A Weighted Tauberian Theorem

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ABSTRACT. We prove a Tauberian theorem of the form $\phi \ast g(x) \sim p(x)w(x)$ as $x \to \infty$, where $p(x)$ is a bounded periodic function and $w(x)$ is a weighted function of power growth. It can be used to study the weighted average of the form $T^{(n, \ln T)^{b-1}} \int_0^T \frac{f(t)}{h(t)} dt$.

1. Introduction

Tauberian theorems concern the asymptotic behavior of functions (or sequences) deduced from the behavior of their averages. The most celebrated Tauberian theorem is due to Wiener [W2] and is as follows.

Theorem 1.1. For $\phi \in L^\infty(\mathbb{R})$, the relation $\lim_{x \to \infty} \phi \ast g(x) = 0$ holds true for all $g \in L^1(\mathbb{R})$ whenever it holds true for some $f \in L^1(\mathbb{R})$ such that the Fourier transformation $\hat{f}(\xi) \neq 0$ for all $\xi \in \mathbb{R}$.

This theorem is an important consequence of the more general treatment of the translation invariant subspaces of $L^1(\mathbb{R})$ (see, e.g., [B], [R], or [T]). It can be reformulated on the multiplicative group $\mathbb{R}^+$ by using the expression $\lim_{\tau \to \infty} \int_0^\tau \phi(Tx)g(x) \, dx$. In particular, if $g(x) = \chi_{(0,1]}$, then the limit becomes

$$\lim_{\tau \to \infty} \int_0^1 \phi(Tx) \, dx = \lim_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau \phi(x) \, dx,$$

which is the most elementary average. This average was actually Wiener’s original motivation to develop his Tauberian theorem [W1, W2], by which he proved the Wiener-Plancherel theorem on the class of functions $F$ with bounded quadratic averages ($\limsup_{\tau \to \infty} \frac{1}{\tau} \int_0^\tau |F(x)|^2 \, dx < \infty$) and their Fourier transformations [W1].

There are interesting cases where a function $\phi(x)$ (or its average) behaves like a periodic function at large $x$. For example, the solution $\phi$ of the renewal equation

$$\phi(x) = \int_0^x (\phi(x-y) \, d\mu(y) + S(x), \quad x > 0$$

is asymptotically a periodic function and the period depends on the support of the probability measure $\mu$ [F, Chapter 11]. Another important class of examples appears in the recent study of “self-similarity”. It is known that the Fourier transformation $\hat{\mu}$ of the Cantor measure behaves chaotically as $|\xi| \to \infty$. On the other hand, Strichartz [S1] proved that the weighted quadratic average

$$\varphi(T) = \frac{1}{T^{1-a}} \int_{-T}^T |\hat{\mu}(\xi)|^2 \, d\xi,$$  \hspace{1cm} (1.1)
where \( \alpha = \ln 2 / \ln 3 \) is the dimension of \( \mu \), is asymptotically a multiplicative periodic function. This phenomenon holds for more general self-similar measures, and the proof is via an extension of the above Tauberian theorem [L, LW, S2]. Further investigation of such averages can be found in [JRS], where numerical solutions and open problems are presented. The self-similarity and the Tauberian theorem also play a role in the study of compactly supported \( L^2 \)-solutions \( \phi \) of the two-scale dilation equation \([D]\)

\[
\phi(x) = \sum_{n=1}^{N} c_n \phi(2^n x - n).
\]

In [LWM] it is proved that (using Corollary 4.5 here)

\[ \varphi(T) = \frac{1}{T^{1-n \alpha \ln T}} \int_{-T}^{T} |\tilde{\varphi}(\xi)|^2 d\xi \tag{1.2} \]

is asymptotically multiplicatively periodic, where the \( \alpha \) is the Sobolev exponent of \( \phi \) and the \( \beta \) is related to the multiplicity of the eigenvalue of a matrix associated with the coefficients \( \{c_n\} \) of this equation.

