply the first identity in the proposition to conclude the second identity, noting that $\mu(\{z_j\})=0$ (as μ is a continuous measure). $\square$

For our later estimations, we will express $(S_j^k(z))^n$ into an exponential form as follows: for $j \in \mathscr{J}_1,\ k \in \mathbb{N}^+$ and for any $z \in K \backslash S_j K$,

$$\begin{aligned}(S_j^k(z))^n &= z_j^n(1 + r_j^k(\bar{z}_j z - 1))^n = z_j^n e^{n \log(1+r_j^k(\bar{z}_j z-1))} \\ &= z_j^n e^{-nr_j^k(1-\bar{z}_j z)+O(nr_j^{2k})},\end{aligned} \tag{2.5}$$

where $O(nr_j^{2k})$ is uniform with respect to $z \in K \backslash S_j K$. In the sequel we make some considerations on the two expressions $(1 - \bar{z}_j z)$ and nr_j^k in the above exponent.

For $|z_0| = 1$, a (symmetric) closed *Stolz angle* with vertex z_0 and angle $2\theta < \pi$ is a set of the form

$$A_{z_0}(\theta) = \{z : |z| \leqslant 1,\ |\arg(1 - \overline{z_0} z)| \leqslant \theta\} \cup \{z_0\},$$

i.e., a sector at vertex z_0 with an angle 2θ symmetric to $[0, z_0]$. Note that $\gamma = \arg(1 - \bar{z}_0 z) = \arg(z_0 - z) - \arg z_0$ is the angle $(<\pi/2)$ of z_0 and $z_0 - z$; $\mathrm{Re}(1 - \bar{z}_0 z) = |z_0 - z| \cos \gamma$ is the projection of $z_0 - z$ along z_0.

Lemma 2.3. *For $j \in \mathscr{J}_1$, we have*

(i) *K is contained in the Stolz angle $A_{z_j}(\theta_j)$ for some $0 \leqslant \theta_j < \pi/2$;*
(ii) $\eta_j := \inf\{\mathrm{Re}(1 - \bar{z}_j z) : z \in K \backslash S_j K\} > 0$;
(iii) *there exists $\lambda > 0$ such that $|z| < 1 - \lambda r_j^n$ for $z \in S_j K \backslash S_j^{n+1} K,\ n \geqslant 1$.*

Remark. We do not need the OSC in $\{S_j\}_{j=1}^m$ here.

Proof. By a rotation of $\bar{z}_j$, we can assume for simplicity that $z_j = z_1 = 1$. Let G be the convex hull of $\{z_1, \ldots, z_m\}$. Then $G \subseteq \{|z| \leqslant 1\}$ and $S_i G \subseteq G$ for each $1 \leqslant i \leqslant m$. Hence $K \subseteq \bigcap_{n=1}^{\infty} \bigcup_{|J|=n} S_J G$ [F]. The strict convexity of the boundary of the unit disc yields a $0 \leqslant \theta_1 < \pi/2$ such that G lies in the Stolz angle $A_1(\theta_1)$. This implies (i).

For (ii), we observe that for $2 \leqslant i \leqslant m,\ 1 \notin S_i G$, hence $1 \notin S_i K$. Therefore

$$|1 - z| \geqslant \mathrm{dist}\left(1, \bigcup_{i=2}^{m} S_i K\right) = \delta > 0, \quad \forall\ z \in K \backslash S_1 K. \tag{2.6}$$

This implies that $\mathrm{Re}(1 - z) \geqslant \delta \cos \theta_1 > 0$ for $z \in K \backslash S_1 K$.

To prove (iii), we apply the isometric contractive property of $S_1(z) = 1 + r_1(z - 1)$ to (2.6) inductively and obtain $|1 - z| \geqslant \delta r_1^n$ for $z \in S_1^n K \backslash S_1^{n+1} K$ so that

$$|1 - z| \geqslant \delta r_1^n, \quad \forall\ z \in K \backslash S_1^{n+1} K.$$

Next note that by (i), K is contained in the Stolz angle $A_1(\theta_1)$. It is clear that $S_1(A_1(\theta_1))$ is a "fan-shape" set within the $A_1(\theta_1)$ and its circular arc is bounded away from $\{z : |z| = 1\}$. Hence from elementary geometry, there exists $\lambda > 0$ such

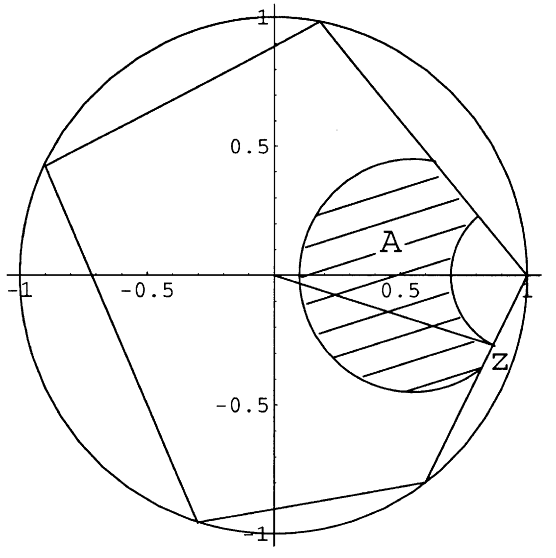

Fig. 1. $|1-z| = ar^n$, $|z|^2 = (1 - ar^n \cos\theta_1)^2 + (ar^n \sin\theta_1)^2$.

that $S_1(A_1(\theta_1))\backslash\{z : |z-1| \geqslant \delta r_1^n\}(:= A)$ is a subset of $\{|z| \leqslant 1 - \lambda r_1^n\}$ (see Fig. 1 for the special z with maximum length in the set A). Statement (iii) follows from this. $\square$

For the sequence $\{nr^k\}$ in (2.5), we proceed as follows. For fixed $0 < r < 1$, and for any $n \geqslant 1$, we can choose a unique sequence

$$N(n) \text{ such that } r \leqslant nr^{N(n)} < 1; \quad \text{let } x_n = nr^{N(n)}. \tag{2.7}$$

Note that $N(n) = 1 + [\log n / |\log r|]$.

Proposition 2.4. *For* $0 < r < 1$, *the sequence* $\{x_n\}_{n=1}^{\infty}$ *defined above is dense in* $[r, 1]$, *but it is not uniformly distributed.*

Proof. Observe that $\{N(n)\}_{n=1}^{\infty}$ is a monotonic increasing sequence with jumps at most 1 at each n. We define i_k to be the n such that the kth jump occurs, i.e., $i_k = n$ where $N(n) = k$. It follows that

$$N(n) = k \quad \text{for } i_k \leqslant n < i_{k+1}.$$

Hence $x_n = nr^k$ for $i_k \leqslant n < i_{k+1}$. This implies that $\{x_{i_k}, x_{i_k+1}, \dots, x_{i_{k+1}-1}\}$ has equal spacing r^k and lies in $[r, 1]$; also $0 \leqslant x_{i_k} - r \leqslant r^k$ and $0 \leqslant 1 - x_{i_{k+1}-1} \leqslant r^k$. Hence $\{x_n\}_{n=1}^{\infty}$ is dense in $[r, 1]$.

For the non-uniform distribution of $\{x_n\}_{n=0}^{\infty}$, it suffices to show that for a function $f(x)\not\equiv 0$ continuous on $[r,1]$, $\int_r^1 f(x)\,dx = 0$ but

$$\lim_{n\to\infty}\frac{1}{n}\sum_{\ell=1}^{n} f(x_\ell) \quad \text{does not exist},$$

[KN, p. 2]. Indeed let $\{i_k\}_{k=0}^{\infty}$ be defined as above, then by using the partition $\{r, x_{i_k}, \dots, x_{i_{k+1}-1}, 1\}$ of $[r,1]$, it is easy to show by the definition of Riemann integral that for $i_k \leqslant n \leqslant i_{k+1}-1$,

$$r^k \sum_{\ell=i_k}^{n} f(x_\ell) - \int_r^{x_n} f(x)\,dx = o(1) \quad \text{as } k\to+\infty,$$

in particular, $r^k \sum_{\ell=i_k}^{i_{k+1}-1} f(x_\ell) = o(1)$ as $k\to\infty$. Consequently, for $i_k\leqslant n\leqslant i_{k+1}-1$ and for $k\to\infty$,

$$\begin{aligned}
\frac{1}{n}\sum_{\ell=1}^{n} f(x_\ell) &= \frac{r^k}{x_n}\left(\sum_{\ell=1}^{i_{k_0}-1} + \sum_{q=k_0}^{k}\ \sum_{\ell=i_q}^{i_{q+1}-1} + \sum_{\ell=i_k}^{n}\right) f(x_\ell) \\
&= O(r^k) + o\left(\sum_{q=k_0}^{k} r^{k-q}\right) + \frac{r^k}{x_n}\sum_{\ell=i_k}^{n} f(x_\ell) \\
&= \frac{1}{x_n}\int_r^{x_n} f(x)\,dx + o(1).
\end{aligned}$$

Noting that $\{x_n\}_{n=1}^{\infty}$ is dense in $[r,1]$, it is clear that the last term does not have limit. $\square$

3. Estimations of the coefficients

Let $\{p_j\}_{j=1}^{m}$ be a set positive probability weights associated with $\{S_j\}_{j=1}^{m}$. Let α be defined by (1.6). We are aiming to show that $a_{n+1} = O(n^{-\alpha})$. Let $\lambda > 0$ be defined by Lemma 2.3(iii), i.e., $|z| \leqslant 1-\lambda r_j^k$ for all $z\in S_jK\backslash S_j^k K$ and $j\in\mathscr{J}_1$. For each $j\in\mathscr{J}_1$, $n\geqslant 2$, we define a positive integer $j_n(:=\ell_j(n))$ by

$$\gamma\lambda^{-1}\log n \leqslant n r_j^{j_n} < \gamma\lambda^{-1}(\log n)/r_j, \tag{3.1}$$

where the positive constant γ is to be chosen later (see (4.1)); write

$$\begin{aligned}
a_{n+1} &= \int_K w^n\,d\mu(w) = \left(\sum_{j\in\mathscr{J}\backslash\mathscr{J}_1}\int_{S_jK} + \sum_{j\in\mathscr{J}_1}\int_{S_jK\backslash S_j^{j_n}K} + \sum_{j\in\mathscr{J}_1}\int_{S_j^{j_n}K}\right) w^n\,d\mu(w) \\
&:= A(n) + B(n) + C(n)
\end{aligned} \tag{3.2}$$

(note that the measure separation property (2.4) is applied to the decomposition of K into the sums).

