Metric Subregularity for proximal generalized equations in Hilbert Spaces☆

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Abstract

In this paper, we introduce and consider the concept of the prox-regularity of a multifunction. We mainly study the metric subregularity of a generalized equation defined by a proximal closed multifunction between two Hilbert spaces. Using proximal analysis techniques, we provide sufficient and/or necessary conditions for such a generalized equation to have the metric subregularity in Hilbert spaces. We also establish results of Robinson-Ursescu theorem type for prox-regular multifunctions.

Keywords: metric subregularity, metric subregularity, prox-regularity, coderivative, normal cone

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1. Introduction

As an extension of the convexity, the prox-regularity expresses a variational behavior of "order two" and plays an important role in optimization and variational analysis (see [2-6, 11, 12, 30, 31, 34] and references therein). In this paper we first discuss the prox-regularity of a multifunction, and we observe that the class of prox-regular multifunctions is much larger than the class of convex multifunctions. The main aim of this paper is to study the metric

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subregularity for a generalized equation defined by a prox-regular multifunction between two Hilbert spaces.

Recall that a closed multifunction F (between two Banach spaces) is metrically regular at $(a, b) \in \mathrm{Gr}(F) := \{(x, y) :\ y \in F(x)\}$ if there exist $\tau, \delta \in (0,\ +\infty)$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad \forall (x, y) \in B(a, \delta) \times B(b, \delta),$$

where $B(a, \delta)$ denotes the open ball with center a and radius δ. As it is well recognized that the notion of the metric regularity plays an important role in nonlinear analysis and variational analysis, and it has been well studied by many authors with a lot of valuable results (for details see [13,14,19,21,22,24-26]). In particular, the following Robinson-Ursescu theorem is a cornerstone in this field.

Theorem RU. *Let F be a closed convex multifunction between Banach spaces X and Y. Let $a \in X$ and $b \in F(a)$. Then the following statements are equivalent:*

(i) $b \in \mathrm{int}(F(X))$.

(ii) There exists $\eta > 0$ such that $B(b, \eta) \subset F(B(a, 1))$.

(iii) There exist $\eta, r \in (0,\ +\infty)$ such that

$$B(y, t\eta) \subset F(B(x, t)) \quad \forall (x, y) \in \mathrm{Gr}(F) \cap (B(a, r) \times B(b, r)) \text{ and } t \in (0,\ 1).$$

(iv) F is metrically regular at (a, b).

In this paper, in the Hilbert space setting, we address the corresponding issue in Section 4 for a large class of (possibly nonconvex) prox-regular multifunctions.

A weaker property (than the metric regularity of F) is that of the metric subregularity concerning generalized equations of the form

$$\text{(GE)} \qquad b \in F(x),$$

where and throughout we assume that $b \in Y$ is a given point. Recall (cf. [14]) that (GE) is metrically subregular at $a \in F^{-1}(b)$ if there exists $\tau \in [0,\ +\infty)$ such that

$$d(x, F^{-1}(b)) \leq \tau d(b, F(x)) \ \text{ for all } x \text{ close to } a. \tag{1.1}$$

This property provides an estimate of how far a candidate x (in a neighborhood of a) can be from the solution set $F^{-1}(b)$ of generalized equation (GE). A multifunction $M : Y \rightrightarrows X$ is said to be calm at $(b, a) \in \mathrm{Gr}(M)$ if there exists $L \in (0,\ +\infty)$ such that

$$d(x, M(b)) \leq L\|y - b\| \text{ for all } (y, x) \in \mathrm{Gr}(M) \text{ close to } (b, a).$$

It is known and is easy to verify that (GE) is metrically subregular at $a \in F^{-1}(b)$ if and only if $M = F^{-1}$ is calm at (b, a). The metric subregularity and calmness have been already studied by many authors under various names (see [15-17, 20, 23, 24, 28, 38-41] and therein references).

A special case of interest is the following one:

$$Y = \mathbb{R},\ b = \lambda \text{ and } F(x) := [f(x),\ +\infty) \quad \forall x \in X. \tag{1.2}$$

where $f : X \to \mathbb{R} \cup \{+\infty\}$ is a proper lower semicontinuous bounded below function and $\lambda := \inf\limits_{x \in X} f(x)$. In this case, generalized equation (GE) reduces to optimziation problem

$$\min_{x \in X} f(x)$$

while metric subregularity (1.1) reduces to

$$d(x, S) \leq \tau(f(x) - \lambda) \text{ for all } x \text{ close to } a, \tag{1.3}$$

where $S := \{x \in X : \ f(x) = \lambda\}$. Usually a is said to be a weak sharp minimum of f if there exists $\tau \in (0,\ +\infty)$ such that (1.3) holds. Weak sharp minima have important applications in sensitivity analysis and convergence analysis of mathematical programming. In recent years, weak sharp minima have been extensively studied (cf. [7, 23, 28, 38] and references therein). In terms of the subdifferentials of f outside the solution set S, Ioffe [18] first studied weak sharp minima (under a different name) when f is locally Lipschitz and proved the following result:*if f is locally Lipschitz at $a \in S$ and there exist $\eta, \delta \in (0,\ +\infty)$ such that*

$$d(0, \partial f(x)) \geq \eta \ \forall x \in B(a, \delta) \setminus S$$

then a is a weak sharp minimum of f. His work has been followed by many others, and it is now well known that *Ioffe's result is still true when f is a general proper semicontinuous function on X.* In this line, in terms of the coderivatives, the authors [41] further extended the Ioffe's result to the case when F is a general closed multifunction and established the following result:

