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Asymptotic Behavior of Multiperiodic Functions $G(x) = \prod_{n=1}^{\infty} g(x/2^n)$

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ABSTRACT. Let $0 \le g$ be a dyadic Hölder continuous function with period 1 and $g(0) = 1$, and let $G(x) = \prod_{n=0}^{\infty} g(x/2^n)$. In this article we investigate the asymptotic behavior of $\int_0^T |G(x)|^q dx$ and $\frac{1}{n}\sum_{k=0}^{n} \log g(2^k x)$ using the dynamical system techniques: the pressure function and the variational principle. An algorithm to calculate the pressure is presented. The results are applied to study the regularity of wavelets and Bernoulli convolutions.

1. Introduction

The equation

$$G(x) = g\left(\frac{x}{\beta}\right) G\left(\frac{x}{\beta}\right),$$

where $\beta > 1$ and g is a periodic function of period 1 with $g(0) = 1$, arises naturally as the Fourier transform of self-similar objects such as Bernoulli convolution measures, scaling functions, etc. Iteration of the equation yields the infinite product

$$G(x) = \prod_{k=1}^{\infty} g\left(\frac{x}{\beta^k}\right)$$

which we call a *multiperiodic function* [16]. The simplest multiperiodic function is defined by $g(x) = \cos(2\pi x)$, $\beta = 2$:

$$G(x) = \prod_{k=1}^{\infty} \cos\left(\frac{2\pi x}{2^k}\right) = \frac{\sin 2\pi x}{2\pi x}$$

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(the Euler-Vieta formula). In general there is no closed form formula and the behavior of G is rather complicated.

Our main interest is on the asymptotic rate of G at infinity. There are many ways to estimate such a rate. The most common one is the Sobelev exponent defined by the supremum of the α so that

$$\int_R \left(1+|x|^2\right)^\alpha |G(x)|^2 dx < \infty\,.$$

In wavelet theory ($\beta = 2$), this exponent and some other similar types of exponents had been studied by Daubechies [5, 6], Cohen and Daubechies [3], Lau et al. [20], Lau and Ma [21], and Villemoes [25, 26]. For self-similar measures μ, Strichartz [24] and Lau and Wang [19] showed that under the open set condition on the iterated function system of μ, (β is not necessarily an integer), if $G = \hat{\mu}$, then

$$\int_0^T |G(x)|^2 dx \sim Q(T)T^{1-\alpha}, \quad T \to \infty$$

where α is the L^2-dimension of μ and Q is a positive continuous, multiplicatively periodic function of period β. This estimation was extended to L^2-scaling functions in [20] and a sharper estimate than the ones in [3, 5, 6] was obtained. In [16] Strichartz et al. made a head start study on the asymptotic behavior of

$$I_q(T) = \int_0^T |G(x)|^q dx, \quad T \to \infty\,.$$

While it is difficult to obtain a precise estimation like the case $q = 2$, they raised a weaker question: under what condition does

$$\lim_{T\to\infty} \frac{\log I_q(T)}{\log T} \quad \text{exist} \tag{1.1}$$

In the same paper they also initiated another direction to study the behavior of G at infinity by considering $G(\beta^n x)$ for $x \in [0, 1)$. Observe that $\log G(\beta^n x) - \log G(x) = \sum_{j=0}^{n-1} \log g(\beta^j x)$. Therefore one can study the convergence of

$$h_n(x) = \frac{1}{n}\sum_{j=0}^{n-1} \log g\left(\beta^j x\right)\,. \tag{1.2}$$

This will imply $G(\beta^n x) \sim G(x)e^{nh_n(x)}$ as $n \to \infty$. For almost all x, $\beta^j x$ (mod 1) is uniformly distributed on $[0, 1)$ [17], and it follows that if $g > 0$, $h_n(x)$ converges to $\int_0^1 \log g(t)dt$ for almost all x. However, the numerical experiments in [16] revealed a more complicated structure of the limit of $\{h_n\}$ on the exceptional set.

In this article we will study the asymptotic behavior of $G(x)$ at infinity in the setup of (1.1) and (1.2). The basic idea comes from the following result which can be derived easily from the theory of Ruelle operator on symbolic dynamical systems [1]. Let $g(x)$ be a period 1 dyadic Hölder continuous function with $g(0) = 1$ and $\inf\{g(x) : x \in [0, 1)\} > 0$. For any $q, s \in R$, let

$$\alpha = s + \frac{P(q)}{\log 2}\,.$$

Then,

$$\int_1^T x^s G^q(x)dx \approx \begin{cases} T^\alpha & \text{if } \alpha > 0 \\ \log T & \text{if } \alpha = 0 \\ O(1) & \text{if } \alpha < 0 \end{cases}$$

where $P(q)$ is the *pressure function* associated with g (see Section 2). Moreover,

$$\dim_H \left\{x : \lim_{n\to\infty} h_n(x) = \alpha\right\} = -\frac{P^*(\alpha)}{\log 2}$$

where P^* is the Legendre transformation (convex conjugate) of P and $\dim_H E$ denotes the Hausdorff dimension of E.

The positivity of g is a major restriction. In Section 3 we actually prove some more general theorems that include cases where $g(x)$ is proximal (see Section 1), or g contains a factor $|\cos \pi x|^N$ which is common for wavelets and Bernoulli convolutions. For these we make use of Hennion's [13] quasi-compactness approach to the Ruelle operator.

The other aim of this article is to obtain explicit calculations of the pressure function $P(q)$. We obtain an algorithm for the class of functions constant on the dyadic intervals of $[0, 1)$, i.e., dyadic step functions (Theorem 7). This can be used to approximate the pressure function for more general g. For example, we apply this to estimate the modulus of continuity of the Daubechies scaling functions and significantly improve the known estimate. The pressure of g can also be calculated explicitly in some special cases (see Proposition 4 and Section 5).

The material is organized as follows. In Section 2, we recall some notations and results concerning the dynamical system, and Hennion's approach of the Ruelle operator. The main theorems are proved in Section 3. In Section 4, we obtain a matrix method which gives an exact calculation of the pressure function if g is a dyadic step function. We also present different calculation techniques for some other important special cases. Section 5 is devoted to the distribution solution of the dilation equation of which we can apply our results. Specifically, we examine the Cantor measures, the Bernoulli convolutions, and the wavelets, as well as some other illustrating examples.

We point out that here we are only dealing with the dyadic case; similar results can be stated directly for β-adic case; higher dimension cases can be handled with certain modifications and will be presented in a forthcoming paper. The main unsettled case is that $G(x) = g(x/\beta)G(x/\beta)$ when $\beta > 1$ is not an integer.

2. The Transfer Operators

Let $\Sigma_2 = \{0, 1\}^N$ be the symbolic space and equipped with the usual metric $d(x, y) = \min\{2^{-i} : x_i \neq y_i\}$. Let $\sigma : \Sigma_2 \to \Sigma_2$ be the shift transformation defined by $\sigma(x_1, x_2, \cdots) = (x_2, x_3, \cdots)$. For a continuous function on Σ_2 (i.e., $g \in C(\Sigma_2)$), let

$$var_k g = \sup\{|g(y_1) - g(y_2)| : y_1, y_2 \in I_k(x), x \in \Sigma_2\} \tag{2.1}$$

where $I_k(x) = \{y : y_i = x_i,\ i = 1, \cdots, k\}$ is the cylinder set with base $(x_1, \cdots, x_k)$. A function $g \in C(\Sigma_2)$ is said to be *Hölder continuous* if there exists $c > 0$ and $0 < \alpha \leq 1$ such that

$$var_k g \leq c 2^{-\alpha k} \quad \text{for all } k \geq 1 . \tag{2.2}$$

Denote by $H(\Sigma_2)$ the class of all such functions and by $H_\alpha(\Sigma_2)$ $(0 < \alpha \leq 1)$ the subspace of f satisfying (2.2). Let $|f|_\alpha$ be the infimum of the c in (2.2) and let

$$|||f|||_\alpha = \|f\| + |f|_\alpha$$

($\|f\|$ being the uniform convergence norm of $C(\Sigma_2)$). Then the space $H_\alpha(\Sigma_2)$ equipped with the norm $|||\cdot|||_\alpha$ becomes a Banach space.

For $g \in C(\Sigma_2)$, define the *transfer operator* $L_g : C(\Sigma_2) \to C(\Sigma_2)$ by

$$L_g f(x) = \sum_{y \in \sigma^{-1}(x)} g(y) f(y) .$$

(see [1, 4, 11, 13, 23, 27]). We shall need the following theorem, which is a special case of a theorem of Hennion [13] (see also [15] and [22]).

Theorem 1.

Suppose $0 \le g \in H_\alpha(\Sigma_2)$ *for some* $0 < \alpha \le 1$.