Lemma 3.1. *With the above notations and the choice of j_n for a given $\gamma > 0$, we have for some* $0<\rho<1$,

$$|A(n)| \leqslant \rho^n, \qquad |B(n)| \leqslant n^{-\gamma}.$$

Proof. Note that for $j \in \mathscr{J} \backslash \mathscr{J}_1$, $|r_j|<1$. Hence there exists $0<\rho<1$ such that $S_j K \subset \{|w| \leqslant \rho\}$. This yields $|A(n)| \leqslant \int_K |w|^n \, d\mu(w) \leqslant \rho^n$. The estimation of $|B(n)|$ follows from (3.1):

$$|B(n)| \leqslant \sum_{j \in \mathscr{J}_1} \int_{S_j K \backslash S_j^{j_n} K} |w|^n \, d\mu(w) \leqslant (1 - \lambda r_j^{j_n})^n \leqslant e^{-\lambda n r_j^{j_n}} \leqslant n^{-\gamma}. \qquad \square$$

The main difficulty is to handle $C(n)$. For this we first establish an elementary estimation: for $b > 1, \alpha > 0$ and for sufficiently large a, we have

$$\sum_{\ell=1}^{\infty} b^{\alpha \ell} e^{-ab^{\ell}} \leqslant \frac{2}{a^{\alpha} \log b} e^{-\frac{1}{2} a}. \tag{3.3}$$

In fact, note that $\varphi(x) = b^{\alpha x} e^{-ab^x}$ is a decreasing function on x for large a, the left-hand side of (3.3) is less than

$$\int_0^{\infty} b^{ax} e^{-ab^x} \, dx = \frac{1}{a^{\alpha} \log b} \int_a^{\infty} y^{\alpha - 1} e^{-y} \, dy.$$

By using $y^{\alpha-1} e^{-y} \leqslant e^{-y/2}$, we get (3.3). In the following proof, we see that the choice of j_n is to make the a large enough to apply (3.3).

For simplicity, we slightly abuse the notation by writing $\sum_\ell c_\ell + O(\epsilon_n) \sum_\ell |c_\ell|$ as $(1 + O(\epsilon_n)) \sum_\ell c_\ell$ in Lemma 3.2 and Theorem 4.1.

Lemma 3.2. *For $j \in \mathscr{J}_1$, let j_n be defined as in* (3.1). *Then for sufficiently large n,*

$$\int_{S_j^{j_n} K} w^n \, d\mu(w) = \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) z_j^n \sum_{\ell \in \mathbb{Z}} r_j^{\alpha_j (j_n + \ell)} \int_{K \backslash S_j K} e^{-n r_j^{j_n + \ell} (1 - \bar{z}_j z)} \, d\mu(z)$$
$$+ \; O(n^{-(\alpha_j + \frac{1}{2} \eta \gamma \lambda^{-1})}),$$

where $\eta = \min\{\eta_j : j \in \mathscr{J}_1\}$, $\eta_j > 0$ is given by Lemma 2.3(ii), *λ by Lemma* 2.3(iii) *and γ by* (3.1).

Proof. We will make use of the expression of $(S_j^k(z))^n$ in (2.5). From the choice of j_n we have $e^{O(nr_j^{2k})} = (1 + O(\log^2 n/n))$ for $k \geqslant j_n$, hence

$$(S_j^k(z))^n = z_j^n e^{-nr_j^k(1-\bar{z}_j z)}(1 + O(\log^2 n/n)) \quad \text{for } k \geqslant j_n.$$

By Proposition 2.2 and noting the uniformity of $O(\log^2 n/n)$ with respect to $z \in K \backslash S_j K$ and $k \geqslant j_n$, we have

$$\begin{aligned}\int_{S_j^{j_n} K} w^n \, d\mu(w) &= \sum_{\ell=0}^{\infty} p_j^{j_n+\ell} \int_{K \backslash S_j K} (S_j^{j_n+\ell} w)^n \, d\mu(w) \\ &= \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) z_j^n \sum_{\ell=0}^{\infty} r_j^{\alpha_j(j_n+\ell)} \int_{K \backslash S_j K} e^{-nr_j^{j_n+\ell}(1-\bar{z}_j z)} \, d\mu(z). \quad (3.4)\end{aligned}$$

To put it in the sum for $\ell \in \mathbb{Z}$ as in the statement of the lemma, we need to estimate

$$R_n := z_j^n \sum_{\ell \leqslant -1} r_j^{\alpha_j(j_n+\ell)} \int_{K \backslash S_j K} e^{-nr_j^{j_n+\ell}(1-\bar{z}_j z)} \, d\mu(z).$$

By making use of the choice of j_n in (3.1),

$$|R_n| \leqslant \sum_{\ell=1}^{\infty} r_j^{\alpha_j(j_n-\ell)} e^{-\eta n r_j^{j_n-\ell}} \leqslant C_1 \left(\frac{\log n}{n}\right)^{\alpha_j} \sum_{\ell=1}^{\infty} r_j^{-\alpha_j \ell} e^{-a r_j^{-\ell}},$$

where $C_1 = (\gamma \lambda^{-1} r_j^{-1})^{\alpha_j}$ and $a = \eta \gamma \lambda^{-1} \log n$, $\eta = \min\{\eta_j : j \in \mathscr{J}_1\}$, $\eta_j > 0$ is defined in Lemma 2.3(ii). By (3.3)

$$|R_n| \leqslant C_2 n^{-(\alpha_j + \frac{1}{2}\eta\gamma\lambda^{-1})}$$

and the lemma follows. □

In view of the expression in the above lemma, we introduce two auxiliary functions: for $j \in \mathscr{J}_1$, let

$$\Phi_j(z) = \sum_{n \in \mathbb{Z}} (z r_j^n)^{\alpha_j} \int_{K \backslash S_j K} e^{-z r_j^n (1-\bar{z}_j w)} \, d\mu(w), \quad (3.5)$$

$$\Psi_j(t) = \sum_{n \in \mathbb{Z}} (t r_j^n)^{\alpha_j} \int_{K \backslash S_j K} e^{-t r_j^n \mathrm{Re}(1-\bar{z}_j w)} \, d\mu(w). \quad (3.6)$$

Lemma 3.3. *For $j \in \mathscr{J}_1$, let θ_j be given by Lemma* 2.3(i), *then*

(i) *$\Psi_j(t)$ is bounded above and bounded away from* 0 *on* $\mathbb{R}^+$;
(ii) *$\Phi_j(z)$ is analytic in $|\arg z| < \pi/2 - \theta_j$;*
(iii) *$\Phi_j(r_j z) = \Phi_j(z)$ for $|\arg z| < \pi/2 - \theta_j$ and $\Psi_j(r_j t) = \Psi_j(t)$ for all $t \in \mathbb{R}^+$.*

Proof. Write $r = r_1$ for simplicity. In view of Lemma 2.3(ii), it suffices to prove (i) by showing that for $0<r<1, \alpha > 0, \eta > 0$, there exists $C_1, C_2 > 0$ such that

$$C_1 \leqslant \sum_{n\in\mathbb{Z}} (tr^n)^\alpha e^{-\eta tr^n} \leqslant C_2, \quad \forall t \in (0, \infty). \tag{3.7}$$

Indeed for each $t > 0$ we can choose an integer N (depends on t) such that $r \leqslant tr^N < 1$. Let $k = [\alpha] + 2$. Then

$$\sum_{n=-\infty}^{N} (tr^n)^\alpha e^{-\eta tr^n} \leqslant \sum_{n=-\infty}^{N} (tr^n)^\alpha \leqslant \frac{k!}{(\eta tr^n)^k} \leqslant C_1 \sum_{n=0}^{\infty} (tr^{-n+N})^{\alpha-k} \leqslant C_1 \sum_{n=0}^{\infty} r^n$$

(the first inequality follows from $e^y \geqslant y^k/k!, y > 0$). Also $\sum_{n=N+1}^{\infty} (tr^n)^\alpha e^{-\eta tr^n} \leqslant \sum_{n=0}^{\infty} r^n$. Hence the upper bound in (3.7) follows.

To prove the lower bound, we observe that the sum in (3.7) $\geqslant (tr^N)^\alpha e^{-\eta tr^N} \geqslant r^\alpha e^{-\eta}$.

For (ii), we let $\delta > 0$ be sufficiently small and let $D^*_\delta = \{z : |\arg z| \leqslant \pi/2 - (\theta_j + \delta), \delta \leqslant |z| \leqslant \delta^{-1}\}$. It follows from Lemma 2.3(i), (ii) that

$$|\arg(z(1 - \bar{z}_j w))| \leqslant \pi/2 - \delta \quad \text{and} \quad |1 - \bar{z}_j w| \geqslant \eta_j$$

for $z \in D^*_\delta$ and $w \in K \backslash S_j K$. Hence

$$\mathrm{Re}(z(1 - \bar{z}_j w)) \geqslant \delta\eta_j \cos\left(\frac{\pi}{2} - \delta\right) := \delta\eta.$$

Similar to the above estimate, we have for $z \in D^*_\delta$,

$$\begin{aligned} &\sum_{|\ell| \geqslant n} |zr^\ell|^\alpha \int_{K\backslash S_j K} e^{-\mathrm{Re}\{zr^\ell(1-\bar{z}_j w)\}}\, d\mu(w) \\ &\quad = \sum_{\ell=n}^{+\infty} + \sum_{\ell=-\infty}^{-n} (\cdots) \leqslant C(\delta)(r^{\alpha n} + r^{(k-\alpha)n}) \to 0 \quad \text{as } n \to +\infty. \end{aligned}$$

Hence as a function term series $\Phi_j(z)$ converges uniformly on D^*_δ, so $\Phi_j(z)$ is analytic in $|\arg z| < \pi/2 - \theta_j$.