Result I. *Let F be a closed multifunction between Banach spaces X and Y. Suppose that there exist $\eta, \delta \in (0, +\infty)$ and $\varepsilon \in (0, 1)$ such that*

$$d(0, D_c^* F(x, y)(J_\varepsilon(y - b))) \geq \eta \;\; \forall x \in B(a, \delta) \setminus F^{-1}(b) \text{ and } y \in P^{\varepsilon}_{F(x)}(b) \cap B(b, \delta), \tag{1.4}$$

where $J_\varepsilon(y - b) := \{y^* \in \partial\| \cdot \|(y - b) + \varepsilon B_{Y^*} : \|y^*\| = 1\}$ and $P^{\varepsilon}_{F(x)}(b) := \{y \in F(x) : \|b - y\| < d(b, F(x)) + \varepsilon\}$. Then (GE) is metrically subregular at a.

In general, the converse of Ioffe's result and that of Result I are not necessarily true. But, under the convexity assumption, the converse of each of these results does hold. Indeed, the authors [41] proved the following characterization: *if F is a closed convex multifunction between Banach spaces X and Y then (GE) is metrically subregular at $a \in F^{-1}(b)$ if and only if there exist $\eta, \delta \in (0, +\infty)$ and $\varepsilon \in (0, 1)$ such that (1.4) holds.* It is a natural problem to ask whether the above characterization can been extended to a larger class of possibly nonconvex functions. In Theorem 5.1, we provide an answer to this problem for the class of prox-regular multifunctions. Moreover, under the prox-regularity assumption and in terms of the normal cone of the solution set as well as some properties of the concerned multifunction on the solution set, we provide several characterizations for the metric subregularity in Theorem 5.2. In particular, we extend some existing results on weak sharp minima to the prox-regularity case from the convex one.

2. Preliminaries

Let X be a Banach space and B_X (resp. Σ_X) denote the closed unit ball (resp. the unit sphere) of X. For $x \in X$ and $r > 0$, we denote by $B(x, r)$ the open ball with center a and radius r. For a closed subset A of X and a point a in A, let $T_c(A, a)$ and $T(A, a)$ denote respectively the Clarke tangent cone and the contingent (Bouligand) cone of A at a (cf. [10, 26, 35]); they are defined by

$$T_c(A,a) := \liminf_{x \xrightarrow{A} a, t \to 0^+} (A-x)/t \quad \text{and} \quad T(A,a) := \limsup_{t\to 0^+}(A-a)/t,$$

where $x \xrightarrow{A} a$ means that $x \to a$ with $x \in A$. Thus, $v \in T_c(A,a)$ if and only if, for each sequence $\{a_n\}$ in A converging to a and each sequence $\{t_n\}$ in $(0,\ \infty)$ decreasing to 0, there exists a sequence $\{v_n\}$ in X converging to v such that $a_n + t_n v_n \in A$ for all n, while $v \in T(A,a)$ if and only if there exist a sequence $\{v_n\}$ converging to v and a sequence $\{t_n\}$ in $(0,\ \infty)$ decreasing to 0 such that $a + t_n v_n \in A$ for all n. We denote by $N_c(A,a)$ the Clarke normal cone of A at a, that is,

$$N_c(A,a) := \{x^* \in X^* | \langle x^*, h\rangle \le 0 \ \text{ for all } h \in T_c(A,a)\}.$$

Let $\hat{N}(A,a)$ denote the Fréchet normal cone of A at a; thus $x^* \in \hat{N}(A,a)$ if and only if for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\langle x^*, x-a\rangle \le \varepsilon\|x-a\| \quad \forall x \in A \cap B(a,\delta). \tag{2.1}$$

A relate but distinct notion of normal cone is that of proximal normal cone $N_P(A,a)$. This later notion is particularly relevant for the investigation regarding variational behavior of ”order two” (cf [11,12,30,35]). We recall that $x^* \in N_P(A,a)$ if and only if there exist $\sigma, \delta \in (0,\ +\infty)$ such that

$$\langle x^*, x-a\rangle \le \sigma\|x-a\|^2 \quad \forall x \in A \cap B(a,\delta). \tag{2.2}$$

It is known that

$$N_P(A,a) \subset \hat{N}(A,a) \subset N_c(A,a).$$

The Mordukhovich (limiting) normal cone is denoted by $N(A,a)$ and is defined by

$$N(A,a) := \limsup_{x \xrightarrow{A} a} N_P(A,x).$$

Thus, $x^* \in N(A,a)$ if and only if there exist sequences $\{x_n\}$ and $\{x_n^*\}$ with each $x_n^* \in N_P(A,x_n)$ such that $x_n \xrightarrow{A} a$ and $\{x_n^*\}$ weak*-converges to x^*. It is known (cf. [12, Ch.2, Theorem 6.1]) that if X is a Hilbert space then

$$N_c(A,a) = \overline{\mathrm{co}}N(A,a), \tag{2.3}$$

where $\overline{\mathrm{co}}(\cdot)$ denotes the closed convex hull. If A is convex, then $T_c(A,a) = T(A,a)$ and $N_c(A,a) = N_P(A,a)$ are respectively the tangent cone and the normal cone in the sense of convex analysis; in this case,

$$N_c(A,a) = N_P(A,a) = \{x^* \in X^* |\ \langle x^*, x\rangle \le \langle x^*, a\rangle \ \text{ for all } x \in A\}.$$