(i) *The operators* L_g *and* $L_g|_{H_\alpha(\Sigma_2)}$ *have the same spectral radius* ρ.

(ii) $L_g : H_\alpha(\Sigma_2) \to H_\alpha(\Sigma_2)$ *is quasi-compact.*

(iii) ρ *is an eigenvalue of maximal order among all eigenvalues* λ *with* $|\lambda| = \rho$.

(iv) *If* g *is irreducible, then* ρ *is the only eigenvalue with modulus* ρ *and the eigenfunction space corresponding to* ρ *is generated by one strictly positive function.*

Let us recall the two notions involved in the statement of the theorem. The quasi-compactness of L_g means that there exists a positive $0 \le r < \rho$ and two closed subspaces E and F of $H_\alpha(\Sigma_2)$ such that

$$H_\alpha(\Sigma_2) = E \bigoplus F, \qquad L_g(E) \subseteq E, \quad L_g(F) \subseteq F$$

and that $1 \le \dim E < \infty$, the eigenvalues of $L_g|_E$ are of modulus $\ge r$, the spectral radius of $L|_F$ is strictly smaller than r [13]. The irreducibility of g (or of L_g) means that for any x and any continuous $f \ge 0$ not identically zero, there is an n such that $L_g^n f(x) > 0$. There is a geometrical way to describe the irreducibility. For $g \ge 0$, we define a *path* of $x \in \Sigma_2$ of length n to be a finite sequence $\{x_k\}_{k=1}^n$ such that $x_k \in \sigma^{-1}(x_{k-1})$ and $g(x_k) > 0$ for all $1 \le k \le n$ (with the convention that $x_0 = x$); the *orbit* $O(x)$ of x is defined to be the closure of the union of all the paths $\{x_n\}$. Then g is irreducible iff $O(x) = \Sigma_2$ for any $x \in \Sigma_2$. To show this equivalence, it suffices to note that

$$L_g^n f(x) = \sum g(x_1) \cdots g(x_n)\, f(x_n)$$

where the sum is taken over all possible paths $\{x_k\}$ of x of length n. A still weaker condition concerning the positivity of g is the *proximality*, i.e., $O(x) \bigcap O(y) \neq \emptyset$ for any x and y.

Theorem 2.

Suppose $0 \le g \in H_\alpha(\Sigma_2)$. *Let* ρ *be the spectral radius of* L_g.

(i) *Suppose that* g *is proximal and that there is a strictly positive* ρ*-eigenfunction* $h \in C(\Sigma_2)$. *Then there is a probability measure* ν *with* $\langle h, \nu\rangle = 1$ *such that for every* $f \in C(\Sigma_2)$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \rho^{-j} L_g^j f(x) = \langle f, \nu\rangle h(x) \qquad \textit{uniformly for } x \in \Sigma_2 .$$

(ii) *Suppose* g *is irreducible. Then the above strictly positive* h *exists and for every* $f \in C(\Sigma_2)$,

$$\lim_{n\to\infty} \rho^{-n} L_g^n f = \langle f, \nu\rangle h(x) \qquad \textit{uniformly for } x \in \Sigma_2 .$$

Proof. (i) It is known that if g is proximal and $L_g 1 = 1$, then there exists a unique probability measure μ which satisfies $L_g^*(\mu) = \mu$ and for each $f \in C(\Sigma_2)$,

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} L_g^n(f)(x) = \langle f, \mu\rangle \qquad \text{uniformly on } x \tag{2.3}$$

([4, p. 287]). For the present case with a positive h, we let

$$\tilde g(x) = \frac{h(x)g(x)}{\rho h(\sigma x)} .$$

It is easy to check that $\tilde g \in H_\alpha(\Sigma_2)$ is proximal, $L_{\tilde g} 1 = 1$ and

$$\rho^{-n} L_g^n f(x) = h(x) L_{\tilde g}^n \left(h^{-1} f\right)(x) .$$

Note that if h is changed to ch, $\tilde{g}$ does not change. So, by changing the scale c, we can get a probability measure $\nu = h^{-1}\mu$. By applying (2.3) to $\tilde{g}$ and $h^{-1}f$, the assertion in (i) follows

To prove (ii) we observe that ρ is the only eigenvalue with modulus ρ. Using the quasi-compactness, it is then easy to see that $\rho^{-n}L_g^n f$ converges uniformly for every $f \in H_\alpha(\Sigma_2)$. In fact, every $f \in H_\alpha(\Sigma_2)$ can be decomposed into $f_1 + f_2$ with $f_1 \in E$ and $f_2 \in F$ where E and F are as in the definition of quasi-compactness. Moreover, suppose E is the ρ-eigenfunction space. So,

$$L_g^n f = \rho^n f_1 + L_g^n f_2$$

which implies the announced convergence because the spectral radius of L_g restricted on F is strictly smaller than ρ. The limit is the same as in (i). Observe that $H_\alpha(\Sigma_2)$ is dense in $C(\Sigma_2)$. So, the restriction $f \in H_\alpha(\Sigma_2)$ can be reduced to $f \in C(\Sigma_2)$. $\square$

To be able to apply (i), we should verify that there is a strictly positive eigenfunction for ρ. Under the condition that g has a finite number of zeros and that $L_g 1(x) > 0$ for all x, Hervé [14] gave a necessary and sufficient condition for this: either there is no invariant periodic cycle or there is a (unique) invariant periodic cycle C with

$$\prod_{y \in C} g(y) = \rho .$$

and the order of ρ equals to 1. Recall that a point x is *periodic* of order p if there exists p such that $x \in \sigma^{-p}(x)$, the *periodic cycle* determined by the periodic point x is by definition the set $\{x_0 = x, x_1, \cdots, x_{n-1}\}$ where $x_i \in \sigma^{-1}(x_{i-1})$, $i = 1, \cdots, n$ and a compact set F is said to be *invariant* if F contains all orbits of points in F. The reader can refer to [4, 13] for all these notions.

The measure ν in (i) is continuous if g has no invariant periodic cycle, and ν is discrete and supported by the cycle if g has a unique periodic cycle [4, p. 287–295]. In the case that g is irreducible, it is easy to see that the support of μ_g is the whole Σ_2 because for any $0 \le f \in C(\Sigma_2)$ not identically zero, there exists n such that $\{x : L_g^n f(x) \neq 0\} \cap \text{supp}(\mu) \neq \emptyset$. The invariance of μ_g implies that

$$\langle f, \mu_g \rangle = \langle L_g^n f, \mu_g \rangle > 0 .$$

Therefore, $\text{supp}(\mu_g) \cap \text{supp}(f) \neq \emptyset$. Since f is arbitrary, $\text{supp}(\mu_g)$ equals Σ_2.

We call

$$P_g = \log \rho .$$

the *pressure* of g ([1, 23] where it is called the pressure of $\log g$). Since the operator L_g is positive, the operator norm of L_g^n equals to $\|L_g^n 1\|$. So we have

$$P_g = \lim_{n\to\infty} \frac{1}{n} \log \left\| L_g^n 1 \right\| .$$

For a fixed g we use $P(q)$ to denote the one parameter family $P_{g^q}, q \in R$. It is easy to show that $P(q)$ is a convex function (may attain ∞).

If $g > 0$, Theorem 2 corresponds to the Ruelle-Perron-Frobenius theorem [1, 23] (see also [11, 27]). In this case, P_g is well related to the entropy via the variational principle. Recall that the *entropy* of a measure μ is defined as

$$h_\sigma(\mu) = \lim_{n\to\infty} \sum_{I_n} -\mu(I_n) \log \mu(I_n) .$$

where the I_ns are the cylinder sets of Σ_2 with bases $(x_1, \cdots x_n)$. The variational principle states that

$$P_g = \sup_{\nu \in \mathcal{I}} \left\{ h_\sigma(\nu) + \int \log g(x) d\nu(x) \right\}$$

where $\mathcal{I}$ denotes the set of σ-invariant measures μ on Σ_2 (i.e., $\mu = \mu \circ \sigma^{-1}$), and the supremum is uniquely attained by $\mu_g = h^{-1}\nu$, called the *Gibbs measure* of g, where h and ν are defined as in Theorem 2 [1, 28]. Moereover μ_g shares the following *Gibbs property*. There exists $\gamma > 0$ such that for all $x \in \Sigma_2$ and $n \geq 1$,

$$\gamma^{-1} \leq \frac{\mu_g\left(I_n(x)\right)}{\exp\left(-nP_g + \sum_{k=0}^{n-1} \log g\left(\sigma^k x\right)\right)} \leq \gamma . \tag{2.4}$$

The Gibbs measure μ_g is ergodic and even mixing.