In this note our main purpose is to provide a general Tauberian theorem that covers all the above cases, namely, a Tauberian theorem of the form

\[
\lim_{x \to \infty} \left( \frac{\phi * g(x)}{w(x)} - \rho(x) \right) = 0, \tag{1.3}
\]

where \( p(x) \) is a bounded periodic function and \( w(x) \) is certain weighted function; this will include the cases of \( x^\alpha \) or \( x^\alpha \log x \beta \), \( \alpha, \beta \geq 0 \). To prove such a theorem (Theorem 3.3), we adapt the traditional approach \([R]\) by first obtaining a Tauberian theorem on the translation invariant subspaces of the weighted space \( L^1(w) \) (Theorem 3.2). Once (1.3) is established we can easily derive corollaries that include convolutions with measures and on the multiplicative group \( \mathbb{R}^+ \).

An example at the end of §3 shows that some restrictions on the growth of \( w \) are necessary.

We remark that in [BBE], Wiener’s Tauberian Theorem was extended to \( \mathbb{R}^d \) and used to prove the Wiener–Plancherel theorem on \( \mathbb{R}^d \). It is likely that the present weighted consideration can be carried to such a setting. We also remark that there is another kind of weighted Tauberian theorem that was investigated in [Bi] and [F] (Beurling’s Tauberian theorem) and has important applications to the central limit theorem.

2. The Weighted Functions

Let \( \Omega \) be the class of continuous functions \( w : \mathbb{R} \to \mathbb{R}^+ \) such that for any \( x, y \in \mathbb{R} \),

i. \( w(0) \geq 1, w(x) = w(-x) \);

ii. \( w(x + y) \leq w(x)w(y), w(xy) \leq w(x)w(y) \);

iii. \( \lim_{x \to \infty} \frac{w(x)}{w(x + 1)} = 1 \) and there exist \( K > 0 \) and an integer \( n > 0 \) such that \( x^{-n} w(x) \) is decreasing for \( x > K \).

Some typical examples of this class of functions are

\[
w(x) = a + |x|^\alpha \quad \text{and} \quad a + \log^+ |x| \beta ,
\]

where \( \alpha, \beta \geq 0 \) and \( a > 1 \) is sufficiently large. It is easy to check that \( w(x) \geq 1 \) for \( x \in \mathbb{R} \), and if \( w_1, w_2 \in \Omega \), then \( w_1 w_2 \in \Omega \).

Proposition 2.1.

Suppose \( u \) is a continuous function on \( \mathbb{R} \) that satisfies i and iii and there exists an \( M > 0 \) such that ii holds for all \( |x|, |y| > M \). Then there exists \( w \in \Omega \) such that \( \lim_{x \to \infty} u(x)/w(x) = 1 \).
Proof. Let $a > 1$ be large enough so that $u(x + y), u(x)y) \leq a^2 - a$ for all $|x|, |y| \leq M$. Let $w(x) = a + u(x)$. Then it is straightforward to show that $w \in \Omega$ and both $u$ and $w$ have the same property. □

We use $L^1(w)$ to denote the class of $f$ such that $\|f\|_1 := \int |f(x)|w(x)\,dx < \infty$ and $L^\infty(w^{-1})$ the class of real-valued $f$ such that $\|f\|_\infty := \text{ess sup}_x |f(x)/w(x)| < \infty$.

**Proposition 2.2.**

Let $w \in \Omega$. Then $L^1(w)$ is a Banach algebra and its dual is $L^\infty(w^{-1})$. Moreover if $f \in L^1(w)$ and $\phi \in L^\infty(w^{-1})$, then $\|\phi \ast f\|_\infty \leq \|\phi\|_\infty \|f\|_1$.

**Proof.** The first statement depends upon the fact that $w(x + y) \leq w(x)w(y)$. The last inequality follows from

$$|\phi \ast f(x)| \leq \int |\phi(x - y)f(y)|\frac{w(x)w(y)}{w(x - y)}\,dy \leq w(x)\|\phi\|_\infty \|f\|_1.$$ □

The following is the key lemma of our Tauberian theorems. It is a modification of [R, Lemma 9.2].

**Lemma 2.3.**

Let $w \in \Omega$ and $f \in L^1(w)$. Then for any $\epsilon > 0$ and any fixed $\xi_0$, there exists an $h \in L^1(w)$ such that $\|h\|_1 < \epsilon$ and

$$\hat{h}(\xi) = \hat{f}(\xi) - \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}f(k\xi - \xi_0) + \xi_0$$

for all $\xi$ in some neighborhood of $\xi_0 \in \mathbb{R}$. Here $n$ is the integer associated with $w$ in (ii).