Let $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function,

$$\mathrm{dom}(\phi) := \{x \in X |\ \phi(x) < +\infty\} \ \text{ and } \ \mathrm{epi}(\phi) := \{(x,t) \in X \times \mathbb{R} |\ \phi(x) \le t\}.$$

For $a \in \mathrm{dom}(\phi)$, let $\partial_P\phi(a)$ denote the proximal subdifferential of f at a, that is, $\partial_P\phi(a)$ is the set of all $x^* \in X^*$ satisfying the property that there exist $\sigma, \delta \in (0,\ +\infty)$ such that

$$\langle x^*, x - a\rangle \le \phi(x) - \phi(a) + \sigma\|x - a\|^2 \quad \forall x \in B(a,\delta). \tag{2.4}$$

For $a \in \mathrm{dom}(\phi)$ and $h \in X$, let $\phi^\uparrow(a,h)$ denote the generalized directional derivative introduced by Rockafellar, that is,

$$\phi^\uparrow(a,h) := \lim_{\varepsilon\downarrow 0} \limsup_{z \xrightarrow{\phi} a, t\downarrow 0} \inf_{w \in h+\varepsilon B_X} (\phi(z+tw) - \phi(z))/t,$$

where the expression $z \xrightarrow{\phi} a$ means that $z \to a$ and $\phi(z) \to \phi(a)$. Let $\partial_c\phi(a)$ denote the Clarke-Rockafellar subdifferential of ϕ at a, that is,

$$\partial_c\phi(a) := \{x^* \in X^* | \langle x^*, h\rangle \le \phi^\uparrow(a,h) \quad \forall h \in X\}.$$

For a closed subset set A of X, let δ_A denote the indicator function of A, that is, $\delta_A(x) = 0$ if $x \in A$ and $\delta_A(x) = +\infty$ otherwise. It is known that

$$N_P(A,a) = \partial_P\delta_A(a) \quad \text{and} \quad N_c(A,a) = \partial_c\delta_A(a).$$

For a closed multifunction F between Banach spaces X an Y and $(x, y) \in \mathrm{Gr}(F)$, let $DF(x, y)$ denote the tangent derivative of F at (x, y), that is, $DF(x, y) : X \rightrightarrows Y$ is a multifunction from X into Y defined by

$$DF(x, y)(u) = \{v \in Y : (u, v) \in T(\mathrm{Gr}(F), (x, y)\} \quad \forall u \in X.$$

We also need the coderivative $D_P^* F(x, y)$ of F at (x, y), which is a multifunction from Y^* to X^* and defined by

$$D_P^* F(x, y)(y^*) := \{x^* \in X^* : (x^*, -y^*) \in N_P(\mathrm{Gr}(F), (x, y))\} \quad \forall y^* \in Y^*.$$

The following two lemmas (cf. [12, Ch.1, Proposition 2.11 and Theorem 6.1]) are useful for us.

Lemma 2.1. *Let X be a Hilbert space and $f, g : X \to \mathbb{R} \cup \{+\infty\}$ be proper lower semicontinuous and suppose that g is twice continuously differentiable at $x_0 \in \mathrm{dom}(f)$. Then*

$$\partial_P (f + g)(x_0) = \partial_P f(x_0) + g'(x_0),$$

where $g'(x_0)$ denotes the derivative of g at x_0.

Lemma 2.2. *Let A be a nonempty closed subset of a Hilbert space X and let $x \in X \setminus A$ be such that $\partial_P d(\cdot, A)(x) \neq \emptyset$. Then there exists $a \in A$ satisfying the following properties:*
(i) The set $P_S(x)$ of closest points in A to x is the singleton $\{a\}$.
(ii) $d(\cdot, A)$ is Fréchet differentiable at x, and

$$\partial_P d(\cdot, A)(x) = \{d'(\cdot, A)(x)\} = \big\{(x - a)/\|x - a\|\big\}.$$

(iii) $x - a \in N_P(A, a)$.

We also need the following Hilbert space version of the famous Borwein-Preiss smooth variational principle.

Lemma 2.3. *Let X be a Hilbert space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function. Let $x_0 \in X$ and $\varepsilon \in (0, +\infty)$ be such that $f(x_0) < \inf\limits_{x \in X} f(x) + \varepsilon$. Then for* *any* $\lambda > 0$ *there exist* $y, z \in X$ *such that*

$$\|z - x_0\| < \lambda,\ \|y - z\| < \lambda,\ f(y) \le f(x_0)$$

and

$$f(y) + \frac{\varepsilon}{\lambda^2}\|y - z\|^2 \le f(x) + \frac{\varepsilon}{\lambda^2}\|x - z\|^2 \quad \forall x \in X.$$

In contrast with the approximate projection theorem established in [42] in terms of the Clarke normal cone, we use Lemma 2.3 to establish the following approximate projection result in a Hilbert space in terms of the proximal normal cone.