All the preceding results can be translated onto the interval $[0, 1)$ instead of Σ_2. Each $x \in [0, 1)$ has a dyadic representation. In the case where x has two representations $x = 0.x_1 \cdots x_n 100 \cdots = 0.x_1 \cdots x_n 011 \cdots$, we will use the first representation only. Let $\iota : [0, 1) \to \Sigma_2$ be the natural embedding and let $\Sigma_2' = \iota([0, 1))$. Then we can identify $[0, 1)$ with Σ_2'. A Hölder continuous function φ on Σ_2' [an obvious adjustment of (2.1)] can be extended to a Hölder continuous function on Σ_2 and hence we can identify $H(\Sigma_2')$ and $H(\Sigma_2)$. Furthermore, Σ_2' equals Σ_2 except for countably many points, and the non-atomic property of the Gibbs measure μ_g enables us to restrict μ_g on Σ_2' without any change.

A function g on $[0, 1)$ is said to be *dyadic Hölder continuous* if $g \circ \iota^{-1} \in H(\Sigma_2)$. Equivalently there exist $K > 0$ and $0 < \alpha \leq 1$ such that

$$|g(y_1) - g(y_2)| \leq K 2^{-n\alpha} \qquad \forall\, y_1, y_2 \in I_n(x),\ x \in [0, 1),\ n \geq 1$$

where $I_n(x)$ is the dyadic interval of length 2^{-n} containing x. Let $H([0, 1))$ denote the class of all such functions, $H_\alpha([0, 1))$ the subspaces of $H([0, 1))$ corresponding to $H_\alpha(\Sigma_2)$, and $H_0([0, 1))$ the class of functions g on $[0, 1)$ corresponding to $C(\Sigma_2)$, i.e., $g \in H_0([0, 1))$ if and only if $g \circ \iota^{-1}$ is the restriction of a continuous function on Σ_2. For convenience we extend these functions periodically on the real line and call them dyadic Hölder continuous (dyadic continuous respectively). It is clear that a dyadic continuous function is bounded; it is continuous on the non-dyadic points and is right continuous and left-hand limits exist on the dyadic points.

Since $\sigma^{-1}(x) = \{S_0(x), S_1(x)\}$ where $S_0(x) = \frac{1}{2}x$, $S_1(x) = \frac{1}{2}x + \frac{1}{2}$, the transfer operator L_g is translated to

$$L_g f(x) = g\left(\frac{x}{2}\right) f\left(\frac{x}{2}\right) + g\left(\frac{x}{2} + \frac{1}{2}\right) f\left(\frac{x}{2} + \frac{1}{2}\right) .$$

The following two propositions explain why the transfer operator is useful in the study of multiperiodic functions.

Proposition 1.

Suppose $0 \leq g \in H_0([0, 1))$ *and is periodic on* R. *Then for any* $f \in H_0([0, 1))$,

$$\int_0^1 L_g^n f(x)dx = 2^n \int_0^1 f(x) \prod_{j=0}^{n-1} g(2^j x) dx .$$

Proof. For $n = 1$, by making use of a change of variables we have

$$\begin{aligned}
\int_0^1 L_g f(x)dx &= \int_0^1 g(x/2)f(x/2)dx + \int_0^1 g(x/2 + 1/2)f(x/2 + 1/2)dx \\
&= 2\int_0^{1/2} g(y)f(y)dy + 2\int_{1/2}^1 g(y)f(y)dy \\
&= 2\int_0^1 g(x)f(x)dx .
\end{aligned}$$

Suppose now the result is true for n. Then by using the same change of variable technique,

$$\int_0^1 L_g^{n+1} f(x)dx = 2^n \int_0^1 \left(L_g f(x)\right) \prod_0^{n-1} g\left(2^j x\right) dx = 2^{n+1} \int_0^1 g(x) f(x) \prod_{j=0}^{n-1} g\left(2^{j+1} x\right) dx$$

which proves the proposition. □

Proposition 2.

Suppose $0 \le g \in H([0,1))$. Let $P(q)$ be the pressure function corresponding to g.

(i) If g is irreducible, then

$$P(q) = \lim_{n\to\infty} \frac{1}{n} \log \int_0^1 \left(\prod_{j=0}^{n} g\left(2^j x\right)\right)^q dx + \log 2 .$$

(ii) If $g(x) > 0$ and g is not a constant, then $P(q)$ is analytic, strictly convex, and strictly positive for all $q \in R$.

Proof. Assertion (i) follows by taking $f = 1$ in Theorem 2 (ii) and making use of Proposition 1. To prove (ii), we first note that if

$$g(x) = Cu(2x)/u(x) \tag{2.5}$$

for some constant C and some continuous function $u > 0$, then $P(q)$ is affine; otherwise $P(q)$ is strictly convex and analytic [23, p. 63, p. 91].

Now under the assumption that g satisfies (ii), we need only see that g is not of the form (2.5) and the strict convexity and analyticity of $P(q)$ follows. Assume otherwise, then the C in (2.5) must be 1 (check $x = 0$) and the equation on g reduces to

$$\prod_{j=0}^{n} g\left(2^{-j} x\right) = \frac{u(2x)}{u\left(2^{-n+1} x\right)} .$$

Without loss of generality we can assume that $u(0) = 1$, then $G(x) = \prod_{j=0}^{\infty} g(2^{-j}x) = u(2x)$. On the other hand, from $G(x) = g(x/2)G(x/2)$, it follows that $u(2x) = g(x/2)u(x)$ and hence $g(x) = u(4x)/u(2x) = g(2x)$. We have inductively $g(x) = g(2^n x)$ for all n. It is well known that for almost all x, $(2^n x)$ is uniformly distributed, *a priori* dense in the interval $[0, 1)$, from which we deduce that g is a constant, which is a contradiction to the assumption.

For the positivity of P, we observe that $\delta_0 \in \mathcal{I}$ and the variational principle implies that $P(q) \ge h_\sigma(\delta_0) = 0$. Moreover, the above inequality must be strict by the fact that the maximality of $P(q)$ in the variational principle is not attained at δ_0 which, being discrete, is not the Gibbs measure. □

We conclude this section by giving an illustrating example. Let $g(x) = \cos^2 \pi x$, then $g(1/2) = 0$ and g is not proximal because the orbits of 0 and 1 are the points themselves and they are disjoint. On the other hand, if we consider $g_1(x) = \cos^2 2\pi x$, then $O(0) = \{0, \frac{1}{2}\}$, $O(1) = \{1, \frac{1}{2}\}$, and $O(\frac{1}{2}) = \{\frac{1}{2}\}$. It is easy to show that $O(x) \cap O(y) \neq \emptyset$ for any $x, y \in [0, 1]$. Consequently, g_1 is proximal but not irreducible.

It is easy to see that L_g has maximal eigenvalue 1 and the corresponding eigenfunction is the constant function 1. L_{g_1} also has maximal eigenvalue 1 in view of the calculation (for P(1)) in the following, but there is no simple form for the eigenfunction.

The pressure function of g (and also of g_1) is given by

$$P(q) = \begin{cases} (1-2q)\log 2 & \text{if } \ 0 \le q \le 1/2 \\ 0 & \text{if } \ q \ge 1/2 . \end{cases} \tag{2.6}$$

To see this, we use $P(q) = \lim_{n\to\infty} \|L^n_{g^q} 1(x)\|^{1/n}$. Note that

$$L^n_{g^q} 1(x) = \sum_{y\in\sigma^{-n}(x)} \left(\prod_{k=0}^{n-1} g\left(2^k y\right)\right)^q = \sum_{y\in\sigma^{-n}(x)} \left(\frac{\sin \pi 2^n y}{2^n \sin \pi y}\right)^q .$$

Since $y \in \sigma^{-n}(x)$ if and only if $y = k/2^n + x/2^n$ for some $0 \le k < 2^q$, if $x = 1/2$, $\sin \pi 2^n y = 1$ for every $y \in \sigma^{-n}(x)$, then

$$L^n_{g^q} 1(x) = \sum_{y\in\sigma^{-1}(x)} \left(\frac{1}{2^n \sin \pi y}\right)^{2q} .$$

If $x \neq 1/2$, $L^n_{g^q} 1(x)$ is always bounded by the right-hand member in the above expression. By using $\sin y = \sin(k/2^n + x/2^n) \approx k/2^n$, we have

$$\left\| L^n_{g^q} 1(x) \right\| \approx \sum_{k=1}^{2^n} \left(\frac{1}{k}\right)^{2q} \begin{cases} \approx 2^{(1-2q)n} & \text{if} \quad 0 \le q < 1/2 \\ \approx n & \text{if} \quad q = 1/2 \\ \text{converges} & \text{if} \quad q > 1/2 \end{cases}$$

and (2.6) follows. This example also shows that in general $P(q)$ is neither strictly convex nor analytic.