**Proof.** Without loss of generality we assume that $\xi_0 = 0, \hat{f}(0) = 1$. We can choose a rapid decreasing $C^\infty$-function $g$ such that $\hat{g}(\xi) = 1$ for all $\xi$ in some neighborhood of 0. For $\lambda > 0$, let

$$g_\lambda(x) = \frac{1}{\lambda}g\left(\frac{x}{\lambda}\right) \quad \text{and} \quad h_\lambda(x) = g_\lambda(x) - g_\lambda \ast \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}f_k(x).$$

Then

$$\hat{h_\lambda}(\xi) = \hat{g}(\lambda\xi)\left(1 - \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}\hat{f}(k\xi)\right),$$

which satisfies (2.1) in some neighborhood of 0.

We claim that $\|h_\lambda\|_1 \to 0$ as $\lambda \to \infty$. Once this is established, the lemma then follows by taking $h = h_\lambda$ for $\lambda$ large enough. To prove the claim, we observe that

$$\|h_\lambda\|_1 = \int |g_\lambda(x) - \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}g_\lambda(x - ky)|w(x)\,dx$$

$$= \int \left|\int f(y)(g_\lambda(x) - \sum_{k=1}^{n+1}(-1)^{k+1}\binom{n+1}{k}g_\lambda(x - ky))\,dy\right|w(x)\,dx$$

$$\leq \int |f(y)|\left(\int \sum_{k=0}^{n+1}(-1)^k\binom{n+1}{k}g\left(x - \frac{ky}{\lambda}\right)w(\lambda x)\,dx\right)\,dy$$

$$= I_1 + I_2,$$
where $I_1$ is the integral over $\{ y : |y| < \delta \lambda \}$ and $I_2$ is the integral over $\{ y : |y| \geq \delta \lambda \}$. If $0 < \delta < 1$ and $|y| < \delta \lambda$, then the mean value theorem implies

$$\left| \sum_{k=0}^{n+1} \frac{(n+1)}{k} g \left( x - \frac{k y}{\lambda} \right) \right| \leq \left| \frac{y}{\lambda} \right|^{n+1} \langle \tau, x \rangle,$$

where $\langle \tau, x \rangle = \max \{ |g^{n+1} u| : x - 1 \leq u \leq x + 1 \}$. Note that $\tau$ is still rapidly decreasing and belongs to $L^1(w)$. The decreasing property of $x^{-n} w(x)$ in iii implies that for $|y| > K, \lambda^{-n} w, \lambda \leq y^{-n} w, y$; hence,

$$I_1 = \int_{|y| < \delta \lambda} |f(y)| \left( \int \langle \tau, x \rangle w(x) \, dx \right) \left| \frac{y}{\lambda} \right|^{n+1} w(x) \lambda \, dy \leq \| \tau \|_1 \left( \int_{|y| < \delta \lambda} |f(y)| \right) \left( \int_{|y| < \delta \lambda} \left| \frac{y}{\lambda} \right|^{n+1} w(x) \lambda \, dy \right) \leq \delta \| \tau \|_1 \left( \delta^2 w, K \right) \int_{|y| < K} |f(y)| \, dy + \max_{\lambda > K} \left\{ \frac{w(x)}{\lambda^n} \right\} \int_{|y| < K} |f(y)| y^n \, dy \leq C \delta.$$

To establish $I_2$ we note that for $|y| \geq \delta \lambda$ and any $x$

$$\sum_{k=0}^{n+1} \binom{n+1}{k} \lambda y \lambda w(x) + k |y| \leq \sum_{k=0}^{n+1} \binom{n+1}{k} \left( w \left( \frac{1}{\delta} \right) w(x) \lambda w(x) + w(k) w(y) \right) \leq C w(x) w(y),$$

where $C = \sum_{k=0}^{n+1} \binom{n+1}{k} \left( w \left( \frac{1}{\delta} \right) + w k \right)$. Thus

$$I_2 \leq \int_{|y| \geq \delta \lambda} |f(y)| \left( \int |g(x)| \sum_{k=0}^{n+1} \binom{n+1}{k} \lambda y \lambda w(x) + k |y| \right) \, dx \, dy \leq C \| g \|_1 \int_{|y| \geq \delta \lambda} |f(y)| w(y) \, dy,$$

which converges to 0 as $\lambda \to \infty$. The claim now follows from the two estimates on the integrals $I_1$ and $I_2$. 