Proposition 2.1. *Let* A *be a closed nonempty subset of a Hilbert space* X *and* $\gamma \in (0,\ 1)$. *Then for any* $x \in X \setminus A$ *there exist* $a \in A$ *and* $a^* \in N_P(A, a)$ *with* $\|a^*\| = 1$ *such that*

$$\gamma\|x - a\| \le \min\{\langle a^*, x - a\rangle, d(x, A)\}.$$

Proof. Define $f : X \to \mathbb{R} \cup \{+\infty\}$ by

$$f(u) = \delta_A(u) + \|u - x\| \quad \forall u \in X.$$

For each $n \in \mathbb{N}$, take $a_n \in A$ such that

$$\|a_n - x\| < d(x, A) + 1/n^3. \tag{2.5}$$

Then $d(x, A) = \inf_{u\in X} f(u)$ and $f(a_n) < \inf_{u\in X} f(u) + 1/n^3$. By Lemma 2.3, there exist $u_n \in X$ and $\bar{a}_n \in A$ (so $\bar{a}_n \neq x$) such that

$$\|u_n - a_n\| < 1/n,\ \ \|\bar{a}_n - u_n\| < 1/n \tag{2.6}$$

and $\bar{a}_n$ is a minimizer of the function $u \mapsto f(u) + \|u - u_n\|^2/n$. Noting that the function $u \mapsto \|u - x\| + \|u - u_n\|^2/n$ is twice continuously differentiable at $\bar{a}_n$, it follows from the optimality condition and Lemma 2.1 that

$$\begin{aligned} 0 &\in \partial_P\delta_A(\bar{a}_n) + (\bar{a}_n - x)/\|\bar{a}_n - x\| + 2(\bar{a}_n - u_n)/n \\ &= N_P(A, \bar{a}_n) + (\bar{a}_n - x)/\|\bar{a}_n - x\| + 2(\bar{a}_n - u_n)/n. \end{aligned}$$

Thus $z_n^* \in N_P(A, \bar{a}_n)$, where $z_n^* := (x - \bar{a}_n)/\|x - \bar{a}_n\| + 2(u_n - \bar{a}_n)/n$. Moreover, by (2.5) and (2.6), one can verify easily that $\|x - \bar{a}_n\| \to d(x, A)$ and $\langle z_n^*, x - \bar{a}_n\rangle \to d(x, A)$. Since $\gamma \in (0,\ 1)$, noting $\|z_n^*\| \to 1$ and letting $a_n^* := z_n^*/\|z_n^*\|$, it follows that

$$\gamma\|x - \bar{a}_n\| < \min\{\langle a_n^*, x - \bar{a}_n\rangle,\ d(x, A)\}$$

for all sufficiently large n. The proof is completed.

3. Prox-regularity of a multifunction

For theoretical interest as well as for applications, many generalization notions have been introduced in the literature to replace the convexity. Among them, prox-regularity is a useful and important one. Following [31], we say that a closed subset A of a Banach space X is prox-regular at $a \in A$ if there exist $\sigma, \delta > 0$ such that

$$\langle x^* - u^*, x - u\rangle \geq -\sigma\|x - u\|^2$$

whenever $x, u \in B(a, \delta) \cap A$, $x^* \in N_c(A, x) \cap B_{X^*}$ and $u^* \in N_c(A, u) \cap B_{X^*}$. Readers can find some interesting properties of the prox-regularity in [2-6, 12, 30, 31, 35]. Since $0 \in N_c(A, x) \cap N_c(A, u)$, it is easy to verify that A is prox-regular at a if and only if there exist $\sigma, \delta \in (0,\ +\infty)$ such that

$$\langle u^*, x - u\rangle \leq \sigma\|x - u\|^2 \tag{3.1}$$

whenever $x, u \in A \cap B(a, \delta)$ and $u^* \in N_c(A, u) \cap B_{X^*}$.

In this paper, we adopt the following notions which are motivated by (3.1).

Definition 3.1 *Let A be a closed subset of a Banach space X and F be a closed multifunction between Banach spaces X and Y. We say that*
(a) A is sub-prox-regular at $a \in A$ if there exist $\sigma, \delta \in (0,\ +\infty)$ such that

$$\langle x^*, x - a\rangle \leq \sigma\|x - a\|^2 \quad \forall x \in A \cap B(a, \delta) \text{ and } x^* \in N_c(A, a) \cap B_{X^*}.$$

(b) F is prox-regular at $(a,b)\in \mathrm{Gr}(F)$ if there exist $\sigma,\delta\in(0,\ +\infty)$ such that

$$\langle x_1^*, x_2-x_1\rangle \le \langle y_1^*, y_2-y_1\rangle + \sigma(\|x_2-x_1\|^2+\|y_2-y_1\|^2)$$

whenever $(x_i,y_i)\in \mathrm{Gr}(F)\cap(B(a,\delta)\times B(b,\delta))$ $(i=1,2)$, $y_1^*\in B_{Y^*}$ and $x_1^*\in D_c^*F(x_1,y_1)(y_1^*)\cap B_{X^*}$.

(c) generalized equation (GE) is prox-regular at $a\in F^{-1}(b)$ if there exist $\sigma,\delta\in(0,\ +\infty)$ such that

$$\langle u^*, x-u\rangle \le \langle v^*, y-b\rangle + \sigma(\|x-u\|^2+\|y-b\|^2)$$

whenever $x\in B(a,\delta)$, $u\in F^{-1}(b)\cap B(a,\delta)$, $y\in F(x)\cap B(b,\delta)$, $v^*\in B_{Y^*}$ and $u^*\in D_c^*F(u,b)(v^*)\cap B_{X^*}$.

The following properties are immediate from the related definitions:

(i) If A is sub-prox-regular at $a\in A$ then

$$N_P(A,a)=N_c(A,a)\quad\text{and}\quad T(A,a)=T_c(A,a).$$

(ii) If F is prox-regular at (a,b) then (GE) is prox-regular at $a\in F^{-1}(b)$.