3. Asymptotics of Multiperiodic Functions

Let $g \in H([0, 1))$ with $g(0) = 1$. Then the product

$$G(x) = \prod_{k=1}^{\infty} g\left(\frac{x}{2^k}\right) \tag{3.1}$$

converges uniformly on compact subsets of R and $G > 0$ provided that $g > 0$. The function G defined by the product (3.1) is called a *multiperiodic function* in [16]. Observe that on every bounded interval G is Hölder continuous and is of the same order as g.

Theorem 3.

Let $0 \le g \in H([0, 1))$ with $g(0) = 1$. Let $P(q)$ be the pressure function of g^q. For any $s \in R$ and any $q > 0$, we have the inequality

$$\limsup_{T\to\infty} \frac{\log \int_1^T x^s G^q(x)dx}{\log T} \le \max\left\{0, s + \frac{P(q)}{\log 2}\right\} .$$

If in addition g is proximal without an invariant cycle and has at most finitely many zeros, and if $L_g 1 > 0$, then the limit exists and equality holds in the above expression.

Proof. By using $G(x) = G(2^{-n}x)\prod_{k=1}^n g(2^{-k}x)$, we have

$$\begin{aligned} \int_{2^n}^{2^{n+1}} G(x)^q dx &= \int_{2^n}^{2^{n+1}} G\left(2^{-n}x\right)^q \prod_{k=1}^{n} g\left(2^{-k}x\right)^q dx \\ &= 2^n \int_1^2 G(x)^q \prod_{k=0}^{n-1} g(2^k x)^q dx \qquad \text{(change of variable)} \end{aligned}$$

$$= 2^n \int_0^1 G(1+x)^q \prod_{k=0}^{n-1} g\left(2^k x\right)^q dx \qquad (g \text{ has period } 1)$$

$$= \int_0^1 L_{g^q}^n f(x) dx \qquad \text{(Proposition 1)}$$

where $f(x) = G(1+x)^q$. For $T = 2^N$, we have

$$\int_1^T x^s G(x)^q dx \approx \sum_{n=0}^{N-1} 2^{sn} \int_{2^n}^{2^{n+1}} G(x)^q dx = \sum_{n=0}^{N-1} 2^{n(s+\frac{P(q)}{\log 2})} \rho_{g^q}^{-n} \int_0^1 L_{g^q}^n f(x) dx \,. \tag{3.2}$$

where ρ_q is the spectral radius of L_{g^q}. Note that the eigenvalue ρ^q corresponding to L_{g^q} is of finite multiplicity (by Theorem 1). It follows that $||L_{g^q}^n f|| = O(n^{m-1}\rho_q^n)$, so that $\int_0^1 L_g^n f(x)dx = O(n^{m-1}\rho_q^n)$ and there is a constant C such that

$$\int_1^T x^s G(x)^q dx \leq C \sum_{n=0}^{N-1} n^{m-1} 2^{n\left(s+\frac{P(q)}{\log 2}\right)}$$

According to $s + \frac{P(q)}{\log 2} > 0, = 0$, or < 0, the last sum is bounded by $N^{m-1}2^{N(s+\frac{P(q)}{\log 2})}$, N^m, or $O(1)$. Thus, the inequality in the theorem follows.

To prove the reverse inequality, we only need to treat the case $s + \frac{P(q)}{\log 2} \geq 0$. We first observe that the hypotheses implies that L_{g^q} has a strictly positive eigenfunction [14]. For any $0 < \delta < 1$, Theorem 2 (i) implies that for large N,

$$\int_0^1 \sum_{n=\delta N}^{N-1} \rho_q^{-n} L_{g^q}^n f(x) dx = N \int_0^1 \frac{1}{N} \sum_{n=\delta N}^{N-1} \rho_q^{-n} L_{g^q}^n f(x) dx \geq C N \langle f, \nu \rangle > 0$$

where $C = (1-\delta)\int h d\nu$. Since g has finitely many zeros, $f(x) = G^q(1+x)$ has at most countably many zeros. Also the additional assumption that g has no invariant periodic cycle implies that μ is a continuous measure. Hence, $\langle f, \nu \rangle \neq 0$. By (3.2) there exists a constant $C_\delta > 0$ such that

$$\int_1^T x^s G(x)^q dx \geq 2^{N\delta\left(s+\frac{P(q)}{\log 2}\right)} \int_0^1 \sum_{n=\delta N}^{N-1} \rho_q^{-n} L_{g^q}^n f(x) dx \geq C_\delta N 2^{N\delta\left(s+\frac{P(q)}{\log 2}\right)} \,.$$

It follows that

$$\liminf_{T\to\infty} \frac{\log \int_1^T x^s G^q(x) dx}{\log T} \geq \delta \left(s + \frac{P(q)}{\log 2}\right)$$

which gives the reverse inequality as $\delta \to 1$. □

We remark that the proximality alone is not sufficient for the equality of the identity in the theorem. An easy example is to check the $P(g)$ of $g(x) = \cos^2 2\pi x$ in the last section.

If g is irreducible, by Theorem 2 (ii), $\int_0^1 L_{g^q} f \approx \rho_q^n$. It follows from (3.2) that

$$\int_1^{2^N} x^s G(x)^q dx \approx \sum_{n=0}^{N-1} 2^{n\left(s+\frac{P(q)}{\log 2}\right)} \,.$$

Consequently, we have the following.

Theorem 4.

Suppose $0 \le g \in H([0,1))$ *is irreducible and* $g(0) = 1$. *Let*

$$\alpha = s + \frac{P(q)}{\log 2} .$$

Then for $q \in R$,

$$\int_1^T x^s G^q(x)dx \approx \begin{cases} T^\alpha & \text{if} \quad \alpha > 0 \\ \log T & \text{if} \quad \alpha = 0 \\ O(1) & \text{if} \quad \alpha < 0 \end{cases} \tag{3.3}$$

Corollary 1.

Let $g \in H([0,1))$ *with* $g(0) = 1$. *Suppose* $\inf_x g(x) > 0$, *then we have*

$$\int_0^T G(x)^q dx \approx T^{\frac{P(q)}{\log 2}} .$$

In particular $G \notin L^q(R)$ *for any* $q > 0$.

Proof. The last statement follows from the fact that $P(q) > 0$ for $q > 0$, which is a consequence of the variational principle (take ν to be δ_0). □

Theorem 4 gives a partial answer to Problem 1.3 in [16]. Note that the zeros of g actually play a crucial rule in the behavior of $G(x)$ as $x \to \infty$. Besides the above two theorems, we can see this more explicitly from the following theorem. Such g appears frequently in wavelet theory (see Section 5).

Theorem 5.

Let $g(x) = |\cos \pi x|^N g_1(x)$ *where* $g_1 \in H([0,1))$ *is irreducible and* $g_1(0) = 1$. *Let* $P_1(q)$ *be the pressure of* g_1^q *and let*

$$\alpha = s + \frac{P_1(q)}{\log 2} - Nq .$$

Then (3.3) holds the same.

Proof. The proof will be based on Theorem 4 and the formula

$$\prod_{k=1}^{\infty} \cos \frac{\pi x}{2^k} = \frac{\sin \pi x}{\pi x} .$$

The Hölder continuity of g at 0 implies that if $2^n \le x \le 2^{n+1}$,

$$C^{-1} \le \prod_{k=n+1}^{\infty} g_1\left(2^{-k}x\right) \le C$$

for some constant $C > 0$ depending only on g_1. Hence,

$$\begin{aligned} \int_1^{2^n} x^s G^q(x)dx &= \int_1^{2^n} x^s G_1^q(x) \frac{|\sin \pi x|^{Nq}}{\pi x^{Nq}} dx \\ &\approx \sum_{k=1}^{n} 2^{k(1+s-Nq)} \int_0^1 \left(\prod_{j=0}^{k} g_1\left(2^j y\right) \right)^q \left|\sin \pi 2^k y\right|^{Nq} dy . \end{aligned}$$

Let $I = [i/2^k, (i+1)/2^k)$ be a dyadic interval of $[0, 1)$ and y_I an arbitrary point in this interval. The dyadic Hölder condition of $\log g_1$ implies that

$$1 \leq \frac{\max_I \prod_{j=0}^{k} g_1\left(2^j y\right)}{\min_I \prod_{j=0}^{k} g_1\left(2^j y\right)} \leq C$$

where C is a constant independent of k and I. By the mean value theorem for integral,

$$\begin{aligned}\int_0^1 \left(\prod_{j=0}^{k} g_1\left(2^j y\right)\right)^q \left|\sin \pi 2^k y\right|^{Nq} dy &= \sum_I \left(\prod_{j=0}^{k} g_1\left(2^j y_I\right)\right)^q \int_I \left|\sin \pi 2^k y\right|^{Nq} dy \\ &= C' 2^{-k} \sum_I \left(\prod_{j=0}^{k} g_1\left(2^j y_I\right)\right)^q \\ &\approx \int_0^1 \left(\prod_{j=0}^{k} g_1\left(2^j y\right)\right)^q dy .\end{aligned}$$

If we denote G_1 the multiperiodic function defined by g_1, then

$$\int_1^T x^s G^q(x) dx \approx \sum_{k=1}^{n} 2^{k(1+s-Nq)} \int_0^1 \left(\prod_{j=0}^{k} g_1\left(2^j x\right)\right)^q dx \approx \int_1^T x^{(s-Nq)} G_1^q(x) dx$$

(the last approximation is in the proof of Theorem 3). By applying Theorem 4 to G_1, the assertion of the theorem follows. □

Remark 1.