\section{The Tauberian Theorems}

We first formulated the Tauberian theorem in terms of the spectra of $\phi$ and $f$.

\textbf{Theorem 3.1.}

Let $w \in \Omega$. Let $\phi \in L^\infty(\mathbb{R}^n, w^{-1})$ and $Y$ be a subspace of $L^1(w)$. If $\phi \ast f = 0$ for all $f \in Y$, then

$$\supp \phi \subseteq \bigcap \{ \xi : \hat{f} \in \chi_n = 0 \} \text{ for all } f \in Y,$$

where $\supp \phi$ is the support of the tempered distribution $\phi$.

The proof is the same as in [R, Theorem 9.3], using Lemma 2.3 (replacing [R, Lemma 9.2]) to localize $\hat{f}$ on a given neighborhood and that $\hat{f}$ has only small perturbation. By using the same argument as in [R, Theorem 9.4], we have the following Tauberian theorem expressed in translation invariant subspaces.
Theorem 3.2.
Let \( w \in \Omega \). Suppose \( Y \) is a closed translation invariant subspace in \( L^1(w) \) generated by the translates of \( f \). Then \( Y = L^1(w) \) if and only if \( \hat{f}(\xi) \neq 0 \) for all \( \xi \in \mathbb{R} \).

Let \( P_a \) denote the class of bounded periodic functions with period \( a \).

Theorem 3.3.
Let \( w \in \Omega \). Let \( \phi \in L^\infty(w^{-1}) \), and assume \( f \) in \( L^1(w) \) is such that \( \hat{f}(\xi) \neq 0 \) for all \( \xi \in \mathbb{R} \). Suppose

\[
\lim_{x \to \infty} \left( \frac{1}{w(x)} \phi * f(x) - p(x) \right) = 0
\]

for some \( p \in P_a \). Then for any \( g \in L^1(w) \), there exists \( q \in P_a \) such that

\[
\lim_{x \to \infty} \left( \frac{1}{w(x)} \phi * g(x) - q(x) \right) = 0. \tag{3.1}
\]

Moreover, the Fourier coefficients \( \{a_k\} \) and \( \{b_k\} \) of \( p, q \), respectively, are related by

\[
b_k = a_k \frac{\hat{g}(2\pi k/a)}{\hat{f}(2\pi k/a)}, \quad k \in \mathbb{N}. \tag{3.2}
\]

Proof. For convenience we assume that \( a = 2\pi \). Let

\[
Y = \{ g \in L^1(w) : \lim_{x \to \infty} \left( \frac{1}{w(x)} \phi * g(x) - q(x) \right) = 0 \text{ for some } q \in P_a \}.
\]

Clearly \( Y \) is translation invariant. To show that \( Y \) is closed, let \( \{g_n\} \subset Y \) with \( g_n \to g \in L^1(w) \), and let \( \{q_n\} \) be the corresponding periodic functions in \( P_a \). Then for any \( \epsilon > 0 \) and any \( m, n \) there exists \( k_0 \) such that for \( k > k_0 \) and for any \( x \) in \([0, 2\pi]\)

\[
|q_n(x) - q_m(x)| \leq \frac{1}{w(x) + 2\pi k} |\phi * g_m(x + 2\pi k) - \phi * g_n(x + 2\pi k)| + \epsilon
\]

\[
\leq \|\phi\|_\infty \|g_m - g_n\|_1 + \epsilon.
\]

This implies that \( \{q_n\} \) is a Cauchy sequence in \( L^\infty([0, 2\pi]) \) and converges to a bounded periodic function \( q \). It is straightforward to show that \( \lim_{x \to \infty} \left( \frac{1}{w(x)} \phi * g(x) - q(x) \right) = 0 \) so that \( g \in Y \). Then Theorem 3.2 implies that \( Y = L^1(w) \) and the first part of the theorem holds.