(iii) If (GE) is prox-regular at $a\in F^{-1}(b)$ then $\mathrm{Gr}(F)$ is sub-prox-regular at (u,b) for all $u\in F^{-1}(b)$ close to a.

We will show that the class of prox-regular multifunctions is larger than that of convex multifunctions. To do this, we need the primal-lower-nice property for proper lower semicontinuous functions. Let $f: X\to\mathbb{R}\cup\{+\infty\}$ be a proper lower semicontinuous function and recall (cf. [9]) that f is primal-lower-nice at $\bar{x}\in\mathrm{dom}(f)$ (with respect to the Clarke subdiferential) if there exist $\delta, c, T\in(0,\ +\infty)$ such that

$$\langle u^*, x-u\rangle \le f(x)-f(u)+\frac{ct}{2}\|x-u\|^2$$

whenever $x,u\in B(\bar{x},\delta)$, $t\in[T,\ +\infty)$ and $u^*\in\partial_c f(u)\cap tB_{X^*}$. It is clear that A is prox-regular at a if and only if the indicator δ_A is primal lower nice at a. The primal-lower-nice property has been found to have important applications in variational analysis and optimization. Several authors proved that some important kinds of proper lower semicontinuous functions have the primal-lower-nice property (cf. [9, 29, 35]). In particular, Combari et al [9] proved the following interesting result which will help us to prove that many composite-convex multifuntions are prox-regular.

Proposition 3.1. *Let Y be a Banach space, $g : X \to Y$ be a continuously differentiable mapping with g' being locally Lipschitz at $\bar{x} \in X$ and $\phi : Y \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function with $g(\bar{x}) \in \mathrm{dom}(\phi)$. Suppose that the following Robinson qualification holds:*

$$\mathbb{R}_+(\mathrm{dom}(\phi) - g(\bar{x})) - g'(\bar{x})(X) = Y.$$

Then the composite function $\phi \circ g$ is primal-lower-nice at $\bar{x}$.

The following proposition can be found in [31] and shows that the Clarke normal cone in (3.1) can be replaced by the proximal normal cone in the Hilbert space setting.*Here and throughout a Hilbert space and its dual space are identified as usual.*

Proposition 3.2. *Let X be a Hilbert space and A be a closed subset of X. Then A is prox-regular at $a \in A$ if and only if there exist $\sigma, \delta \in (0, +\infty)$ such that (3.1) holds whenever $x, u \in A \cap B(a, \delta)$ and $u^* \in N_P(A, u) \cap B_X$. Consequently, a closed multifunction F between two Hilbert spaces X and Y is prox-regular at $(\bar{x}, \bar{y}) \in \mathrm{Gr}(F)$ if and only if there exist $\sigma, \delta \in (0, +\infty)$ such that*

$$\langle x_1^*, x_2 - x_1 \rangle \leq \langle y_1^*, y_2 - y_1 \rangle + \sigma(\|x_2 - x_1\|^2 + \|y_2 - y_1\|^2)$$

whenever $(x_i, y_i) \in \mathrm{Gr}(F) \cap (B(\bar{x}, \delta) \times B(\bar{y}, \delta))$ $(i = 1, 2)$, $y_1^* \in B_{Y^*}$ and $x_1^* \in D_P^* F(x_1, y_1)(y_1^*) \cap B_{X^*}$.

The following proposition provides a characterization of the prox-regularity.

Proposition 3.3. *Let A be a closed subset of X and $a \in A$. Then A is prox-regular at a if and only if there exist $\sigma, \delta \in (0, +\infty)$ such that*

$$\langle u^*, x - u \rangle \leq d(x, A) + \sigma \|x - u\|^2 \quad \forall x \in B(a, \delta)$$

whenever $u \in A \cap B(a, \delta)$ and $u^* \in N_c(A, u) \cap B_{X^*}$.

Proof. The sufficiency part is trivial. To prove the necessity part, suppose that there exist $\sigma, \delta \in (0, +\infty)$ such that

$$\langle u^*, z-u\rangle \le \frac{\sigma}{4}\|z-u\|^2 \tag{3.2}$$

whenever $z, u \in A \cap B(a, 2\delta)$ and $u^* \in N_c(A, u) \cap B_{X^*}$. Let $x \in B(a, \delta)$, $u \in A \cap B(a, \delta)$ and $u^* \in N_c(A, u) \cap B_{X^*}$. Then there exists a sequence $\{u_n\}$ in $A \cap B(a, 2\delta)$ such that $\|x - u_n\| \to d(x, A)$. It follows from (3.2) that

$$\begin{aligned}\langle u^*, x-u\rangle &= \langle u^*, x-u_n\rangle + \langle u^*, u_n - u\rangle \\ &\le \|x-u_n\| + \frac{\sigma}{4}\|u_n - u\|^2 \\ &\le \|x-u_n\| + \frac{\sigma}{2}(\|u_n - x\|^2 + \|x-u\|^2).\end{aligned}$$

Letting $n \to \infty$, one has

$$\langle u^*, x-u\rangle \le d(x,A) + \frac{\sigma}{2}(d(x,A)^2 + \|x-u\|^2) \le d(x,A) + \sigma\|x-u\|^2.$$

This completes the proof.

Remark. The referee pointed out that Proposition 3.3 has been obtained in the paper "Prox-regular sets and applications" by Colombo and Thilbault. However as we cannot locate the paper (which, we guess, has not yet appeared in print). For the sake of completeness, the proof and the proposition are kept here.

The following propositions show that the class of prox-regular multifunctions is much larger than the class of convex multifunctions.