Theorem 3 implies that $x^s G(x)^q$ is integrable on $[1, \infty)$ if $s + \frac{P(q)}{\log 2} < 0$. In particular, $G \in L^q$ if $P(q) < 0$. Corollary 1 shows that for G^q to be integrable, g must be zero somewhere. Under the condition $g(1/2) = 0$, Hervé [14] proved that if $P(q) = 0$ and the spectral radius of L_{g^q} is a simple eigenvalue, then $G \in L^q$. He also proved that this condition is necessary if, moreover, every invariant compact set associated with g contains 0 *or* 1. □

Remark 2.

For $q > 0$ and $s \in R$, we define $L_s^q(R)$ to be the space of F on R such that

$$\int_R (1 + |\xi|)^{qs} |F(\xi)|^q d\xi < \infty .$$

When $q = 2$, $L_s^2(R)$ is just the Fourier transformation of the Sobolev space $H_s^2(R)$. If G is a multiperiodic function as in Theorem 5, then $G \in L_s^q(R)$ if and only if $s < N - \frac{P_1(q)}{q \log 2}$. □

Now we are going to study the limit (1.2). For an (extended value) convex function f on R, we define the *convex conjugate* (also called the *Legendre transformation*) of f by

$$f^*(\alpha) = \sup\{\alpha x - f(x) : x \in R\} .$$

Note that f^* is a convex function and if f is differentiable, then $f^*(\alpha) = \alpha x - f(x)$ where $\alpha = f'(x)$. For a fixed $\alpha \in R$, let

$$K_\alpha = \left\{ x \in [0, 1) : \lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \log g\left(2^j x\right) = \alpha \right\} .$$

Theorem 6.

Let $g \in H([0,1))$. *Suppose* $\inf\{g(x) : x \in [0,1)\} > 0$, $g(0) = 1$ *and* g *is not identically constant. Then for* $\alpha \in R$ *such that* $-\infty < P^*(\alpha) < \infty$, *we have*

$$\dim K_\alpha = -\frac{P^*(\alpha)}{\log 2}$$

where $\dim K_\alpha$ *means the Hausdorff dimension of* K_α.

Proof. If we take $\alpha = P'(q)$, we claim that μ_q is concentrated in K_α. Indeed since μ_q is ergodic we can apply the ergodic theorem to the Gibbs measure μ_q:

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log g\left(2^j x\right) = \int_0^1 \log g(x) d\mu_q(x) \tag{3.4}$$

for μ_g almost all $x \in [0, 1]$. From the variational principle, μ_q is the unique invariant measure which attains the supremum

$$P(q) = h_\sigma\left(\mu_q\right) + q \int_0^1 \log g(x) d\mu_q(x) .$$

This combines with the property of the convex conjugate that $P(q) = -P^*(\alpha) + q\alpha$ implies that $h_\sigma(\mu_q) = -P^*(\alpha)$ and $\alpha = \int_0^1 \log g(x) d\mu_q(x)$. Hence, $\mu_q(K_\alpha) = 1$.

Next we note that the Gibbs property of μ_q implies that

$$\lim_{n\to\infty} \frac{\log \mu_q\left(I_n(x)\right)}{n} = P(q) - q \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log g\left(2^j x\right)$$

provided that either one of the limits exists. Hence, we have for $x \in K_\alpha$,

$$\lim_{n\to\infty} \frac{\log \mu_q\left(I_n(x)\right)}{\log 2^{-n}} = \frac{P(q) - q\alpha}{\log 2} = \frac{-P^*(\alpha)}{\log 2} .$$

We have seen that μ_q is concentrated on the set

$$K_\alpha = \left\{x \in [0,1] : \lim_{n\to\infty} \frac{\log \mu_q\left(I_n(x)\right)}{\log 2^{-n}} = \frac{-P^*(\alpha)}{\log 2}\right\} .$$

The mass distribution theorem (Proposition 4.9 in [10]) implies that $\dim K_\alpha = -P^*(\alpha)/\log 2$ which completes the proof of the theorem. □

Remark 3.

By using the uniformly distribution property of $2^n x$ *(mod 1), it is easy to show that if* $g(x) > 0$, *then*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log g\left(2^j x\right) = \int_0^1 \log g(t) dt$$

for almost all $x \in [0, 1)$. *See also [16] for more general cases.* □

To illustrate this theorem, we let $g(x) = (1 + 2\sin^2 \pi x)$. Note that the multiperiodic function defined by $\cos^2 \pi x \sqrt{g(x)}$ is the modulus of the Fourier transformation of the well-known Daubechies scaling function D_4 (denoted by f_2 in Section 5). Figure 1 is the graph of $G(x) = \prod_{k=0}^{\infty} g(\frac{x}{2^k})$.

Figure 2 is the graph of $h_n(x) = \frac{1}{n}\sum_{j=0}^{m-1}\log(1+2\sin^2\pi x)$ for $n = 5, 7, 9, 11$. Figure 3 is the graph of $P(q)$ and $P^*(\alpha)$. In view of the above remark, $\alpha_0 = P'(0) = \int_0^1 \log g(t)dt = 0.62381$ and $P^*(\alpha_0) = 1$ as in the picture. Also, in the picture the domain of $P^*(\alpha)$ is in-between 0 and 0.85 which agrees with the values of $\{h_n(x)\}$. The set of x that assumes such values are becoming more rare as the values are off from the mean α_0.

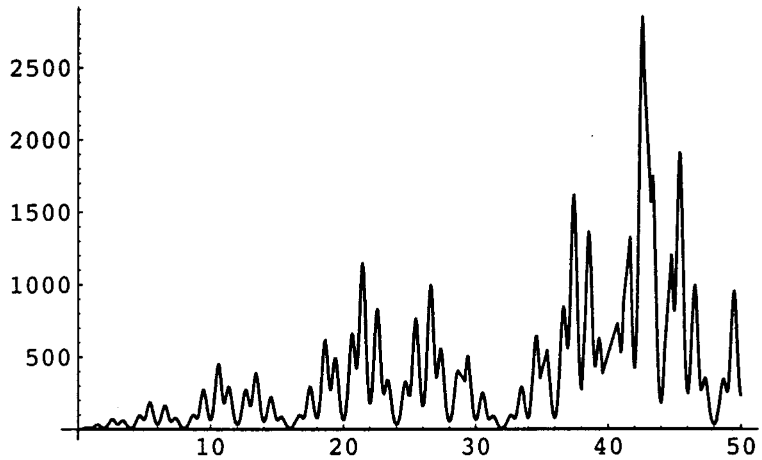


FIGURE 1. $G(x) = \prod_{k=0}^{\infty}\left(1+2\sin^2\left(\frac{\pi x}{2^k}\right)\right)$.

4. Calculation Techniques

In view of applications of the previous results, we should calculate the spectral radius of a transfer operator. However, it is in general difficult. We present here an approximation approach.

Let us first consider the case where g is a dyadic step function. If $g > 0$ takes only two values on $[0, 1/2)$ and $[1/2, 1)$, i.e., $g(x) = a_0\chi_{[0,1/2)} + a_1\chi_{[1/2,1)}$ with $a_0, a_1 > 0$. Then for $x = 0.x_1x_2\cdots$,

$$\prod_{k=0}^{n-1} g\left(2^k x\right) = a_{x_1}\cdots a_{x_n} .$$

Hence,

$$\int_0^1\left(\prod_{k=0}^{n-1} g\left(2^k x\right)\right)^q dx = 2^{-n}\sum_{x_1,\cdots,x_n}\left(a_{x_1}\cdots a_{x_n}\right)^q = \left(\frac{a_0^q + a_1^q}{2}\right)^n .$$

Consequently by Proposition 2,

$$P(q) = \log\left(a_0^q + a_1^q\right) .$$

Theorem 7.