To determine the Fourier coefficients of the periodic function \( q \), we first observe that

\[
b_k = \frac{1}{2\pi} \int_0^{2\pi} q(x) e^{-iks} dx = \lim_{l \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \phi * g(x + 2\pi l) e^{-iks} dx.
\]

Since \( L^1(w) \) equals the closed subspace spanned by the translates of \( f \), we can find a sequence \( \{h_n\} \subset L^1(w) \) such that \( \{h_n * f\} \) converges to \( g \) in \( L^1(w) \). Hence for \( x \in [0, 2\pi] \) and \( n, l \) positive integers,

\[
\left| \int_0^{2\pi} \frac{\phi \ast h_n \ast f - g}{w(x) + 2\pi l} e^{-iks} dx \right| \leq 2\pi \|\phi\|_\infty \|h_n * f - g\|_1.
\]
We apply this to interchange the limit in the following calculation and thus complete the proof as

\[
\theta_k = \lim_{n \to \infty} \lim_{l \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{\phi \ast h_n \ast f)(x + 2\pi l)}{w(x + 2\pi l)} e^{-ikx} \, dx
\]

\[
= \lim_{l \to \infty} \lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} \frac{\phi \ast h_n \ast f)(x + 2\pi l)}{w(x + 2\pi l)} e^{-ikx} \, dx
\]

\[
= \lim_{n \to \infty} \frac{1}{2\pi} \int_0^{2\pi} h_n \ast p(x) e^{-ikx} \, dx
\]

\[
= \lim_{n \to \infty} \hat{h}_n(k) a_k = \frac{\hat{g}(k)}{\hat{f}(k)} a_k. \quad \square
\]

**Remark.** The referee suggested the following proof. Assume \( a = 2\pi \) and let \( h_n(t) = \phi \frac{2\pi n + t}{w \cdot 2\pi n} \). Since

\[
\left| \frac{h_n(t)}{w(t)} \right| = \frac{\phi \frac{2\pi n + t}{w \cdot 2\pi n}}{\left| \frac{2\pi n + t}{w \cdot 2\pi n + t} \right|} < \infty \quad \text{uniformly on } n \text{ and } t,
\]

\( \{h_n\}_n \) is a bounded family in \( L^\infty(w^{-1}) \) and has a weak* limit \( \psi \) in \( L^\infty(w^{-1}) \) as \( n \to \infty \). It follows from the limit assumption that

\[
\int_{-\infty}^{\infty} \psi(x - t)f(t) \, dt = p(x).
\]

Since \( p \) has spectrum in \( \mathbb{Z} \), we can apply the same argument as in Theorem 3.1 (i.e., [R, Theorem 9.3]), again using Lemma 2.3 and \( \int \hat{x} \neq 0 \) to show that the spectrum of \( \psi \) is contained in \( \mathbb{Z} \). Thus \( \psi \) is a bounded periodic function. Let \( F(s) = \sum_{k=-\infty}^{\infty} f(s + 2\pi k) \) be the periodization of \( f \). Then

\[
\int_{-\infty}^{\infty} \psi(x - t)F(t) \, dt = p(x)
\]

and \( \hat{\psi}(k) = \hat{F}(k) \hat{\psi}(k) \), \( k \in \mathbb{Z} \). This implies that \( \psi \) is the unique limit point of \( \{h_n\}_n \) as \( n \to \infty \). Now for any \( g \in L^1(w) \) we have

\[
\int_{-\infty}^{\infty} \psi(x - t)g(t) \, dt = q(x),
\]

which implies (3.1). That \( \hat{F}(k) \) (Fourier coefficient) = \( \hat{f}(k) \) (Fourier transformation) yields the relationship of the Fourier coefficients in (3.2). \( \square \)

**Corollary 3.4.**

Let \( w \in \Omega \). Suppose \( \phi \in L^\infty(w^{-1}) \) and \( f \in L^1(w) \) is such that \( \hat{f}(\xi) \neq 0 \) for all \( \xi \in \mathbb{R} \). If

\[
\lim_{x \to \infty} \frac{1}{w(x)} \phi \ast f(x) = \hat{f}(0)c
\]

for some \( c \in \mathbb{R} \), then

\[
\lim_{x \to \infty} \frac{1}{w(x)} \phi \ast g(x) = \hat{g}(0)c \quad \text{for all } g \in L^1(w).
\]