Proposition 3.4. *Let X, Y, Z be Banach spaces. Let $G : Z \rightrightarrows Y$ be a closed convex multifunction and $g : X \to Z$ be a continuously differentiable function such that g' is locally Lipschitz at a and the following qualification holds:*

$$\mathbb{R}_+(\mathrm{dom}(G) - g(a)) - g'(a)(X) = Z. \tag{3.3}$$

Then, for any $b \in G(g(a))$, the composite $G \circ g$ is prox-regular at (a, b).

Proof. Let $T : X \times Y \to Z \times Y$ be such that

$$T(x, y) := (g(x), y) \quad \forall (x, y) \in X \times Y.$$

Then T is continuously differentiable and

$$T'(u, v)(x, y) = (g'(u)(x), y) \quad \forall (u, v), (x, y) \in X \times Y.$$

Hence T' is locally Lipschitz at (a, b) and

$$T'(a, b)(X \times Y) = g'(a)(X) \times Y. \tag{3.4}$$

It is easy to verify that $\mathrm{Gr}(G \circ g) = T^{-1}(\mathrm{Gr}(G))$, that is, $\delta_{\mathrm{Gr}(G \circ g)} = \delta_{\mathrm{Gr}(G))} \circ T$. It follows from (3.3) and (3.4) that

$$\mathbb{R}_+(\mathrm{dom}(\delta_{\mathrm{Gr}}) - T(a, b)) - T'(a, b)(X \times Y) = Z \times Y.$$

This and Proposition 3.1 imply that the composite function $\delta_{\mathrm{Gr}(G))} \circ T$ is primal-lower-nice at (a, b), and so the indicator $\delta_{\mathrm{Gr}(G \circ g)}$ is primal-lower nice at (a, b). This means that the composite $G \circ g$ is prox-regular at (a, b). The proof is completed.

Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous function and recall that f is weakly convex at $\bar{x} \in \mathrm{dom}(f)$ if there exist $\sigma, \delta \in (0, +\infty)$ such that

$$tf(x_1) + (1 - t)f(x_2) \leq f(tx_1 + (1 - t)x_2) + \sigma t(1 - t)\|x_1 - x_2\|^2$$

for all $(t, x_1, x_2) \in [0, 1] \times B(\bar{x}, \delta) \times B(\bar{x}, \delta)$ (cf. [37, 1]). In the case when X is a Hilbert space, it is known (cf. [1, Theorem 4.1]) that f is weakly convex at $\bar{x} \in \mathrm{dom}(f)$ if and only if there exist $\sigma_0, \delta_0 \in (0, +\infty)$ such that

$$\langle x^*, x_2 - x_1 \rangle \leq f(x_2) - f(x_1) + \sigma_0\|x_2 - x_1\|^2 \quad \forall x_1, x_2 \in B(\bar{x}, \delta_0) \text{ and } x^* \in \partial f(x_1).$$

This motivates us to introduce the following concepts.

For a closed multifunction F between Banach spaces X and Y, we say that F is weakly convex at $(\bar{x}, \bar{y}) \in \mathrm{Gr}(F)$ if there exist $\sigma, \delta \in (0, +\infty)$ and a neighborhood V of $\bar{y}$ such that for all $(t, x, u) \in [0, 1] \times B(\bar{x}, \delta) \times B(\bar{x}, \delta)$,

$$t(F(x) \cap V) + (1 - t)(F(u) \cap V) \subset F(tx + (1 - t)u) + \sigma t(1 - t)\|x - u\|^2 B_Y. \tag{3.5}$$

The weak convexity is closely related the paraconvexity introduced by Rolewicz (cf. [32, 33]).

Proposition 3.5. *Let F be a closed multifunction between Hilbert spaces X and Y. Let $\bar{x} \in X$ and $\bar{y} \in F(\bar{x})$ be such that F is weakly convex at $(\bar{x}, \bar{y})$; more precisely, suppose that there exist $\sigma, \delta \in (0, +\infty)$ and a neighborhood V of $\bar{y}$ such that (3.5) holds for all $(t, x, u) \in [0, 1] \times B(\bar{x}, \delta) \times B(\bar{x}, \delta)$. Let $\delta_0 \in (0, \delta]$ be such that $B(\bar{y}, \delta_0) \subset V$. Then,*

$$\langle x_1^*, x_2 - x_1 \rangle \leq \langle y_1^*, y_2 - y_1 \rangle + \sigma \|x_2 - x_1\|^2 \tag{3.6}$$

whenever $(x_i, y_i) \in \mathrm{Gr}(F) \cap (B(\bar{x}, \delta_0) \times B(\bar{y}, \delta_0))$ $(i = 1, 2)$, $y_1^* \in B_{Y^*}$ and $x_1^* \in D_P^*(x_1, y_1)(y_1^*) \cap B_{X^*}$. Consequently, F is prox-regular at $(\bar{x}, \bar{y})$.