Let $m \geq 0$. Suppose $g > 0$ takes $2^{(m+1)}$ values on the dyadic intervals of length $2^{-(m+1)}$ and let $G_m(x) = \prod_{k=0}^{m-1} g(2^k x)$. Let M_q be the $2^m \times 2^m$ matrix defined by

$$M_q = \left[G_m^q\left(\frac{i2^m + j}{2^{2m}}\right)\right], \qquad 0 \leq i, j \leq 2^m - 1$$

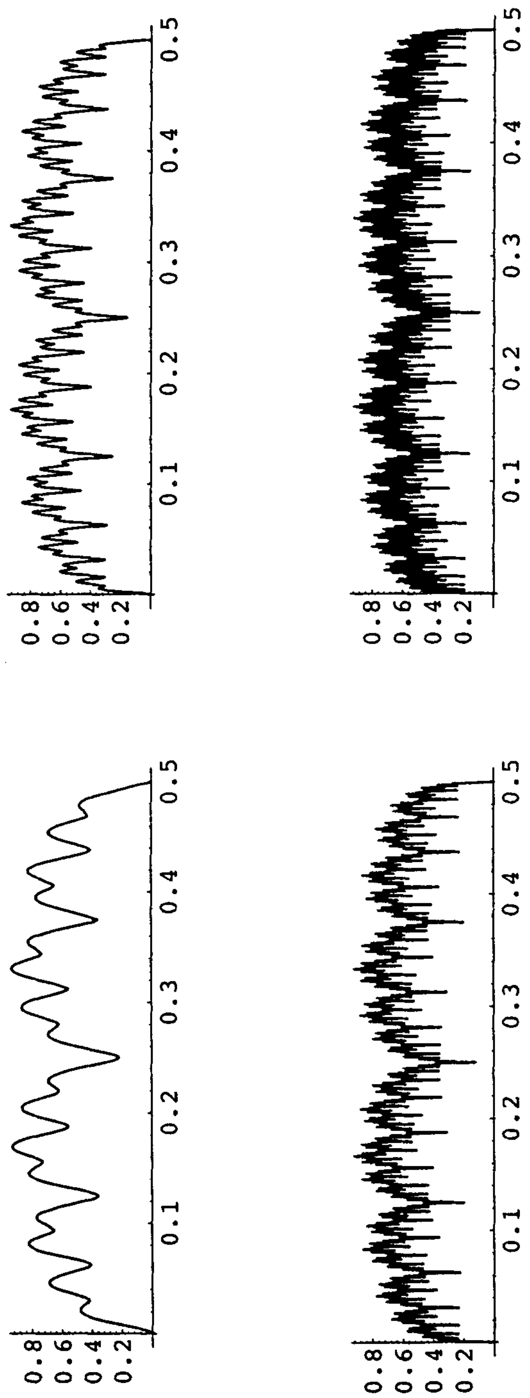
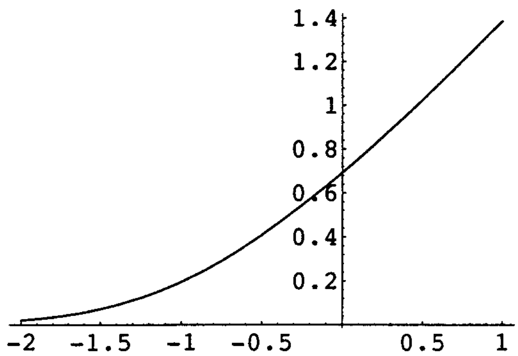


FIGURE 2. $h_n(x) = \frac{1}{n}\sum_{j=0}^{n-1}\log\left(1+\sin^2\left(\frac{\pi x}{2^k}\right)\right)$, $n = 5, 7, 9, 11$.



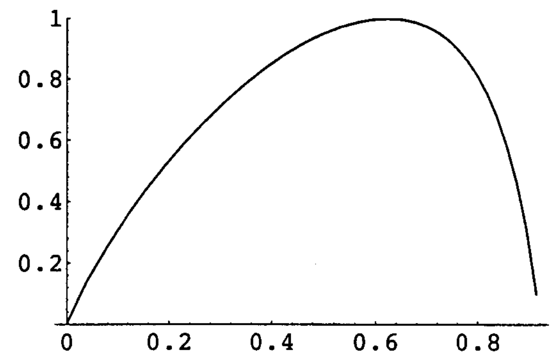


FIGURE 3. $P(q)$ and $P^*(\alpha)$.

and let λ_q is the maximal eigenvalue. Then

$$P(q) = \frac{\log \lambda_q}{m}.$$

Proof. Suppose $g > 0$ takes 2^{m+1} values on the dyadic intervals of length $2^{-(m+1)}$:

$$g(x) = a(x_1, \cdots, x_{m+1}) \qquad \text{if} \quad x = 0.x_1 \cdots x_{m+1} \cdots .$$

Then for $x = 0.x_1 \cdots x_m y_1 \cdots y_m \cdots$, by shifting the coordinates we have

$$G_m(0.x_1 \cdots x_m y_1 \cdots y_m) = \prod_{k=0}^{m-1} g\left(2^k x\right) = \prod_{k=1}^{m} a(x_k, \cdots, x_m, y_1, \cdots, y_k).$$

Inductively if $x = 0.x_1 \cdots x_m y_1 \cdots y_m \cdots w_1 \cdots w_m z_1 \cdots z_m \cdots$ [the first $(n+1)m$ coordinates are regrouped into $(n+1)$ blocks], then

$$\prod_{k=0}^{nm-1} g\left(2^k x\right) = G_m(0.x_1 \cdots x_m y_1 \cdots y_m) \cdots G_m(0.w_1 \cdots w_m z_1 \cdots z_m).$$

It follows that

$$\int_0^1 \left(\prod_{k=0}^{nm-1} g\left(2^k x\right)\right)^q dx = 2^{-(n+1)m} \sum G_m^q(0.x_1 \cdots x_m y_1 \cdots y_m) \cdots G_m^q(0.w_1 \cdots w_m z_1 \cdots z_m)$$

where the sum is taken over all the possible choices of 0s and 1s of the variables in the n-products. To arrange this in a matrix form, we let

$$M_q = \left[G_m^q\left(\frac{i2^m + j}{2^{2m}}\right)\right], \quad 0 \le i, j \le 2^m - 1.$$

Then

$$\int_0^1 \left(\prod_{k=0}^{nm-1} g\left(2^k x\right)\right)^q dx = 2^{-(n+1)m} 1 M_q^n 1^t$$

where $1 = [1, \ldots, 1]$. The statement of the theorem follows from this expression and Proposition 2. □

Example 1. Suppose $g > 0$ takes four values on the four dyadic intervals $[0, 1/4), \cdots, [3/4, 1)$. Let $g(x) = a(x_1, x_2)$ if $x = 0.x_1x_2\cdots$. Then according to the above notation, $G_1(0.x_1y_1) = a(x_1, y_1)$ and

$$M_q = \begin{pmatrix} a^q(0,0) & a^q(0,1) \\ a^q(1,0) & a^q(1,1) \end{pmatrix}$$

and we can calculate the maximal eigenvalue of M_q and then the pressure of g.

Example 2. For $x = 0.x_1x_2x_3\cdots$, let $g(x) = a(x_1, x_2, x_3)$ with

$$a(0,0,0) = a(1,1,1) = 1, \qquad a(0,0,1) = a(1,1,0) = \frac{3}{4}$$
$$a(0,1,0) = a(1,0,1) = \frac{1}{2}, \qquad a(0,1,1) = a(1,0,0) = \frac{1}{4}\,.$$

Then g takes 8 values and $m = 2$ in the above theorem and $G_2(0.x_1x_2y_1y_2) = a(x_1, x_2, y_1)a(x_2, y_1, y_2)$. Write $G_2^q(0.x_1x_2y_1y_2) = b(x_1x_2; y_1y_2)$, the corresponding M_q is the 4×4 matrix defined by

$$\begin{aligned} M_q &= \begin{pmatrix} b(00;00) & b(00;01) & b(00;10) & b(00;11) \\ b(01;00) & b(01;01) & b(01;10) & b(01;11) \\ b(10;00) & b(10;01) & b(10;10) & b(10;11) \\ b(11;00) & b(11;01) & b(11;10) & b(11;11) \end{pmatrix} \\ &= \frac{1}{16^q}\begin{pmatrix} 16^q & 12^q & 6^q & 3^q \\ 2^q & 4^q & 3^q & 4^q \\ 4^q & 6^q & 4^q & 2^q \\ 3^q & 12^q & 12^q & 16^q \end{pmatrix}. \end{aligned}$$

The eigenvalues λ_q can be calculated and the graphs of $P(q) = \log \lambda_q$ and $P^*(\alpha)$ are drawn.