In the following we show that some growth restrictions on the weighted function \( w \) in Theorems 3.1, 2, 3 are necessary.
Let \( w(x) = e^x \). Then \( L^1(w) \) is a Banach algebra, but \( w \notin \Omega \). Consider
\[
\phi(x) = e^x; \quad f(x) = \begin{cases} e^{-2x} & \text{for } x \geq 0, \\ 0 & \text{for } x < 0. \end{cases}
\]
Then
\[
\int_{-\infty}^{\infty} f(x)e^{-itx} \, dx = \frac{1}{.2 + i\xi} \neq 0 \quad \text{for all } \xi \in \mathbb{R}
\]
and
\[
\phi * f(x) = \int_0^\infty e^{-x} e^{-2y} \, dy = \frac{1}{3} e^x. \quad (3.3)
\]
Let \( h \) be a bounded function on \( \mathbb{R} \) with the property that \( h \) vanishes for \( x < 0 \) and
\[
\int h(y)e^{-2y} \, dy \neq 0, \quad \int h(y)e^{-3y} \, dy = 0.
\]
If \( g(x) = h(x)e^{-2x} \), then \( g \in L^1(w) \), \( \hat{g}(0) = \int g(y) \, dy \neq 0 \), and
\[
\phi * g(x) = \int g(y)e^{-y} \, dy = e^x \int h(y)e^{-3y} \, dy = 0. \quad (3.4)
\]
Since (3.3) and (3.4) are inconsistent with Corollary 3.4, it follows that the theorems in this section do not hold for \( L^1(w) \) where \( w \) has the exponential growth.

4. Some Extensions

We can extend Theorem 3.3 to include convolutions of measures as in Wiener's second Tauberian Theorem [W2; T, Theorem 7.6]. Let \( W \) be the class of continuous functions \( f \) on \( \mathbb{R} \) such that
\[
\|f\| := \sum_{k=-\infty}^{\infty} \left( \sup_x |f(x)\frac{k}{w(x,k+1)}| w(x,k) \right) < \infty,
\]
where \( w(x,k) = \int_{k}^{k+1} w(t) \, dt \). It is easy to show that \( W^* \), the dual of \( W \), is the class of regular Borel measures \( \mu \) satisfying
\[
\|\mu\| := \sup_k |\mu(k, k + 1)| w(x,k) < \infty.
\]
Note that by assumption iii for \( w \in \Omega \), we can actually use \( w(k) \) instead of \( w(x,k) \).

**Theorem 4.1.**

If we replace \( L^1(w) \) and \( L^\infty(w^{-1}) \) in Theorem 3.3 by \( W \) and \( W^* \), then the same conclusion holds.

The proof is essentially the same as Theorem 3.3, starting from a straightforward modification of Lemma 2.3 (see [T, Theorem 7.6]).

Next we give a very useful criterion (see Corollaries 4.4 and 4.5) when the measure \( \mu \) is not known to be in \( W^* \) a priori.

**Corollary 4.2.**

Let \( w \in \Omega \) and let \( \mu \) be a positive regular Borel measure on \( \mathbb{R} \) such that \( \lim_{k \to -\infty} \mu[k, k+1] < \infty \). Suppose there exists \( f \in W \), \( f \geq 0 \), such that \( \hat{f}(\xi) \neq 0 \) for all \( \xi \in \mathbb{R} \) and
\[
\lim_{x \to \infty} \left( \frac{\mu * f(x)}{w(x)} - p(x) \right) = 0. \quad (4.1)
\]
for some \( p \in P_a \). Then for any \( g \in W \), there exists \( q \in P_a \) such that
\[
\lim_{x \to \infty} \left( \frac{\mu * g \cdot x}{w \cdot x} - q \cdot x \right) = 0
\]
and the Fourier coefficients of \( p \) and \( q \) are related as in Theorem 4.1.