Property (iii) follows from a direct check of the definition. □

In the following, we show that for some special cases, $\Phi_j(z)$ can be expressed as an infinite product. We will use it at the end of the paper.

Proposition 3.4. *Suppose in addition we assume that $\{S_j\}_{n=1}^m$ satisfies $r_j = r$ for each j. Then for $j \in \mathscr{J}_1$,*

$$\Phi_j(z) = z^{\alpha_j} \lim_{n\to\infty}\left(p_j^{n+1}\prod_{k=-n}^{n} Q_j^{(k)}(z)\right),$$

where $Q_j^{(k)}(z) = p_j^{-1}\sum_{i=1}^m p_i e^{-z(1-r)r^k(1-\bar{z}_j z_i)}$.

Proof. By $r_j^{\alpha_j} = p_j$, we have

$$\Phi_j(z) = z^{\alpha_j}\sum_{n\in\mathbb{Z}} p_j^n \int_{K\setminus S_jK} e^{-zr^n(1-\bar{z}_j w)}\, d\mu(w).$$

Let $I_n(t)$ be the expression inside the sum. Then by the measure separation property of μ in (2.4),

$$I_n(z) = p_j^n e^{-zr^n}\int_{K\setminus S_jK} e^{zr^n\bar{z}_j w}\, d\mu(w) = p_j^n e^{-zr^n}\sum_{i\neq j}\int_{S_iK} e^{zr^n\bar{z}_j w}\, d\mu(w).$$

By Proposition 2.2,

$$\int_{S_iK} e^{zr^n\bar{z}_j w}\, d\mu(w) = p_i\int_K e^{zr^n\bar{z}_j S_i w}\, d\mu(w) = p_j^{-1}p_i\int_{S_jK} e^{zr^n\bar{z}_j S_i\circ S_j^{-1}w}\, d\mu(w),$$

hence (note that $S_i(S_j^{-1}w) = w + S_i(z_j) - z_j$)

$$I_n(z) = p_j^n e^{-zr^n}\sum_{i\neq j} p_j^{-1}p_i e^{-zr^n(1-\bar{z}_j S_i(z_j))}\int_{S_jK} e^{zr^n\bar{z}_j w}\, d\mu(w).$$

By Proposition 2.2 again, we can write

$$\int_{S_jK} e^{zr^n\bar{z}_j w}\, d\mu(w) = \sum_{\ell=1}^{\infty} p_j^{\ell} e^{zr^n}\int_{K\setminus S_jK} e^{-zr^{n+\ell}(1-\bar{z}_j w)}\, d\mu(w) = p_j^{-n}e^{zr^n}\sum_{\ell=1}^{\infty} I_{n+\ell}(z).$$

Let $h_n(z) = \sum_{\ell=n}^{\infty} I_\ell(z)$. Observe that $e^{-zr^n(1-\bar{z}_j S_j(z_j))} = 1$; by combining the above identities, we have

$$\begin{aligned} I_n(z) &= \left(p_j^{-1}\sum_{i\neq j} p_i e^{-zr^n(1-\bar{z}_j S_i(z_j))}\right)\left(\sum_{\ell=n+1}^{\infty} I_\ell(z)\right) \\ &= \left(p_j^{-1}\sum_{i=1}^{m} p_i e^{-zr^n(1-\bar{z}_j S_i(z_j))}\right) h_{n+1}(z) - h_{n+1}(z). \end{aligned}$$

Let $Q_j^{(n)}(z)$ denote the term inside the parantheses. By moving $h_{n+1}(z)$ to the left, the identity reduces to $h_n(z) = Q_j^{(n)}(z)h_{n+1}(z)$. Inductively, we have

$$h_{-n}(z) = \left(\prod_{k=-n}^{n} Q_j^{(k)}(z)\right) h_{n+1}(z). \tag{3.8}$$

For $h_{n+1}(z)$, we have the following estimate:

$$\begin{aligned} h_{n+1}(z) &= \sum_{k=n+1}^{\infty} I_k(z) = \sum_{k=n+1}^{\infty} p_j^k \int_{K\backslash S_jK} e^{-zr^k(1-\bar{z}_j w)}\, d\mu(w) \\ &= p_j^{n+1} \sum_{k=0}^{\infty} p_j^k \int_{K\backslash S_jK} e^{-zr^{k+n+1}(1-\bar{z}_j w)}\, d\mu(w). \end{aligned}$$

We note that the last integral $\int_{K\backslash S_jK} e^{-zr^{k+n+1}(1-\bar{z}_j w)}\, d\mu(w)$ converges uniformly to $\mu(K\backslash S_jK) = 1 - p_j$ as $n \to +\infty$ for $|z| < \delta^{-1}$ where $\delta > 0$ is sufficiently small, hence

$$p_j^{-n-1} h_{n+1}(z) \to 1 \quad \text{uniformly on } |z| < \delta^{-1} \text{ as } n \to +\infty. \tag{3.9}$$

The proposition follows from $\Phi_j(z) = z^{\alpha_j} \lim_{n\to\infty} h_{-n}(z)$ and (3.8), (3.9). □

Corollary 3.5. *With the hypotheses of Proposition* 3.4, *then* $\prod_{k=1}^{\infty} Q_j^{(-k)}(z)$ *and* $\prod_{k=0}^{\infty} (p_j Q_j^{(k)}(z))$ *converge uniformly on each compact subset of* $|\arg z| < \pi/2 - \theta_j$. *Hence*

$$\Phi_j(z) = z^{\alpha_j} \prod_{k=1}^{\infty} Q_j^{(-k)}(z) \prod_{k=0}^{\infty} (p_j Q_j^{(k)}(z)), \quad |\arg z| < \pi/2 - \theta_j.$$

Proof. Let $\epsilon > 0$ be sufficiently small. For $\epsilon \leqslant |z| \leqslant \epsilon^{-1}$ and $|\arg z| < \pi/2 - \theta_j - \epsilon$, noting that $|\arg(z(1-\bar{z}_j z_i))| \leqslant |\arg z| + |\arg(1-\bar{z}_j z_i)| \leqslant \pi/2 - \epsilon$ (Lemma 2.3(i)), we have

$$\begin{aligned} |Q_j^{(-n)}(z) - 1| &= \left| p_j^{-1} \sum_{i\neq j} p_i e^{-zr^{-n}(1-r)(1-z_i\bar{z}_j)} \right| \\ &\leqslant p_j^{-1} \sum_{i\neq j} p_i e^{-r^{-n}(1-r)|z(1-\bar{z}_j z_i)|\cos(\pi/2-\epsilon)} \\ &\leqslant \frac{1-p_j}{p_j} e^{-b(\epsilon)(1-r)r^{-n}} < Cr^n, \end{aligned}$$

where $b(\epsilon) > 0$ is defined in an obvious way. This implies that

$$1 - Cr^n \leqslant |Q_j^{(-n)}(z)| \leqslant 1 + Cr^n \quad \text{for } n > 0. \tag{3.10}$$

On the other hand, $Q_j^{(n)}(z) = p_j^{-1}(1 + O(r^n))$ uniformly for $\epsilon \leqslant |z| \leqslant \epsilon^{-1}$ as $n \to +\infty$. These show that

$$\prod_{n=1}^{\infty} |Q_j^{(-n)}(z)| \quad \text{and} \quad \prod_{n=0}^{\infty} (p_j |Q_j^{(n)}(z)|) \text{ converge uniformly} \tag{3.11}$$

on each compact subset of $|\arg z| < \pi/2 - \theta_j$. The corollary follows by Proposition 3.4 and (3.11). □

A similar argument as Proposition 3.4 and Corollary 3.5, we have

Proposition 3.6. *With the hypotheses of Proposition 3.4 and let $\Psi_j(t)$ be defined by* (3.6), *then*

$$\Psi_j(t) = t^{\alpha_j} \prod_{k=1}^{\infty} q_j^{(-k)}(t) \prod_{k=0}^{\infty} \Big(p_j q_j^{(k)}(t) \Big), \quad t \in \mathbb{R}^+$$

where $q_j^{(k)}(t) = p_j^{-1} \sum_{i=1}^{m} p_i e^{-tr^k(1-r)(1-\mathrm{Re}(\bar{z}_j z_i))}$.

It is easy to see that if $z_j = e^{2j\pi i/m}$ and $p_j = \frac{1}{m}$ for $j = 1, 2, \ldots, m$ in Proposition 3.4, then all the Φ_j are equal. Write $\alpha = \log m / |\log r|$.

Example 1. Let $m = 2,\ z_j = e^{2j\pi i/2},\ p_j = \frac{1}{2},\ j = 1, 2$ and the contraction ratio $r = \frac{1}{2}$. Then the attractor K is $[-1, 1]$, $\mu = \frac{1}{2}\mathscr{L}$ (Lebesgue measure), $\alpha = 1$ and

$$\Phi_j(z) = z \prod_{k=1}^{\infty} (1 + e^{-2^k z}) \prod_{k=0}^{\infty} \frac{1 + e^{-2^{-k} z}}{2} \equiv \frac{1}{2} \quad \text{for } |\arg z| < \frac{\pi}{2}.$$

(The last identity follows from $\prod_{k=1}^{\infty}(1 + e^{-2^{-k}z}) = (1 - e^{-2z})^{-1}$ and $\prod_{k=0}^{\infty}(1 + e^{-2^{-k}z})/2 = e^{-z} \prod_{k=1}^{\infty} \cosh(2^{-k}z) = e^{-z} \sinh z / z$.)