Proof. Let $x_1 \in B(\bar{x}, \delta_0)$, $y_1 \in F(x_1) \cap B(\bar{y}, \delta_0)$, $y_1^* \in B_{Y^*}$ and $x_1^* \in D_P^* F(x_1, y_1)(y_1^*) \cap B_{X^*}$. Then there exist $\sigma_1, \delta_1 \in (0, +\infty)$ such that

$$\langle x_1^*, x - x_1 \rangle \leq \langle y_1^*, y - y_1 \rangle + \sigma_1 (\|x - x_1\|^2 + \|y - y_1\|^2) \tag{3.7}$$

for any $(x, y) \in \mathrm{Gr}(F) \cap (B(x_1, \delta_1) \times B(y_1, \delta_1))$. Let $(x_2, y_2) \in \mathrm{Gr}(F) \cap (B(\bar{x}, \delta_0) \times B(\bar{y}, \delta_0))$ and $t \in (0, 1)$ sufficiently close to 1 be such that

$$\max\{(1-t)\|x_2 - x_1\|, \ (1-t)(\|y_2 - y_1\| + \sigma t \|x_2 - x_1\|^2)\} < \delta_1.$$

Then, by (3.5), there exists $e_t \in B_Y$ such that

$$t y_1 + (1-t) y_2 + \sigma t (1-t) \|x_2 - x_1\|^2 e_t \in F(t x_1 + (1-t) x_2)$$

and it follows from (3.7) (applied to $x = t x_1 + (1-t) x_2$ and $y = t y_1 + (1-t) y_2 + \sigma t (1 - t) \|x_2 - x_1\|^2 e_t$) that

$$\begin{aligned} \langle x_1^*, x_2 - x_1 \rangle \ &\leq \ \langle y_1^*, y_2 - y_1 + \sigma t \|x_2 - x_1\|^2 e_t \rangle + \\ &\qquad \sigma_1 (1-t) \big(\|x_2 - x_1\|^2 + \big\|y_2 - y_1 + \sigma t \|x_2 - x_1\|^2 e_t \big\|^2 \big). \end{aligned}$$

Letting $t \to 1^-$, one sees that (3.6) holds. This and Proposition 3.1 imply that F is prox-regular at $(\bar{x}, \bar{y})$.

4. Metric regularity for a prox-regular multifunction

Ursescu [36] and Robinson [34] proved independently that if F is a closed convex multifunction between Banach spaces Z and Y and $(a, b) \in \mathrm{Gr}(F)$ then $b \in \mathrm{int}(F(X))$ if and only if

$$B(b, \eta) \subset F(a + B_Z) \tag{4.1}$$

for some $\eta > 0$. This equivalence can be regarded as an extension of the classical open mapping theorem on a bounded linear operator between Banach spaces. In an earlier paper than [34] and [36], in the topological linear space case, Ng [27] had established an open mapping theorem for a multifunction whose graph is a closed convex cone. In [34], Robinson further proved the metric regularity result in the Robinson-Ursescu theorem (namely Theorem RU mentioned in Section 1). In this section, we will address the corresponding issue for a possibly nonconvex prox-regular multifunction between two Hilbert spaces.

Under the the convexity assumption on F, it is clear that

$$(4.1) \Longleftrightarrow \big[B(b, t\eta) \subset F(a + tB_X) \ \ \forall t \in (0,\ 1]\big].$$

Thus, the following theorem can be regarded as a result of the Robinson-Ursescu theorem type in the prox-regularity setting.

Theorem 4.1. *Let F be a closed multifunction between Hilbert spaces X and Y and suppose that F is prox-regular at $(a, b) \in \mathrm{Gr}(F)$ with the corresponding constants $\sigma, \delta \in (0,\ +\infty)$, namely*

$$\langle x^*, u - x\rangle \le \langle y^*, v - y\rangle + \sigma(\|u - x\|^2 + \|v - y\|^2) \tag{4.2}$$

for all $(u, v), (x, y) \in \mathrm{Gr}(F) \cap (B(a, \delta) \times B(b, \delta))$, $y^* \in B_Y$ and $x^* \in D_c^* F(x, y)(y^*) \cap B_X$. Then, the following statements are equivalent:

(i) There exist $\eta, r \in (0,\ +\infty)$ such that

$$B(y, t\eta) \subset F(B(x, t)) \quad \forall x \in B(a, r),\ y \in F(x) \cap B(b, r) \ \text{ and } \ t \in (0,\ 1).$$

(ii) There exist $\gamma \in (0,\ \frac{\delta}{3})$ and $\beta \in (0,\ \delta)$ with $\beta > \sigma(4\gamma^2 + \beta^2)$ such that

$$B(b, \beta) \subset F(B(a, \gamma)). \tag{4.3}$$

(iii) There exist $\tau, \lambda \in (0,\ +\infty)$ such that

$$d(x, F^{-1}(y)) \leq \tau d(y, F(x)) \quad \forall (x, y) \in B(a, \lambda) \times B(b, \lambda).$$

Proof. (i)$\Rightarrow$(ii). Suppose that (i) holds. Then $B(b, t\eta) \subset F(B(a, t))$ for all $t \in (0,\ 1)$. Let $t_0 \in (0,\ 1)$ be sufficiently small such that $\eta > \sigma(4t_0 + t_0\eta^2)$. Thus, one can see that (ii) holds with $\beta := t_0\eta$ and $\gamma = t_0$.