**Proof.** We need only show that \( \lim_{k \to \infty} \mu(k, k + 1)/w.k < \infty \); this combined with the given condition \( \lim_{k \to \infty} \mu(k, k + 1) < \infty \) implies that \( \mu \in W^* \) and then we can apply Theorem 3.5. To prove this we let \( h \cdot x = e^{-x^2} \) and \( f_1 = f * h \). Then \( f_1 \in W \) and \( f_1 \cdot \xi = \hat{f} \cdot \xi \hat{h} \cdot \xi \neq 0 \) for all \( \xi \in \mathbb{R} \). By the dominated convergence theorem we obtain
\[
\lim_{x \to \infty} \left( \frac{\mu \cdot f_1 \cdot x}{w \cdot x} - p \cdot x \cdot x \right) = 0.
\]
It follows that \( \mu \cdot f_1 \cdot x/w \cdot x \), \( x > 0 \), is bounded, say by \( C \). Hence for \( k > 0 \),
\[
C \geq \frac{1}{w.k} \int_{-\infty}^{\infty} f_1 \cdot (k - y) \cdot d(\mu \cdot y) \geq \frac{1}{w.k} \int_{-k}^{k+1} f_1 \cdot (k - y) \cdot d(\mu \cdot y)
\]
\[
\geq \inf \{ f_1 \cdot y : -1 \leq y \leq 0 \} \frac{\mu(k, k + 1)}{w.k}
\]
\[
\geq C' \frac{\mu(k, k + 1)}{w.k},
\]
where \( C' = \inf \{ f_1 \cdot y : -1 \leq y \leq 0 \} > 0 \). This implies that \( \lim_{k \to \infty} \mu(k, k + 1)/w.k < \infty \) and the proof is complete. \( \square \)

For many applications it is useful to include discontinuous functions \( f \) and \( g \). A way to handle this case is to use the space \( \tilde{W} \) of locally Riemann integrable functions \( f \) such that
\[
\sum_{k=-\infty}^{\infty} \sup_{x} |f(x, k+1) \cdot x)|w.k < \infty \tag{4.2}
\]
(see [W2, T, Chapter 7]).

**Corollary 4.3.**

The \( f \) and \( g \) in Corollary 4.2 can be replaced by \( f \) and \( g \) in \( \tilde{W} \).

**Proof.** Let \( f \in \tilde{W} \) be as in Corollary 4.2. By convolving with \( e^{-x^2} \), we can actually assume that \( f \) is continuous, hence in \( W \) (see the proof in the last corollary) and so by Theorem 4.1,
\[
\lim_{x \to \infty} \left( \frac{\mu \cdot g \cdot x}{w \cdot x} - q \cdot x \right) = 0
\]
for all \( g \in W \). To extend this to all the \( g \in \tilde{W} \), we make use of an equivalent definition of (4.2) [T, Chapter 7]. If \( g \in \tilde{W} \), then there exist \( \{ g_k \}, \{ h_k \} \subseteq W \) such that the sequences \( g_k \searrow g, h_k \not\rightarrow g \) and \( \lim_{k \to \infty} \int (g_k \cdot x - h_k \cdot x)|w.x| dx = 0 \). If \( |q_j| \) and \( |r_j| \) denote the corresponding periodic functions, then \( q_j \searrow q, r_j \not\rightarrow r \) for some periodic functions \( q, r \) of period \( a \). We observe that \( q \geq r \). An application of Corollary 4.2 yields that
\[
\frac{1}{2\pi} \int_0^{2\pi} |q \cdot x - r \cdot x| e^{i k x} dx = \frac{1}{2\pi} \lim_{j \to \infty} \int_0^{2\pi} |q_j \cdot x - r_j \cdot x| e^{i k x} dx
\]
\[
= \lim_{j \to \infty} \frac{\hat{g}_j(2\pi k/a) - \hat{h}_j(2\pi k/a)}{\hat{f}_j(2\pi k/a)}
\]
\[
= 0.
\]
This implies that \( q = r \). For such a \( q \) it is easy to show directly that \( \lim_{x \to \infty} \frac{\mu_k(x) - q}{x^r} = 0 \).

Finally we like to express the Tauberian theorem on the multiplicative group \( \mathbb{R}^+ \). For simplicity we just write down a special case. Let

\[
W_{\alpha, \beta}(\mathbb{R}^+) = \{ f : f \text{ continuous on } \mathbb{R}^+, \sum_{k=-\infty}^{\infty} \sup_{2^k \leq t < 2^{k+1}} t^\beta \ln(t) | f(t) | < \infty \},
\]

where \( \alpha, \beta \in \mathbb{R} \) and \( \tilde{W}_{\alpha, \beta}(\mathbb{R}^+) \) is the class of locally Riemann integrable functions on \( \mathbb{R}^+ \) satisfying the same growth condition.