If $r = \frac{1}{3}$ instead. Then the μ is the Cantor measure, $\alpha = \log 2 / \log 3$ and

$$\Phi_j(z) = z^{\alpha} \prod_{k=1}^{\infty} (1 + e^{-4.3^k z}) \prod_{k=0}^{\infty} (1 + e^{-4.3^{-k} z})/2 \quad \text{for } |\arg z| < \frac{\pi}{2}.$$

Example 2. Let $m = 3,\ z_j = e^{2j\pi i/3},\ p_j = \frac{1}{3},\ j = 1, 2, 3,\ r = \frac{1}{2}$. Then the attractor K is the Sierpinski gasket, μ is the Hausdorff measure $\mathscr{H}^{\alpha}$ restricted and normalized on K with $\alpha = \log 3 / \log 2$ and

$$Q_j^{(k)}(z) = 1 + e^{-z\left(\frac{1}{2}\right)^{k+1}(1 - e^{2\pi i/3})} + e^{-z\left(\frac{1}{2}\right)^{k+1}(1 - e^{-2\pi i/3})},$$

hence

$$\Phi_j(z) = z^\alpha \prod_{k=1}^{\infty} (1 + 2e^{-3\cdot 2^{k-2}z}\cos(\sqrt{3}\cdot 2^{k-2}z)) \prod_{k=0}^{\infty} \frac{1 + 2e^{-3\cdot 2^{-k-2}z}\cos(\sqrt{3}\cdot 2^{-k-2}z)}{3}$$

for $|\arg z| < \pi/3$.

Example 3. Let $m = 4,\ z_j = e^{2j\pi i/4},\ p_j = \frac{1}{4},\ j = 1,2,3,4,\ r = \frac{1}{2}$. Then the attractor K is the square with vertices $z_j, \mu = \frac{1}{2}\mathscr{L}^2,\ \alpha = 2$ and

$$\begin{aligned} Q_j^{(k)}(z) &= 1 + e^{-z\left(\frac{1}{2}\right)^k} + e^{-z\left(\frac{1}{2}\right)^{k+1}(1-i)} + e^{-z\left(\frac{1}{2}\right)^{k+1}(1+i)} \\ &= (1 + e^{-z\left(\frac{1}{2}\right)^{k+1}(1+i)})(1 + e^{-z\left(\frac{1}{2}\right)^{k+1}(1-i)}), \end{aligned}$$

hence

$$\begin{aligned} \Phi_j(z) &= z^2 \prod_{k=1}^{\infty} (1 + e^{-2^k z} + 2e^{-2^{k-1}z}\cos(2^{k-1}z)) \prod_{k=0}^{\infty} \frac{1 + e^{-2^{-k}z} + 2e^{-2^{-k-1}z}\cos(2^{-k-1}z)}{4} \\ &= 2P\left(\frac{1+i}{2}z\right)P\left(\frac{1-i}{2}z\right) \equiv \frac{1}{2}, \quad |\arg z| < \pi/4 \end{aligned}$$

where $P(z)$ is the product $z\prod_{k=1}^{\infty}(1 + e^{-2^k z})\prod_{k=0}^{\infty}(1 + e^{-2^{-k}z})/2$ in Example 1.

4. The theorems

In this section we will prove the theorems of the Laurent coefficients.

Theorem 4.1. *Let* $\{S_j\}_{j=1}^m$ *be the IFS as in* (2.3) *and satisfy the OSC*; *let* $\alpha_j = \log p_j / \log r_j$ *and* $\alpha = \min\{\alpha_j : j \in \mathscr{J}_1\}$. *Then*

$$a_{n+1} = \sum_{j \in \mathscr{J}_1} n^{-\alpha_j} z_j^n \Phi_j(n) + O\left(\frac{\log^2 n}{n^{\alpha+1}}\right),$$

where $\Phi_j(t)$ *satisfies* $\Phi_j(r_j t) = \Phi_j(t)$ (*as in* (3.5)).

Proof. We use the expression of a_{n+1} in (3.2). It is clear from Lemma 3.1 that $|A(n)|$ can be absorbed in $O(n^{-\alpha-1})$; the same for $|B(n)|$ if we choose

$$\gamma > \alpha + 1 + 2\lambda\eta^{-1} \tag{4.1}$$

in (3.1). It remains to estimate the term $\int_{S_j^{j_n}K} w^n \, d\mu(w)$ in C_n: for $j \in \mathscr{J}_1$, by Lemma 3.2 and (4.1) we have

$$\begin{aligned}
&\int_{S_j^{j_n}K} w^n \, d\mu(w) \\
&= \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) z_j^n \sum_{\ell \in \mathbb{Z}} r_j^{\alpha_j(j_n+\ell)} \int_{K \setminus S_j K} e^{-nr_j^{j_n+\ell}(1-\bar{z}_j z)} \, d\mu(z) + O(n^{-\alpha-1}) \\
&= \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) z_j^n \sum_{\ell' \in \mathbb{Z}} r_j^{\alpha_j(N_j+\ell')} \int_{K \setminus S_j K} e^{-nr_j^{N_j+\ell'}(1-\bar{z}_j z)} \, d\mu(z) + O(n^{-\alpha-1}) \\
&\quad \text{(where } \ell' = \ell + j_n - N_j \text{ with } N_j := N_j(n) \text{ is defined in (2.7))} \\
&= \left(1 + O\left(\frac{\log^2 n}{n}\right)\right) z_j^n r_j^{\alpha_j N_j} \sum_{\ell' \in \mathbb{Z}} r_j^{\alpha_j \ell'} \int_{K \setminus S_j K} e^{-x_j(n) r_j^{\ell'}(1-\bar{z}_j z)} \, d\mu(z) + O(n^{-\alpha-1}) \\
&= n^{-\alpha_j} z_j^n \Phi_j(x_j(n)) + O\left(\frac{\log^2 n}{n^{\alpha+1}}\right)
\end{aligned}$$

the last identity is by $|\Phi_j(x)| \leqslant \Psi_j(x)$ and Lemma 3.3. This gives the estimate of $C(n)$ and the theorem follows (note that $\Phi_j(x_j(n)) = \Phi_j(n)$). □

By some obvious modifications of the above proof, we have

Proposition 4.2. *With the same notations as in Theorem* 4.1, *then*

$$\int_K |w|^n \, d\mu(w) = \sum_{j \in \mathscr{J}_1} n^{-\alpha_j} \Psi_j(n) + O\left(\frac{\log^2 n}{n^{\alpha+1}}\right),$$

where $\Psi_j(t)$ *satisfies* $\Psi_j(r_j t) = \Psi_j(t)$ *and* $0 < c_1 \leqslant \Psi_j(t) \leqslant c_2 < \infty$ (*as in* (3.6) *and Lemma* 3.3).

To show that α in Theorem 4.1 is the best possible, we need the auxiliary function $H_k(z)$ defined by

$$H_k(z) = \sum_{n \in \mathbb{Z}} r^{\alpha n} \int_{K \setminus K_1} \frac{d\mu(w)}{(z - r^n(w-1))^{k+1}}, \tag{4.2}$$

where $S_1 K = K_1$ and $k \geqslant [\alpha]$ (the largest integer less that α). Let $H_k^+(z)(H_k^-(z))$ be the sum of $n \geqslant 0$ ($n \leqslant -1$, respectively) of the above series. By Proposition 2.2 (with $j = 1$ and $k = 0$ there), we have

$$H_k^+(z) = \int_K \frac{d\mu(w)}{(z+1-w)^{k+1}} = \frac{(-1)^k}{k!} F^{(k)}(z+1). \tag{4.3}$$

Let $A_1(\tau_1\pi/2)$ be the Stolz angle as in Lemma 2.3(i) where $0<\tau_1<1$. Note that

$$(1+\tau_1)\frac{\pi}{2}\leqslant \arg(w-1)\leqslant(3-\tau_1)\frac{\pi}{2}\quad \text{for } w\in K\backslash\{1\}.$$

Hence $F^{(k)}(z+1)$ is analytic in $D=\{z:|\arg z|<(1+\tau_1)\pi/2\}$ by Lemma 2.3(i).

Lemma 4.3. (i) *$H_k^-(z)$ is analytic on $D\cup\Delta$ where $\Delta=\{z:|z|<r^{-1}d\}$ with* $d=\operatorname{dist}(1,K\backslash K_1)$; (ii) *$H_k(z)$ is analytic on D.*

Proof. Obviously $|w-1|\geqslant d>0$ for $w\in K\backslash K_1$ and

$$H_k^-(z)=\sum_{n=1}^{\infty} r^{(k+1-\alpha)n}\int_{K\backslash K_1}\frac{d\mu(w)}{(r^n z-(w-1))^{k+1}}. \tag{4.4}$$

For small $\delta>0$, let $D_\delta=\{z:|\arg z|<(1+\tau_1-\delta)\pi/2\}$, then a direct check shows that

$$|r^n z-(w-1)|\geqslant d\sin\left(\frac{\delta\pi}{2}\right)$$

for $z\in D_\delta$ and $w\in K\backslash K_1$. This shows that the series in (4.4) is uniformly convergent on D_δ and hence $H_k^-(z)$ is analytic in D. Similarly, we can prove that $H_k^-(z)$ is analytic in $|z|<r^{-1}d$ also and (i) follows.