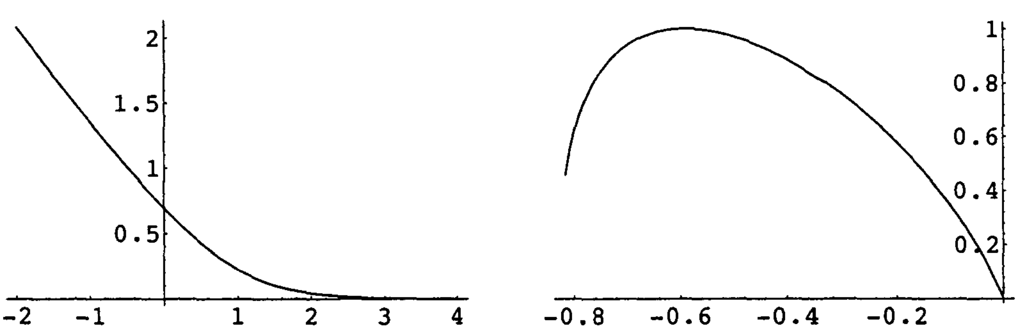


FIGURE 4. $P(q) = \log \lambda_q$ and $P^*(\alpha)$.

The above dyadic functions can be used to approximate any other functions $g \in H([0, 1))$. Specifically, we have the following.

Proposition 3.

Suppose $f, g > 0$ are in $H([0,1))$, $q \in R$,

(i) *If $f \leq g$, then $P_f(q) \leq P_g(q)$.*

(ii) *If $|\log f(x) - \log g(x)| \leq \epsilon$, then $|P_f(q) - P_g(q)| \leq |q|\epsilon$.*

Proof. (i) follows immediately from the definition. (ii) is a simple consequence of the variational principle. It can also be proved directly. In fact, by assumption,

$$\left|\sum_{k=0}^{n-1} \log f\left(2^k x\right) - \sum_{k=0}^{n-1} \log g\left(2^k x\right)\right| \le n\epsilon ,$$

so that

$$e^{-n\epsilon} \le \frac{\prod_{k=0}^{n-1} f\left(2^k x\right)}{\prod_{k=0}^{n-1} g\left(2^k x\right)} \le e^{n\epsilon} .$$

This combined with Proposition 2 yields the desired inequality. □

Remark 4.

The above approximation also holds if f and g are replaced by functions of the form in Theorem 5 (with the same factor $|\cos\pi x|^N$). Furthermore, the approximation holds the same if $\beta = 2$ is replaced by any integer $\beta \ge 2$. □

The next method is especially useful for larger integral β (e.g., $\beta = 3$ in connection with the Fourier transformation of the Cantor measure), we hence formulate it with a more general β. The multiperiodic function is defined by $G(x) = g(x/\beta)G(x/\beta)$ where $\beta \ge 2$ is an integer. Given a function $0 \le g \in H([0, 1))$, the transfer operator becomes

$$L_g f(x) = \sum_{j=0}^{\beta-1} g\left(\frac{x}{\beta} + \frac{j}{\beta}\right) f\left(\frac{x}{\beta} + \frac{j}{\beta}\right) .$$

Proposition 4.

Let $0 \le g \in H([0, 1))$ and $\beta \ge 2$ be an integer. Let ρ be the spectral radius of L_g.

(i) If $m \le L_g^r 1 \le M$ for some integer r and some constants m, M (depending on r), then

$$m \le \rho^r \le M .$$

(ii) Let a_n be the Fourier coefficients of g, then

$$a_0 - \sum_{n\neq 0} |a_{\beta n}| \le \frac{\rho}{\beta} \le a_0 + \sum_{n\neq 0} |a_{\beta n}| .$$

In particular, if $a_{\beta n} = 0$ for all $n \neq 0$, then $\rho = \beta a_0 = \beta \int_0^1 g(x)dx$.

(iii) If g is periodic and continuously differentiable, then

$$\left|\frac{\rho}{\beta} - a_0\right| \le \frac{\max|g'(x)|}{\beta} .$$

Proof. (i) By iterating $L_g^r 1 \le M$, we get $L_g^{nr} 1 \le M^n$ for any $n \ge 1$. That $\rho = \lim_{n\to\infty} \|L_g^{nr} 1\|^{1/n}$ implies that M is the upper bound of ρ^r. In the same way, we can obtain the lower bound of ρ^r.

(ii) By developing g as a Fourier series, we have

$$L_g 1(x) = \sum_{n=-\infty}^{\infty} a_n e^{2\pi i n \frac{x}{\beta}} \left(\sum_{j=0}^{\beta-1} e^{2\pi i n \frac{j}{\beta}}\right) .$$

Observe that the second sum equals β if n is a factor of β and equals 0 otherwise. So,

$$L_g 1(x) = \beta \sum_{m=-\infty}^{\infty} a_{\beta m} e^{2\pi i m x} . \tag{4.1}$$

Then (i) with $r = 1$ applies.

(iii) Let t_β be the best approximation (under the supremum norm) of g by trigonometric polynomial of order $\beta - 1$. Let $\delta = \beta^{-1}\|g'\|_\infty$. Then, by the Jackson theorem [29, p. 115],

$$t_\beta - \delta \le g(x) \le t_\beta + \delta\,.$$

Consequently

$$L_{t_\beta}1(x) - \beta\delta \le L_g 1(x) \le L_{t_\beta}1(x) + \beta\delta\,.$$

By (4.1) $L_{t_\beta}1(x) = a_0$ and the result follows from (i). □

5. Applications

A *dilation equation* is a functional equation of the form

$$f(t) = \sum_n c_n f(2t-n) \tag{5.1}$$

where $\{c_n\}$ is a given finite sequence of real (or complex) numbers. Here the solution f is regarded as a distribution (in the Schwartz sense) and is called a *scaling distribution.* We are interested in the compactly supported solution. It is known that a necessary and sufficient condition for existence of such a solution is $\sum_n c_n = 2^m$ where m is an integer (see, e.g., [7]). The Fourier transform $\hat{f}$ of such f is an entire function of exponential type by the Paley-Wiener Theorem. Its restriction on the real line is a multiperiodic function

$$\hat{f}(x) = p(x/2)\hat{f}(x/2) \quad \text{with} \quad p(x) = \frac{1}{2}\sum_n c_n e^{2\pi i n x}\,.$$

If $\sum c_n = 2$, we have the following expression of $\hat{f}$ in terms of the polynomial p

$$\hat{f}(\xi) = \prod_{n=1}^{\infty} p\left(2^{-n}\xi\right)$$

where $p(0) = 1$. If $\sum c_n = 2^m, m \ne 1$, then

$$f = F^{(m-1)} \quad \text{and} \quad \hat{f}(x) = (ix)^{m-1}\hat{F}(x)$$

where F satisfies $F(t) = 2^{-(m-1)}\sum_n c_n F(2t-n)$. Again the scale 2 in Equation (5.1) can be replaced by an integer $\beta > 2$ and the above holds only with some obvious modification. Next, we give some concrete examples. We can find some other examples in [12].

Example 3. Let $g(x) = \cos^2 \pi x$ and $G(x) = \prod_{k=1}^{\infty} g(x/2^k)$. We can consider this as $g(x) = |p(x)|^2$ where p is the polynomial corresponding to the dilation equation

$$f(t) = f(2t) + f(2t-1)\,.$$

The solution is $f = \chi_{[0,1]}$. As a simple demonstration of Proposition 4 (ii), we note that the Fourier series of $g(x)$ satisfies $\sum_{n\ne 0}|a_{2n}| = 0$, hence

$$\rho = 2a_0 = 2\int_0^1 g(x)dx = 1$$

and the pressure $P(1)$ of g equals to 0. The complete calculation of the pressure function $P(g)$ has been given at the end of Section 2.

Example 4. Let $g(x) = \cos^2 \pi x$ and $G(x) = \prod_{k=1}^{\infty} g(x/3^k)$. The $g(x) = |p(x)|^2$ where p is the polynomial corresponding to the dilation equation

$$f(t) = \frac{3}{2} f(3t) + \frac{3}{2} f(3t-2) .$$

Then f is the (Schwartz) derivative of the classical Cantor measure. Similar to the above example, the Fourier series of $g(x)$ satisfies $\sum_{n \neq 0} |a_{3n}| = 0$. Proposition 4 (ii) implies that $\rho = 3a_0 = 3/2$; the pressure $P(1)$ of g hence is equal to $\log \frac{3}{2}$.