**Corollary 4.4.**

For \( \alpha, \beta \geq 0 \), let \( f \in W_{\alpha, \beta}(\mathbb{R}^+) \) or \( \tilde{W}_{\alpha, \beta}(\mathbb{R}^+) \) be positive and \( \int_0^\infty f(t) t^{\alpha - 1/2 \xi} \, dt \neq 0 \) for all \( \xi \in \mathbb{R} \). Suppose \( \mu \) is a positive regular Borel measure on \( \{ x : x \geq 0 \} \) such that

\[
\lim_{t \to \infty} \left( \frac{1}{T^\alpha \ln(T)^\beta} \int_0^T f \left( \frac{t}{T} \right) d\mu(t) - P(T) \right) = 0 \tag{4.3}
\]

for some bounded multiplicative periodic function of period \( a \), that is \( P(aT) = P(T) \). Then

\[
\lim_{t \to \infty} \left( \frac{1}{T^\alpha \ln(T)^\beta} \int_0^T g \left( \frac{t}{T} \right) d\mu(t) - Q(T) \right) = 0
\]

for all \( g \in W_{\alpha, \beta}(\mathbb{R}^+) \), \( \tilde{W}_{\alpha, \beta}(\mathbb{R}^+) \), respectively, and \( Q(aT) = Q(T) \) for all \( T > 0 \).

**Proof.** By using the transformation \( x = \ln T, y = \ln t, \tilde{f}(x, y) = e^{-y} f(e^{-y}) \), and \( d\tilde{\mu}(x, y) = e^{y} d\mu(e^y) \), (4.3) is transformed into

\[
\lim_{x \to \infty} \left( \frac{1}{x^\beta} \int_{-\infty}^{\infty} \tilde{f}(x - y) d\tilde{\mu}(y) - P(x) \right) = 0,
\]

where \( \tilde{f} \in W = \{ h : h \text{ continuous on } \mathbb{R}, \sum_{k=-\infty}^{\infty} \sup_{x \in [k, k+1)} |h(x)| \ln(x) |k| \} \) and \( P \) is a bounded periodic function of period \( \ln a \). Since \( \tilde{\mu} \) satisfies \( \lim_{k \to -\infty} \tilde{\mu}(k, k+1) < \infty \), one applies Corollary 4.2 and the proof is complete.

**Corollary 4.5.**

Suppose \( \phi \geq 0 \) on \( \mathbb{R}^+ \) and is integrable on \( [0, h] \) for some \( h > 0 \). Let \( f \in W_{\alpha, \beta}(\mathbb{R}^+) \) with \( \alpha \geq 0, \beta \geq 0 \) be such that \( \int_0^\infty f(t) t^{\alpha - 1/2 \xi} \, dt \neq 0 \) for all \( \xi \). Then

\[
\lim_{t \to \infty} \left( \frac{1}{T^\alpha \ln(T)^\beta} \int_0^T \phi(T)f(t) \, dt - P(T) \right) = 0
\]

if and only if

\[
\lim_{t \to \infty} \left( \frac{1}{T^\alpha \ln(T)^\beta} \int_0^T \phi(t) \, dt - Q(T) \right) = 0
\]

for some bounded multiplicative periodic functions \( P \) and \( Q \).

**Proof.** By letting \( d\mu(t) = \phi(t)dt \) and using a change of variables, the first expression reduces to

\[
\lim_{T \to \infty} \left( \frac{1}{T^\alpha \ln(T)^\beta} \int_0^T \phi(T)f \left( \frac{t}{T} \right) \, dt - P(T) \right) = 0,
\]

and the second expression reduces to

\[
\lim_{T \to \infty} \left( \frac{1}{T^\alpha \ln(T)^\beta} \int_0^\infty \phi(T) \chi_{[0,1]} \left( \frac{t}{T} \right) \, dt - Q(T) \right) = 0.
\]
Note that $g(t) = \chi_{[0,1]}(t)$ is in $\mathcal{H}_{a,\alpha}(\mathbb{R}^+)$ for $\alpha \geq 0$ and $\int_0^\infty g(t)t^{a-1+it} \, dt = 1/\alpha + i\xi) \neq 0$ for all $\xi$. Thus Corollary 4.4 can be applied. □

Note that Wiener's third Tauberian theorem is the special case when $\alpha, \beta = 0$ and $P,T$ and $Q,T$ are constants. Corollary 4.5 is used in [LMW] to estimate the Fourier transformation of the compactly supported $L^2$-solution of the two-scale dilation equation (as in (1.2)).

References


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