By (4.3), $H_k^+(z)$ is analytic on D, hence (i) implies (ii). $\square$

Lemma 4.4. *If $\{S_j\}_{j=1}^m$ satisfies the separated OSC, then for any $\xi_0\in K$ and $\epsilon>0$, there exists a simply connected domain Ω with piecewise smooth boundary $\partial\Omega$ such that*

$$\xi_0\in\Omega\subset\{|w-\xi_0|<\epsilon\}\quad\text{and}\quad \partial\Omega\cap K=\emptyset.$$

Proof. Let U be an open set in the separated OSC, then $S_i\bar U\cap S_j\bar U=\emptyset$ for $i\neq j$ and $K\subset\bar U$ [F, p. 115]. It follows that

$$\min_{i<j}\operatorname{dist}(K_i,K_j)>0. \tag{4.5}$$

We choose $S_{J_0}K$ with $|J_0|=\ell<\infty$ such that $\xi_0\in S_{J_0}K\subset\{|w-\xi_0|<\frac{1}{2}\epsilon\}$. By (4.5)

$$\delta_1:=\min_{J\neq J_0;\ |J|=\ell}\operatorname{dist}(S_J K,S_{J_0}K)>0.$$

Let $\delta=\min\{\delta_1,\epsilon\}$. By compactness, we can find open balls O_j $(j=1,2,\dots,p)$ of radius $\frac{1}{4}\delta$ such that $S_{J_0}K\subset\bigcup_{j=1}^p O_j$. Assume that C is a connected component of the union that contains ξ_0, then

$$\xi_0\in C\subset\{|w-\xi_0|<\tfrac{3}{4}\epsilon\}\quad\text{and}\quad K\cap\partial C=\emptyset. \tag{4.6}$$

If C is a multiply connected domain, then the union of C with its holes is also contained in $\{|w - \xi_0| < \frac{3}{4}\epsilon\}$. Hence we may assume that C in (4.6) is a simply connected domain and let it be Ω. By (4.6) the lemma follows. □

Theorem 4.5. *If* $\{S_j\}_{j=1}^m$ *satisfies the OSC, then there exists* $C > 0$ *such that*

$$n^\alpha |a_n| \leqslant C \quad \text{for all } n \geqslant 1. \tag{4.7}$$

Furthermore if $\{S_j\}_{j=1}^m$ *satisfies the separated OSC (see Section 2), then*

$$\overline{\lim_{n\to\infty}}\ n^\alpha |a_n| > 0. \tag{4.8}$$

Proof. Since $|\Phi_j(t)| \leqslant \Psi_j(t)$ and $\Psi_j(t)$ is a bounded function (Lemma 3.3(i)), Theorem 4.1 implies that $n^\alpha |a_n| \leqslant C$ for all $n > 0$. The main proof is for the second assertion.

Let $z_1 = 1$ as before, and write $\alpha = \alpha_1 = \log p_1 / \log r_1$ and $r = r_1$ for simplicity. A technical step is to show that for $k + 1 > \alpha$,

$$H_k(z) \not\equiv 0 \quad \text{for } z \in D = \{z : |\arg z| < (1 + \tau_1)\pi/2\}.$$

If this is proved, let $h(z) = z^{k+1-\alpha} H_k(z)$ (here and throughout this paper $z^{k+1-\alpha}$ is the principal branch in $-\pi < \arg z < \pi$, i.e., $z^{k+1-\alpha}$ is real for $z = x \in \mathbb{R}^+$), then $h(z)$ is a *non-zero* analytic function on D, by (4.2) it satisfies

$$h(rz) = h(z) \quad \text{for } z \in D. \tag{4.9}$$

From Lemma 4.3(i) and (4.2)–(4.4), there exists $C > 0$ such that

$$\left| \frac{(-1)^k}{k!} z^{k+1-\alpha} F^{(k)}(z+1) - h(z) \right| = |z|^{k+1-\alpha} |H_k^-(z)| \leqslant C |z|^{k+1-\alpha} \tag{4.10}$$

for $z \in D$ and small $|z| > 0$. Hence we have

$$\overline{\lim_{t\to 0+}}\ t^{k+1-\alpha} |F^{(k)}(te^{i\theta} + 1)| = \overline{\lim_{t\to 0+}}\ k! |h(te^{i\theta})| > 0 \tag{4.11}$$

for all $|\theta| < (1 + \tau_1)\frac{\pi}{2}$. Let $f(z) = F^{(k)}(z^{-1})$, then

$$f(z) = (-1)^k \sum_{n=1}^\infty n(n+1)\cdots(n+k-1) a_n z^{n+k} := z^k \sum_{n=1}^\infty b_n z^n$$

is analytic in $|z| < 1$. Suppose that (4.8) does not hold, i.e.,

$$\lim_{n\to\infty} n^\alpha a_n = 0, \tag{4.12}$$

then $b_n = o(n^{k-\alpha})$ as $n \to \infty$. By the Bernoulli series expansion we conclude that $f(t) = o((1-t)^{-(k-\alpha+1)})$ as $t \to 1^-$. It follows that

$$F^{(k)}(t+1) = o(t^{-(k-\alpha+1)}) \quad \text{as } t \to 0^+.$$

This contradicts (4.11). Hence (4.12) does not hold and $\overline{\lim}_{n\to\infty} n^\alpha |a_n| > 0$ follows.

It remains to prove that $H_k(z) \not\equiv 0$ on D, we assume the contrary, then (4.2) and (4.3) imply that

$$\frac{(-1)^k}{k!} F^{(k)}(z+1) \equiv -H_k^-(z) = \sum_{n=1}^{\infty} r^{(k+1-\alpha)n} \int_{K \setminus K_1} \frac{-d\mu(w)}{(r^n z - (w-1))^{k+1}} \tag{4.13}$$

for $z \in D$. By Lemma 4.3(i) and (4.13), there exists a function $\varphi(z)$ analytic on $D \cup \Delta$ such that

$$F^{(k)}(z+1) = \varphi(z) \quad \text{for } z \in D. \tag{4.14}$$

Since $D \cup \Delta$ is a simply connected domain, we can find a function $\varphi_k(z)$ analytic on the domain $D \cup \Delta$ such that

$$F(z+1) = \varphi_k(z), \quad z \in D. \tag{4.15}$$

To apply Lemma 4.4, we take $\xi_0 = 0 \in K-1$ and $\epsilon = \frac{1}{2} r^{-1} d$ where $r^{-1}d$ is given by Lemma 4.3(i). Then there exists a simply connected domain Ω such that

$$\Omega \subset 2^{-1}\Delta, \quad 0 \in E := (K-1) \cap \Omega \quad \text{and} \quad (K-1) \cap \partial\Omega = \emptyset. \tag{4.16}$$

By the separated OSC, K is compact, totally disconnected [F, p. 116]; by (4.16) the same conclusion is true for E. It follows that $\Omega \setminus E$ is a connected open set [Mo, p. 93], i.e., $\Omega \setminus E$ is a domain.

Now, note that $F(z+1)$ is analytic on the domain $\Omega \setminus E, \varphi_k(z)$ is analytic on Ω by Lemma 4.3(i) and (4.16), and $D \cap \Omega := I$ is a non-empty open set. It follows from (4.15) and the uniqueness principle for analytic functions that

$$F(z+1) = \varphi_k(z) \quad \text{for } z \in \Omega \setminus E. \tag{4.17}$$

Since E has no interior points, $F(z+1)$ has a unique continuous extension to Ω by (4.17), and it must be analytic as $\varphi_k(z)$ is analytic on Ω. This contradicts Proposition 2.1 as $F(z+1)$ is not analytic at $z \in E$, and $H_k(z) \not\equiv 0$ follows. $\square$

For the second statement in Theorem 4.5, if we only assume the OSC, then we obtain the following partial conclusion.

Proposition 4.6. *If* $\{S_j\}_{j=1}^m$ *satisfies the OSC and if* $0<\alpha<\pi/\theta_0$, *then*

$$\overline{\lim_{n\to\infty}}\, n^\alpha |a_n| > 0, \tag{4.18}$$

where $\theta_0 = \inf_{j\in\mathscr{I}_1,\alpha_j=\alpha}(\sup\{\arg(w-z_j)-\arg(\xi-z_j) : w,\xi\in K\backslash\{z_j\}\})$.

Remark. In the next section, we see that in some cases, (4.18) can still hold without satisfying the condition in Proposition 4.6. In fact our conjecture is that $\overline{\lim}_{n\to\infty} n^\alpha|a_n|>0$ is true without any extra assumption.

Proof. We assume without loss of generality that $z_1=1$ attains θ_0 in the statement of the theorem, hence $\alpha_1=\alpha$. Write $r=r_1$. From the proof of Theorem 4.5, we only need to prove $H_k(z)\not\equiv 0$. For this, we consider

$$\int_r^1 H_k(te^{i\theta})t^{k-\alpha}\,dt.$$

By using a change of variable $t=xr^n$, the above integral reduces to

$$e^{-i(k+1)\theta}\int_{K\backslash K_1}\int_0^\infty \frac{x^{k-\alpha}\,dx}{(x-e^{-i\theta}(w-1))^{k+1}}\,d\mu(w).$$

If α is an integer, then $k=\alpha$, the above improper integral $\int_0^\infty$ is equal to $e^{ik\theta}k^{-1}(w-1)^{-k}$ by a direct calculation; otherwise $k=[\alpha]\neq\alpha$ (in this case $0<k+1-\alpha<1$), the integral equals

$$2\pi i\frac{(-1)^{k+1}\alpha(\alpha-1)\cdots(\alpha-k)e^{\alpha\theta i}}{k!(1-e^{i2\pi(k-\alpha)})(w-1)^\alpha}$$

by the residue theorem. Consequently, we have

$$\int_r^1 H_k(te^{i\theta})t^{k-\alpha}\,dt = Ce^{i(\alpha-k-1)\theta}\int_{K\backslash K_1}\frac{d\mu(w)}{(w-1)^\alpha}$$

for some constant $C\neq 0$ (depends on α). Let θ_0 be the angle in the hypothesis, we can choose τ such that $|\arg(w-1)-\tau|\leqslant\theta_0/2$ for all $w\in K\backslash K_1$. Since $0<\alpha<\pi/\theta_0$,

$$e^{-i\alpha\tau}\int_{K\backslash K_1}\frac{d\mu(w)}{(w-1)^\alpha} = \int_{K\backslash K_1}\frac{e^{i\alpha(\arg(w-1)-\tau)}}{|w-1|^\alpha}d\mu(w)\neq 0.$$

Hence $H_k(z)\not\equiv 0$ on D. $\square$

5. Special cases

In this section we consider the IFS of the form

$$S_j z = \epsilon_j + r(z - \epsilon_j), \quad j = 0, 1, \ldots, m-1, \tag{5.1}$$

where $\epsilon_j = e^{2\pi ij/m}$ and $0<r<1$ (note that we have shifted the indices of j to start from 0 for convenience). Let

$$\mu = \frac{1}{m} \sum_{j=0}^{m-1} \mu \circ S_j^{-1}. \tag{5.2}$$

Our first step is to prove the measure μ defined by (5.2) is invariant under $e^{2\pi i/m}$-rotation and its Cauchy transform F is m-fold symmetric. In fact, consider the new measure $\tilde{\mu}$ defined by

$$\tilde{\mu}(B) = \mu(\epsilon_1 B) = \frac{1}{m} \sum_{k=0}^{m-1} \mu(S_k^{-1}(\epsilon_1 B))$$

for all Borel set B. By making use of (5.1) and a direct calculation, we have $S_k^{-1}(\epsilon_1 w) = \epsilon_1 S_{k-1}^{-1}(w),\ 0 \leqslant k \leqslant m-1$ (where $S_{-1} = S_{m-1}$), hence

$$\tilde{\mu}(B) = \frac{1}{m} \sum_{k=0}^{m-1} \mu(\epsilon_1 S_{k-1}^{-1}(B)) = \frac{1}{m} \sum_{k=0}^{m-1} \mu(\epsilon_1 S_k^{-1}(B)) = \frac{1}{m} \sum_{k=0}^{m-1} \tilde{\mu}(S_k^{-1}(B)),$$

i.e., $\tilde{\mu}$ also satisfies (5.2). The uniqueness implies the $\tilde{\mu} = \mu$ [Hu].