We next claim that g is irreducible (adjusted the definition to $[0, 1]$). Let $S_0 x = \frac{x}{3}$, $S_1 x = \frac{x}{3} + \frac{1}{3}$, and $S_2 x = \frac{x}{3} + \frac{2}{3}$. For $J = (j_1, \cdots, j_n)$, let $S_J = S_{j_n} \cdots S_{j_1}$, then

$$S_J x = \frac{x}{3^n} + \frac{k}{3^n}$$

for some $0 \leq k < 3^n$. By using that $1/2$ is the only zero of g and that $S_J x = 1/2$ implies $x = 1/2$, we deduce that for every $x \neq 1/2$, the orbit is the whole $[0, 1]$. Also the orbit of $1/2$ is the whole interval $[0, 1]$ because $S_0(1/2) = 1/6 \neq 1/2$ and becomes the case $x \neq 1/2$. This proves the claim. Now by Theorem 4 we have for $\alpha = s + \log(3/2)/\log 3$,

$$\int_1^T x^s G(x) dx \approx \begin{cases} T^\alpha & \text{if} \quad \alpha > 0 \\ \log T & \text{if} \quad \alpha = 0 \\ O(1) & \text{if} \quad \alpha < 0 \end{cases}$$

This implies that $f \in H^s(R)$ (Sobolev space) iff $s < \alpha$. In particular, $\alpha = 0$ corresponds to $s = -1 + \log 2/\log 3$. So, $s/2 \approx -1.84535$ is the Sobolev exponent of f.

If we take $q = 2$, then the Fourier coefficients of g^2 are supported by $\{0, \pm 2 \pm 4\}$ and Proposition 4 (ii) can be applied:

$$\rho = 3 \int_0^1 \cos^4 \pi x dx, = \frac{1 \cdot 3}{2 \cdot 4} .$$

Let s_2 be the supremum of s such that $\int_1^T x^s G(x)^2 dx < \infty$, then by Theorem 4,

$$s_2 = -\frac{\log \left(3 \int_0^1 \cos^{2^4} \pi x dx\right)}{\log 3} \approx -0.10721 .$$

It follows that the Sobelev exponent of $f * f$ is approximately -0.053605

Example 5. Consider the random series

$$\sum_{n=1}^{\infty} \frac{\epsilon_n}{2^n}$$

where $\{\epsilon_n\}$ is an i.i.d. random sequence where each ϵ_n takes values 0 or 1 with probability a_0 and a_1, respectively. We are interested in the probability measure μ corresponding to the sum of the series. The variable ϵ_n is called the Bernoulli random variable and the corresponding measure is called the Bernoulli convolution measure. The measure μ is concentrated on the interval $[0, 1]$. We can also put it into the form in (5.1)

$$\mu(E) = a_0 \mu(2E) + a_1 \mu(2E - 1)$$

If $a_j = 1/2$, then μ is the Lebesgue measure on [0, 1]. Otherwise, μ is singular. The Fourier transformation of μ is of the form

$$\hat{\mu}(x) = \prod_{n=1}^{\infty} p\left(2^{-n}x\right) \qquad \text{with} \qquad p(x) = \frac{1}{2}\left(a_0 + a_1 e^{2\pi i x}\right) .$$

Let $g(x) = |p(x)|^2$, then g contains no terms of the form $e^{\pm 2\pi i(2n)x}$, $n > 0$. By Proposition 4 (ii), we have $\rho = \frac{1}{4}(a_0^2 + a_1^2)$ so that

$$P(1) = \frac{\log\left((1/4)\left(a_0^2 + a_1^2\right)\right)}{\log 2} = -2 + \frac{\log\left(a_0^2 + a_1^2\right)}{\log 2} .$$

Since $p(x) \neq 0$, then by Theorem 4 we have

$$s < -1 + \frac{\log\left(a_0^2 + a_1^2\right)}{2\log 2} \quad \text{if and only if} \quad \int_0^{\infty} x^{2s} G(x)dx < \infty .$$

It follows that $\mu \in H^s(R)$ for such s and only for such s.

We remark that in [18], by using an entirely different method we can obtain the exact value of $P(1)$ for the class of self-similar measures satisfies the open set condition which includes the measure considered here.

Example 6. In wavelet theory we are interested in the solution f of (5.1) which is a compactly supported L^2, or continuous or differentiable function. The condition on the coefficients are more conveniently assumed to be $\sum c_n = 2$. A basic question in the theory is to estimate the regularity of f. Very often it reduces to estimate the asymptotic behavior of $\hat{f}(x)$ as $x \to \pm\infty$. Daubechies [5] constructed the following class of scaling functions, which has become classic,

$$\hat{f}_N(x) = \prod_{k=1}^{\infty} m_N\left(\frac{x}{2^k}\right)$$

with $m_N(x) = \left[\frac{1}{2}(1 + e^{2\pi i x})\right]^N Q_N(x)$ where $N \geq 1$ is an integer and Q_N is a polynomial such that

$$|Q_N(x)|^2 = p_N(\sin \pi x), \qquad p_N(x) = \sum_{j=0}^{N-1} \binom{N-1+j}{j} x^{2j} .$$

Let $g_N(x) = |Q_N(x)| = \sqrt{p_N(\sin \pi x)}$, then $|p_N(x)| = |\cos \pi x|^N g_N(x)$ and $g_N(x) > 0$. Theorem 5 implies that

$$\int_R (1 + |x|)^{\alpha q} |\hat{f}_N(x)|^q dx < \infty \Longleftrightarrow \alpha < N - \frac{P_N(q)}{q \log 2} \tag{5.2}$$

where $P_N(q)$ is the pressure of g_N. This is a criterion for $\hat{f} \in L_s^q(R)$ (see Remark 2 after Theorem 5). Hervé [14] showed almost the same result but without discussing the critical value.

We can use Proposition 3 to approximate g_N by a dyadic step function h which takes a value of g_N on each of the 2^m dyadic intervals of length 2^{-m}. By using such an h we can construct a $2^{m-1} \times 2^{m-1}$ matrix M_q as in Section 4. Denote by $\lambda_q^{(m)}$ the largest eigenvalue of M_q, then

$$\left| P_N(q) - \frac{\log \lambda_q^{(m)}}{m-1} \right| \leq \frac{q\pi(N-1)}{2^m} . \tag{5.3}$$

Indeed in view of Proposition 3, we check on $p_N(x)$. For $0 < x < 1$,

$$\frac{d}{dx}\log p_N(x) = \sum_{j=1}^{N-1}\binom{N-1+j}{j}2j\,x^{2j-1}\Big/\sum_{j=0}^{N-1}\binom{N-1+j}{j}x^{2j} \le 2(N-1)\,.$$

The mean value theorem implies that

$$|\log p_N(x) - \log p_N(x+\Delta x)| \le 2(N-1)|\Delta x|\,.$$

As $|\sin x| \le |x|$, it follows that

$$\frac{1}{2}\,|\log p_N(\sin \pi x) - \log p_N(\sin \pi(x+\Delta x))| \le \pi(N-1)|\Delta x|$$

and by Proposition 3, (5.3) follows.

If we take $\overline{h}$ on the dyadic subintervals to be the maximal value of g_N on the subintervals, and let $\underline{h}$ be defined similarly by taking infimum on the subintervals instead. Then according to the monotonicity of the pressure function, we have

$$\frac{\log \underline{\lambda}_q^{(m)}}{m-1} \le P(q) \le \frac{\log \overline{\lambda}_q^{(m)}}{m-1}\,.$$

where the definitions of $\underline{\lambda}_q^{(m)}$ and $\overline{\lambda}_q^{(m)}$ are self-explained. These approximations can be used to give numerical approximations.

We remark that when $q = 2$, $P_N(2)$ [as in (5.2)] corresponds to the Sobolev exponent of f_N in [3, 9, 25] and the L^2-Lipshitz exponent in [20, 21]. These exponents can be calculated exactly by using certain simple matrices obtained from the transfer operator L_g. Our approximation is hence most useful for the case $q \neq 2$. In particular, the values $P_N(1)$ can be used to estimate the modulus of continuity of f_N in [5, Chapter 7]. The following are some numerical estimations of the $\alpha = N - P_N(1)/\log 2$ in (5.2) using $m = 8$. We have $\alpha_{\min} \le \alpha \le \alpha_{\max}$. The third column is an approximation of the α from [6, p. 232]. It is off from the true value significantly when N is large.

N	$\alpha_{\min}$	$\alpha_{\max}$	α [D2]
2	0.5210	0.5216	0.5
3	0.9793	0.9803	0.915
4	1.3911	1.3925	1.275
5	1.7676	1.7694	1.596
6	2.1159	2.1181	1.888
7	2.4407	2.4434	2.158
8	2.7458	2.7488	2.415
9	3.0340	3.0373	2.661
10	3.3080	3.3117	2.902
11	3.5705	3.5745	

N	$\alpha_{\min}$	$\alpha_{\max}$	α [D2]
12	3.8238	3.8281	
13	4.0697	4.0744	
14	4.3100	4.3151	
15	4.5459	4.5514	
16	4.7785	4.7844	
17	5.0084	5.0147	
18	5.2363	5.2431	
19	5.4626	5.4697	
20	5.6875	5.6951	

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