Let K denote the attractor of the IFS, then $e^{2\pi i/m} K = K$ and $\operatorname{supp} \mu = K$. By a change of variable, we have

$$F(z) = \int_K \frac{1}{z - \xi} d\mu(\xi) = \frac{1}{m} \int_K \sum_{k=0}^{m-1} \frac{1}{z - e^{2k\pi i/m} w} d\mu(w).$$

For fixed z, we define for $w \in \mathbb{C} \setminus \{z, e^{-2\pi i/m} z, \ldots, e^{-2(m-1)\pi i/m} z\}$,

$$h(w) = \frac{1}{m} \sum_{k=0}^{m-1} \frac{1}{z - e^{2k\pi i/m} w} - \frac{z^{m-1}}{z^m - w^m}.$$

We remark that $h(w)$ has a simple pole at $w = ze^{-2k\pi i/m}$ and has residue 0, hence $h(w)$ has a removable singularity for $w = e^{-2k\pi i/m} z$. Because $h(\infty) = 0$, we have $h(w) \equiv 0$. This gives

$$F(z) = \int_K \frac{d\mu(w)}{z - w} = \int_K \frac{z^{m-1}}{z^m - w^m} d\mu(w), \tag{5.3}$$

hence $F(\epsilon_1 z) = F(z)$, i.e., F is m-fold symmetric, its Laurent series has the form

$$F(z) = z^{-1} + \sum_{n=1}^{\infty} a_{nm+1} z^{-(nm+1)}, \quad |z| > 1. \tag{5.4}$$

Our next goal is to give a condition on r so that the IFS $\{S_j\}_{j=0}^{m=1}$ in (5.1) satisfies the OSC. Let G be the regular m-polygon defined by the vertices $\{\epsilon_j\}_{j=0}^{m-1}$. If

$$S_0(G)^o \cap S_1(G)^o = \emptyset \quad \text{and} \quad S_0(G) \cap S_1(G) \neq \emptyset,$$

then we have from [St4, p. 1716] that

$$r = \rho_m = \sin\frac{\pi}{m} \Big/ \left(\sin\frac{\pi}{m} + \sin\frac{(2q+1)\pi}{m}\right), \tag{5.5}$$

where $q = [m/4]$. It follows immediately that

Proposition 5.1. *Let* $\{S_j\}_{j=0}^{m-1}$ *be as in* (5.1) *and suppose* $r \leqslant \rho_m$. *Then* $\{S_j\}_{j=0}^{m-1}$ *satisfies the OSC.*

Note that for $r = \rho_m$, the Hausdorff dimension of the attractor K of $\{S_j\}_{j=0}^{m-1}$ is $\alpha_m = \log m / |\log \rho_m|$. Some values of ρ_m and α_m calculated by using Mathematica are given in Table 1.

If $0 < r \leqslant \rho_m$, then the μ in (5.2) is the Hausdorff measure $\mathscr{H}^\alpha$ restricted on the attractor K [F]. By [LSV], the coefficient a_{nm+1} can be determined by the following recursive relation:

$$a_{nm+1} = \frac{(1-r)^{nm}}{1-r^{nm}} \sum_{k=0}^{n-1} \binom{nm}{km} \left(\frac{r}{1-r}\right)^{km} a_{km+1}, \tag{5.6}$$

where $a_1 = 1$. However, the formula is not easy to handle. In the following we will sharpen the asymptotic behavior of a_{nm+1} in Theorem 4.1 for these special cases.

We remark that some of the cases do not satisfy the condition $\alpha \leqslant \pi/\theta_0$ in Proposition 4.6. For example for $m = 5, 6, 7, 8$ and $r = \rho_m$, we have $\pi/\theta_0 = m/(m-2) < \alpha_m = \log m / |\log \rho_m|$ from Table 1.

Table 1
Values of ρ_m and α_m for $2 \leqslant m \leqslant 8$

m	2	3	4	5	6	7	8
ρ_m	0.5	0.5	0.5	0.38196	0.33333	0.30797	0.29289
α_m	1	1.58496	2	1.67228	1.63093	1.65226	1.69343

Theorem 5.2. *For $0<r\leqslant\rho_m$, let $\alpha=\log m/|\log r|$ and let $F(z)$ be the Cauchy transform. Then the Laurent coefficients $\{a_{nm+1}\}_{n=1}^{\infty}$ satisfy*

$$C^{-1}\leqslant(nm+1)^{\alpha}a_{mn+1}\leqslant C,\quad n\geqslant 1$$

for some $C>0$.

Proof. That $z_j=\epsilon_j=e^{2j\pi i/m}, r_j=r$ and $p_j=\frac{1}{m}$ imply that all the $\Phi_j(t)$ in Corollary 3.5 are equal. We write

$$\Phi_0(t)=t^{\alpha}\prod_{k=1}^{\infty}(1+b_{-k}(t))\prod_{k=0}^{\infty}\frac{1+b_k(t)}{m},\tag{5.7}$$

where

$$b_k(t)=\sum_{\ell=1}^{m-1}e^{-tr^k(1-r)(1-e^{2\ell\pi i/m})},\quad k\in\mathbb{Z}.\tag{5.8}$$

We claim that there exists $C>1$ such that

$$C^{-1}\leqslant\Phi_0(t)\leqslant C\quad\text{for } t\in\mathbb{R}^+.\tag{5.9}$$

Note that $\Sigma_{\ell=0}^{m-1}e^{2m\ell\pi i/m}=0$ if $\ell\neq nm$, and $=m$ if $\ell=nm$. By expressing the exponential term in (5.8) as a power series, we have

$$1+b_k(t)=me^{-tr^k(1-r)}\sum_{n=0}^{\infty}\frac{(tr^k(1-r))^{nm}}{(nm)!}.$$

This gives

$$me^{-r^k(1-r)}<1+b_k(t)<m\quad\text{for } r\leqslant t\leqslant 1,\ k\in\mathbb{Z}.\tag{5.10}$$

Hence

$$e^{-1}<\prod_{k=0}^{\infty}\frac{1+b_k(t)}{m}<1\quad\text{for } r\leqslant t\leqslant 1.$$

Also (3.10) and (5.10) imply that there exists $C_1>1$ such that

$$C_1^{-1}\leqslant\prod_{k=1}^{\infty}(1+b_{-k}(t))\leqslant C_1\quad\text{for } r\leqslant t\leqslant 1.$$

Hence the claim follows. The theorem follows from this, $a_{nm+1}>0$ and (Theorem 4.1)

$$(nm+1)^{\alpha}a_{nm+1}=m\Phi_0((nm+1))+O(\log^2 n/n)\quad\text{as } n\to\infty.\qquad\square\tag{5.11}$$

Corollary 5.3. *For any k such that $k+1-\alpha>0$, there exists $C>0$ such that*

$$\frac{C^{-1}t^{k+1}}{(1-t)^{k+1-\alpha}}\leqslant(-1)^kF^{(k)}(t^{-1})\leqslant\max_{|z|=t^{-1}}|F^k(z)|\leqslant\frac{Ct^{k+1}}{(1-t)^{k+1-\alpha}}$$

for $0<t<1$. Moreover, we can find constants $\eta>0, \delta>0$ and $c>0$ such that

$$|F^{(k)}(1+z)|\geqslant c|z|^{\alpha-k-1}\quad\text{for } 0<|z|\leqslant\delta,\ \ |\arg z|\leqslant\eta.$$

Proof. We take the kth derivatives of $F(z)$ in (5.4), then the coefficients of $(-1)^kF^{(k)}(z)$ are $\{(nm+1)\cdots(nm+k)a_{nm+1}\}_{n=0}^{\infty}$ which are bounded above and below by positive multiples of $n^{k-\alpha}$. The inequality follows by comparing with the binomial series expansion of $(1-t^m)^{-(k+1-\alpha)}$.

For the second assertion we use the function $h(z)$ in the proof of Theorem 4.5. By (4.9), (4.10) and the first inequality in the first assertion, there exists $c>0$ such that

$$h(x)\geqslant 3c>0\quad\text{for } x\in\mathbb{R}^+.$$

The continuity and the multiplicative periodicity of h give that $|h(z)|>2c, |\arg z|\leqslant\eta$ for some $\eta>0$. By (4.10) again, there is a constant $\delta>0$ such that

$$|F^{(k)}(1+z)|\geqslant c|z|^{\alpha-k-1}\quad\text{for } 0<|z|\leqslant\delta\quad\text{and}\quad|\arg z|\leqslant\eta.\qquad\square$$

Theorem 5.2 describes the exact "order" of a_{nm+1} at infinity; in the next proposition we like to discuss the "limit behavior" of $(nm+1)^{\alpha}a_{nm+1}$. We will need the $H_k(z)$ (in (4.2)) for the μ given by (5.2) and with $r_j=r\leqslant\rho_m$. Note that $H_k(z)$ is analytic for $|\arg z|<\left(\frac{1}{2}+1/m\right)\pi$ (Lemma 4.3); also note that the main part of the proofs of Theorem 4.5 and Proposition 4.6 are to show that $h(z):=z^{k+1-\alpha}H_k(z)\not\equiv 0$. In the following lemma, we prove that $h(z)$ is again a *non-constant* analytic function.

Lemma 5.4. *For $0<r\leqslant\rho_m$, $m\geqslant 2$, except the cases $m=2,3,4$ with $r=\rho_m=\frac{1}{2}$, and for $k+1-\alpha>0$, then $h(z)=z^{k+1-\alpha}H_k(z)$ is a non-constant analytic function on $D=\{z:|\arg z|<\pi/2+\pi/m\}$.*

Proof. We only need to prove that $h(z)$ is a *non-constant* function. By (5.5), the assumption on r implies that $0<r<\frac{1}{2}$ in all the cases.

We first prove the lemma for the cases $m\geqslant 3$. Let $z_t=(1+t)e^{2\pi i/m}-1$ for $t>0$. Clearly, $z_t\in D$ and $z_0=2\sin(\pi/m)e^{(1/2+1/m)\pi i}\in\partial D\backslash\{0\}$. By (4.2)–(4.4),

$$H_k(z_t)=\frac{(-1)^k}{k!}F^{(k)}(1+z_t)+\sum_{n=1}^{\infty}\int_{K\backslash K_0}\frac{r^{(k+1-\alpha)n}\,d\mu(w)}{(r^nz_t-(w-1))^{k+1}}.\tag{5.12}$$

(We replace the K_1 in $H_k(z)$ in Theorem 4.5 by K_0 because of the change of the indices). To prove $h(z)\not\equiv$ constant, it suffices to prove $H_k(z_t)\to\infty$ as $t\to 0^+$ (note that $z_0 \neq 0$). A simple geometric consideration shows that $|w-1| \geqslant \frac{1}{2}|\varepsilon_1 - 1| = \sin\frac{\pi}{m}$ for $w \in K \backslash K_0$. It follows from $0 < r < \frac{1}{2}$ that

$$|r^n z_0| \leqslant r|z_0| = 2r \sin\frac{\pi}{m} < \sin\frac{\pi}{m} \leqslant |w-1|$$

for $n \geqslant 1, w \in K\backslash K_0$. Hence there are $t_0 > 0, \delta > 0$ such that

$$|r^n z_t - (w-1)| \geqslant |w-1| - r^n|z_t| \geqslant \delta$$

for $0 < t \leqslant t_0, n \geqslant 1, w \in K\backslash K_0$. This together with (5.12) and Corollary 5.3 imply that

$$H_k(z_t) = \frac{(-1)^k}{k!} F^{(k)}(1+z_t) + O(1) \to \infty \quad \text{as } t \to 0^+$$

(here we have used the m-fold symmetry: $|F^{(k)}((1+t)e^{2\pi i/m})| = |F^{(k)}(1+t)|$).

For the case $m = 2$, the above $z_t \notin D$. Alternatively, we choose $z_t^* = -2 - te^{i\eta}$ where η is given in Corollary 5.3, then $z_t^* \in D$ and $z_0^* \in \partial D \backslash \{0\}$. Since $\alpha = \log 2/|\log r| < 1$ and $|F(1+z_t^*)| = |F(-1-te^{i\eta})| = |F(1+te^{i\eta})| \geqslant ct^{\alpha-1}$ (Corollary 5.3), we have

$$F(1+z_t^*) \to \infty \quad \text{as } t \to 0.$$

In view of $0 < r < \frac{1}{2}$ and $K\backslash K_0 \subset [-1, -1+2r]$, we have $|w-1| \geqslant 1 + (1-2r) > 2r = r|z_0^*| \geqslant r^n|z_0^*|$ for $n \geqslant 1$ and $w \in K\backslash K_0$. Similar to the case $m \geqslant 3$, we can use (5.12) to prove that $H_k(z_t^*) \to \infty$ as $t \to 0$. □

Proposition 5.5. *For $0 < r \leqslant \rho_m$, $m \geqslant 2$, except the cases $m = 2, 3, 4$ with $r = \rho_m = \frac{1}{2}$, and for $k + 1 - \alpha > 0$, then*

$$0 < \varliminf_{t \to 1+} (t-1)^{k+1-\alpha}|F^{(k)}(t)| < \varlimsup_{t \to 1+} (t-1)^{k+1-\alpha}|F^{(k)}(t)| < \infty.$$

Proof. By Theorem 5.2 and in view of $F^{(k)}(t)$ preserves sign for $t \in (1, \infty)$, it suffices to show that $\lim_{t \to 1^+}(t-1)^{k+1-\alpha} F^{(k)}(t)$ does not exist. But this follows from $h(t)$ is *non-constant* (Lemma 5.4) and the same reasoning as in (4.11). □

Remark. Proposition 5.5 also holds for $m = 3$, $r = \rho_m = \frac{1}{2}$, but it needs a different proof, the case $k = [\alpha] = 1$ is given in [DL2].

Theorem 5.6. *For $0 < r \leqslant \rho_m$, $m \geqslant 2$, except the cases $m = 2, 4$ with $r = \rho_m = \frac{1}{2}$, then the Laurent coefficients $\{(nm+1)^\alpha a_{nm+1}\}_{n=0}^\infty$ of F is dense in the non-degenerated line segment $m\Phi_0([r, 1])$ where $\Phi_0(t)$ is given by* (5.7) *and* (5.8) *and $\alpha = \log m/|\log r|$.*

Moreover

$$\overline{\lim_{n\to\infty}}\ (nm+1)^{\alpha} a_{nm+1} = \max_{r\leqslant t\leqslant 1}\ m\Phi_0(t),$$
$$\underline{\lim}_{n\to\infty}\ (nm+1)^{\alpha} a_{nm+1} = \min_{r\leqslant t\leqslant 1}\ m\Phi_0(t).$$

Proof. By a slight modification of Proposition 2.4, we know that the sequence $x_n' := (nm+1)r^{N(nm+1)}$ is dense in $[r,1]$. Since Φ_0 is continuous on $\mathbb{R}^+, \Phi_0([r,1])$ is a bounded interval by (5.9). From (5.11) and noting that $\Phi_0(rt) = \Phi_0(t)$, we have

$$(nm+1)^{\alpha} a_{nm+1} = m\Phi_0((nm+1)r^{N(nm+1)}) + O(\log^2 n/n) \quad \text{as } n\to\infty.$$

Proposition 5.5 shows that $\lim_{n\to\infty} (nm+1)^{\alpha} a_{nm+1}$ does not exist (the argument is the same as in (4.12)), with one remaining unjustified case $m=3, r=\rho_m=\frac{1}{2}$. Hence $m\Phi_0(x)$ is not constant and $m\Phi_0([r,1])$ must be a non-degenerated interval; $\{(nm+1)^{\alpha} a_{nm+1}\}_{n=0}^{\infty}$ is dense in the non-degenerated line segment $m\Phi_0([r,1])$.

It remains to show that $\Phi_0(t)$ is not a constant function when $m=3, r=\rho_m=\frac{1}{2}$. We can proceed as Lemma 5.4 to prove this, but there are some added complication because of $r=\frac{1}{2}$ (see [DL2]). We give an alternate proof here. It suffices to find a $t_0 \in [\frac{1}{2}, 1]$ such that $\Phi_0'(t_0) \neq 0$. From Example 2 in Section 3 and noting that (5.9), we have for $t>0$,

$$\begin{aligned}\frac{\Phi_0'(t)}{\Phi_0(t)} = & \frac{\alpha}{t} + \sum_{k=0}^{\infty} \frac{-2\sqrt{3}\sin(\pi/3 + \sqrt{3}2^{k-1}t)}{1+2e^{-3\cdot 2^{k-1}t}\cos(\sqrt{3}2^{k-1}t)} 2^k e^{-3\cdot 2^{k-1}t} \\ & + \sum_{k=1}^{\infty} \frac{-2\sqrt{3}\sin(\pi/3 + \sqrt{3}2^{-k-1}t)}{1+2e^{-3\cdot 2^{-k-1}t}\cos(\sqrt{3}2^{-k-1}t)} 2^{-k} e^{-3\cdot 2^{-k-1}t},\end{aligned} \tag{5.13}$$

where $\alpha = \log 3/\log 2$. Let $R_p^{(1)},\ R_q^{(2)}$ denote the tails $\sum_{k\geqslant p}, \sum_{k\geqslant q}$ of the first and second series in (5.13). For $\frac{1}{2}\leqslant t\leqslant 1$,

$$\begin{aligned}|R_p^{(1)}| &\leqslant \frac{2\sqrt{3}}{1-2e^{-3\cdot 2^{(p-2)}}} \sum_{k=p}^{\infty} 2^k e^{-3\cdot 2^{(k-2)}} \\ &\leqslant \frac{2\sqrt{3}}{1-2e^{-3\cdot 2^{(p-2)}}} \int_{p-1}^{\infty} 2^y e^{-3\cdot 2^{(y-2)}}\, dy \\ &= \frac{8\sqrt{3}e^{-3\cdot 2^{(p-3)}}}{3\log 2(1-2e^{-3\cdot 2^{(p-2)}})} \coloneqq b_1(p),\end{aligned}$$

and for $\frac{1}{2}\leqslant t\leqslant 1$,

$$|R_p^{(2)}| \leqslant \frac{2^{(1-q)}\sqrt{3}\sin(\pi/3 + \sqrt{3}2^{-(q+1)})}{1+2e^{-3\cdot 2^{-(q+1)}}\cos(2^{-(q+1)}\sqrt{3})} \coloneqq b_2(q).$$

It follows that for any $p,\ q>1$ (see (5.13))

$$\frac{\Phi_0'(t)}{\Phi_0(t)} \geqslant \left(\frac{\alpha}{t} + \sum_{k=0}^{p-1} (\cdots) + \sum_{k=1}^{q-1} (\cdots)\right) - b_1(p) - b_2(q).$$

It we take $p=5,\ q=10$ and $t=\frac{3}{5}$, then by using Mathematica, the right-hand side $\approx 0.002265 - 0.000041 - 0.000978 \approx 0.0011$. This implies that $\Phi_0'(3/5)/\Phi_0(3/5)>0$. $\square$

When $m=3$ and $r=\rho_3=\frac{1}{2}$, the case of Sierpinski triangle K, we find numerically

$$\min_{1/2\leqslant t\leqslant 1} 3\Phi_0(t) \approx 1.42668, \qquad \max_{1/2\leqslant t\leqslant 1} 3\Phi_0(t) \approx 1.42676.$$

This shows that $\{(3n+1)^{\alpha} a_{3n+1}\}_{n=0}^{\infty}$ is a sequence with small oscillation in between these two values.

The remaining untreated case in Proposition 5.5 and Theorem 5.6 is for $m=2,4$ and $r=\rho_m=\frac{1}{2}$. In fact they are the simple exceptional cases. For $m=2$, the attractor $K=[-1,1]$ obviously, the self-similar measure $\mu=\frac{1}{2}\mathscr{L}$ and $\alpha=1$, hence its Cauchy transform is

$$F(z)=\frac{1}{2}\int_{[-1,1]} \frac{d\mathscr{L}(x)}{z-x} = \frac{1}{2}\log\frac{z+1}{z-1}, \quad z\in\mathbb{C}\backslash[-1,\ 1], \tag{5.14}$$

where the logarithmic function is the principal branch, i.e., $\log(z+1)$ and $\log(z-1)$ are real for $z=x>1$. It is easy to show from the above that

$$a_{2n+1}=1/(2n+1) \quad \text{for } n\geqslant 0. \tag{5.15}$$

This implies that $\lim_{t\to 1^+}(t-1)^k|F^{(k)}(t)|$ exists for $k\geqslant 1$, and that $\lim_{n\to\infty}(2n+1)a_{2n+1}=1$. By using this, we can directly obtain the interesting identity in Section 3, Example 1:

$$z\prod_{k=1}^{\infty}(1+e^{-2^k z})\prod_{k=0}^{\infty}\frac{1+e^{-2^{-k}z}}{2} \equiv \frac{1}{2} \qquad |\arg z| < \frac{\pi}{2}.$$

It is because from Theorem 4.1 (or (5.11)) and (5.15), we have

$$1=(2n+1)a_{2n+1}=2\Phi_0(2n+1)+O\left(\frac{\log^2 n}{n}\right) \quad \text{for } n\to\infty. \tag{5.16}$$

By the periodicity, $\Phi_0(2n+1)=\Phi_0(x_n)$ where $x_n=(1/2)^{N(2n+1)}(2n+1)\in[\frac{1}{2},1)$ and is dense in $[\frac{1}{2},1]$. Hence (5.16) and the continuity of $\Phi_0(x)$ on $\mathbb{R}^+$ give $\Phi_0(x)\equiv\frac{1}{2}$ for $x\in[\frac{1}{2},1]$. It follows by Lemma 3.3(ii) and the uniqueness principle for analytic functions that $\Phi_0(z)\equiv\frac{1}{2}$ for $|\arg z|<\pi/2$.

For the case $m=4$ and $r=\frac{1}{2}$, the attractor K is the square with vertices $e^{2\pi ij/4}$, $0\leqslant j\leqslant 3$, the self-similar measures $\mu=\frac{1}{2}\mathscr{L}^2$ and $\alpha=2$. Its Cauchy transform is

$$F(z)=\frac{1}{2}\int_K\frac{d\mathscr{L}^2(w)}{z-w}=\frac{1}{2}\int_0^1 dx\int_{x-1}^{1-x}\left[\frac{1}{z+x-yi}+\frac{1}{z-x-yi}\right]dy.$$

A direct calculation gives

$$F(z)=\frac{1}{2}\sum_{n=0}^{3}(-1)^n(z-e^{2\pi in/4})\log\left(1-e^{2\pi in/4}z^{-1}\right) \tag{5.17}$$

for $z\in\mathbb{C}\backslash K$. (For the logarithmic branches, for $\log(1-z^{-1})$ and $\log(1+z^{-1})$, we take real values for $z=x>1$, and for $\log(1+iz^{-1})=\log|1+iz^{-1}|+i\arg(1+iz^{-1})$, we take for $z=x>1$, $0<\arg(1+iz^{-1})<\pi/2$; it is necessary that $\log(1-iz^{-1})=\log|1-iz^{-1}|+i\arg(1-iz^{-1})$ satisfies $-\pi/2<\arg(1-iz^{-1})=-\arg(1+iz^{-1})<0$ for $z=x>1$ since $F(x)>0$ for $x>1$.) It is easy to see that

$$F(z)=\sum_{n=0}^{\infty}\frac{1}{(2n+1)(4n+1)}z^{-(4n+1)}\quad \text{for}|z|\geqslant 1 \tag{5.18}$$

and that

$$F^{(k+2)}(z)=\frac{d^k}{dz^k}\sum_{i=0}^{3}\left(\prod_{\substack{j=0\\ j\neq i}}^{3}\frac{\varepsilon_i}{\varepsilon_i-\varepsilon_j}\right)\frac{2}{z-\varepsilon_i}=\sum_{i=0}^{3}\left(\prod_{\substack{j=0\\ j\neq i}}^{3}\frac{\varepsilon_i}{\varepsilon_i-\varepsilon_j}\right)\frac{2(-1)^k k!}{(z-\varepsilon_i)^{k+1}}$$

where $k\geqslant 0$ and $\varepsilon_j=e^{2j\pi i/4}$. Hence $(4n+1)^\alpha a_{4n+1}=2-1/(2n+1)$ and $\lim_{t\to 1+}(t-1)^{k+1-\alpha}|F^{(k)}(t)|=(k-2)!/2$ where $k\geqslant 2$ and $\alpha=2$.

Note that F is analytic in $\mathbb{C}\backslash K$ and has an analytic extension F_0 in $\mathbb{C}\backslash\{[-1,1]\cup[-i,i]\}$. But $F(z)\not\equiv F_0(z)$ for $z\in K\backslash\{[-1,1]\cup[-i,i]\}$ since F is not analytic for any $z\in K$ by Proposition 2.1.

As a simple consequence of the Laurent series of F and (5.17), we have

$$\sum_{n=0}^{\infty}\frac{1}{(2n+1)(4n+1)}=F(1)=\frac{\ln 2}{2}+\frac{\pi}{4}.$$

Acknowledgments

The authors thank Professor Sze-Man Ngai for providing the reference in Theorem 4.5. They also thank Professor R. Strichartz for various comments to improve the paper.

References

[D] G. David, Unrectifiable 1-sets have vanishing analytic capacity, Rev. Mat. Iberoamericana 14 (1998) 369–479.

[Do] X.H. Dong, Cauchy transforms of self-similar measures, Ph.D Thesis, Department of Mathematics, The Chinese University of Hong Kong, 2002.

[DL1] X.H. Dong, K.S. Lau, Starlikeness and the Cauchy transform of some self-similar measures, in preparation.

[DL2] X.H. Dong, K.S. Lau, The Cauchy transform on the Sierpinski gasket, in preparation.

[Du] P. Duren, Univalent Functions, Springer, New York, 1983.

[F] K. Falconer, Fractal Geometry—Mathematical Foundations and Applications, Wiley, New York, 1990.

[G] J. Garnett, Analytic Capacity and Measure, in: Lecture Notes in Mathematics, Vol. 297, Springer, Berlin, 1972.

[H1] W.K. Hayman, The asymptotic behavior of p-valent functions, Proc. London Math. Soc. 3 (1955) 257–284.

[H2] W.K. Hayman, Multivalent Functions, Second Edition, Cambridge University Press, Cambridge, 1994.

[Hu] J. Hutchinson, Fractals and self-similarity, Indiana Univ. Math. J. 30 (1981) 713–747.

[KN] L. Kuipers, H. Niederreiter, Uniform Distribution of Sequences, Wiley, New York, 1974.

[L1] K.S. Lau, Fractal measures and mean p-variations, J. Funct. Anal. 108 (1992) 427–457.

[L2] K.S. Lau, Dimension of a family of singular Benoulli convolutions, J. Funct. Anal. 116 (1993) 335–358.

[LW] K.S. Lau, J.R. Wang, Mean quadratic variations and Fourier asymptotics of self-similar measures, Monatsh. Math. 115 (1993) 99–123.

[LSV] J. Lund, R. Strichartz, J. Vinson, Cauchy transforms of self-similar measures, Experiment. Math. 7 (1998) 177–190.

[Ma] P. Mattila, Geometry Sets and Measures in Euclidean Spaces, Fractals and Rectifiability, in: Cambridge Studies in Advanced Mathematics, Vol. 44, Cambridge University Press, Cambridge, 1995.

[Mo] E.E. Moise, Geometric Topology in Dimension 2 and 3, Springer, New York, 1977.

[S] A. Schief, Separation properties for self-similar sets, Proc. Amer. Math. Soc. 122 (1994) 111–115.

[St1] R. Strichartz, Fourier asymptotics of fractal measures, J. Funct. Anal. 89 (1990) 154–187.

[St2] R. Strichartz, Self-similar measures and their Fourier transforms, I, II, III, Indiana Univ. Math. J. 39 (1990) 797–817; Trans. Amer. Math. Soc. 336 (1993) 335–361; Indiana. Univ. Math. J. 42 (1993) 367–411.

[St3] R. Strichartz, Self-similarity in Harmonic Analysis, J. Fourier Anal. Appl. 1 (1994) 1–37.

[St4] R. Strichartz, Isoperimetric estimates on Sierpinski gasket type fractals, Trans. Amer. Math. Soc. 351 (1999) 1705–1752.