(ii)$\Rightarrow$(iii). Suppose to the contrary that for any $n \in \mathbb{N}$ there exist $(x_n, y_n) \in X \times Y$ such that

$$\|x_n - a\| < \min\left\{\gamma, 1/n\right\}, \quad \|y_n - b\| < \min\left\{\beta, 1/n\right\} \tag{4.4}$$

and $d(x_n, F^{-1}(y_n)) > nd(y_n, F(x_n))$. Then, by [41, Lemma 3.1] there exist $\bar{x}_n \in X$ and $\bar{y}_n \in F(\bar{x}_n)$ such that

$$\|\bar{x}_n - x_n\| < d(x_n, F^{-1}(y_n)), \ \ 0 < \|\bar{y}_n - y_n\| < d(x_n, F^{-1}(y_n))/n \tag{4.5}$$

and

$$(0, 0) \in \{(0, (\bar{y}_n - y_n)/\|\bar{y}_n - y_n\|)\} + (B_X \times B_Y)/n + N_c(\mathrm{Gr}(F), (\bar{x}_n, \bar{y}_n)).$$

Hence there exists $(x_n^*, -y_n^*) \in N_c(\mathrm{Gr}(F), (\bar{x}_n, \bar{y}_n))$ such that

$$\|x_n^*\| \leq 1/n, \quad y_n^* \in (\bar{y}_n - y_n)/\|\bar{y}_n - y_n\| + B_Y/n, \tag{4.6}$$

and so

$$(x_n^*, -y_n^*)/(1 + 1/n) \in N_c(\mathrm{Gr}(F), (\bar{x}_n, \bar{y}_n)) \cap (B_X \times B_Y). \tag{4.7}$$

By (4.3) and (4.4), for each $n \in \mathbb{N}$ there exists $z_n \in F^{-1}(y_n) \cap B(a, \gamma)$. Then

$$d(x_n, F^{-1}(y_n)) \leq \|x_n - z_n\| \leq \|x_n - a\| + \|a - z_n\| < 1/n + \gamma$$

and it follows from (4.5) that

$$\|\bar{x}_n - a\| < 2/n + \gamma \quad \text{and} \quad \|\bar{y}_n - b\| < (2 + \gamma)/n. \tag{4.8}$$

Consider all large n such that $(2 + \gamma)/n < \beta$ and $3\gamma + 2/n < \delta$. Let $v_n := (\beta - (2 + \gamma)/n)\frac{\bar{y}_n - y_n}{\|\bar{y}_n - y_n\|}$. Then, by (4.8), one has

$$B(a, \gamma) \subset B(\bar{x}_n, 2\gamma + 2/n) \subset B(a, \delta) \ \text{ and } \ \bar{y}_n - v_n \in B(b, \beta).$$

It follows from (4.3) that there exists $w_n \in B(\bar{x}_n, 2\gamma + 2/n)$ such that $\bar{y}_n - v_n \in F(w_n)$. Hence, by (4.7) and $\beta \in (0,\ \delta)$, one can apply (4.2) to $(w_n, \bar{y}_n - v_n)$, $(\bar{x}_n, \bar{y}_n)$ and $\frac{1}{1+\frac{1}{n}}(x_n^*, y_n^*)$ in place of (u, v), (x, y) and (x^*, y^*) respectively, and we get

$$\langle x_n^*, w_n - \bar{x}_n \rangle \le \langle y_n^*, -v_n \rangle + (1 + 1/n)\sigma(\|w_n - \bar{x}_n\|^2 + \|v_n\|^2). \tag{4.9}$$

By (4.6) and our choice of v_n and w_n, we have

$$-1/n(2\gamma + 2/n) \le \langle x_n^*, w_n - \bar{x}_n \rangle \ \text{ and } \ \langle y_n^*, -v_n \rangle \le -\beta + (2 + \gamma)/n + 1/n(\beta - (2 + \gamma)/n).$$

It follows from (4.9) that

$$-1/n(2\gamma+2/n) \le -\beta+(2+\gamma)/n+1/n(\beta-(2+\gamma)/n)+(1+1/n)\sigma\big((2\gamma+2/n)^2+(\beta-(2+\gamma)/n)^2\big).$$

Letting $n \to \infty$, one has $\beta \le \sigma(4\gamma^2 + \beta^2)$, contradicting the given in (ii). This shows that (ii)$\Rightarrow$(iii) holds. Since (iii)$\Rightarrow$(i) is immediate from [19, Proposition 2], the proof is completed.

5. Metric subregularity for generalized equations

In extending several known results in the literature, this section is devoted to provide two characterizations for the metric subregularity of generalized equation (GE): one is in terms of the points of the solution set $F^{-1}(b)$ while the other in terms of the points not belonging to $F^{-1}(b)$. A major novelty here for our consideration is that the usual convexity assumption regarding F is replaced by considerably weaker assumption that F is prox-regular. As in the preceding section, we assume throughout that X, Y are Hilbert spaces; F denotes a closed multifunction from X to Y, $b \in Y$ and $a \in F^{-1}(b)$. In this Hilbert space framework, the implication (ii)$\Longrightarrow$(i) in the following theorem is a strengthened version of Result I (as $D_P^*F(x, y)(y^*) \subset D_c^*F(x, y)(y^*)$ for any y^*).

Theorem 5.1. *Consider the following statements for the data F, a, b specified above:*
(i) (GE) is metrically subregular at a.
(ii) There exist $\eta, \delta \in (0,\ +\infty)$ and $\varepsilon \in (0,\ 1)$ such that

$$d\big(0, D_P^*F(x, y)(J_\varepsilon(y - b))\big) \ge \eta \tag{5.1}$$

for all $x \in B(a,\delta) \setminus F^{-1}(b)$ *and all* $y \in P^{\varepsilon}_{F(x)}(b) \cap B(b,\delta)$.

(iii) There exist $\eta, \delta \in (0,\ +\infty)$ *and* $\varepsilon \in (0,\ 1)$ *such that*

$$d\left(0, D^*_P F(x,y)\big((y-b)/\|y-b\| + \varepsilon B_Y\big)\right) \geq \eta \tag{5.2}$$

for all $x \in B(a,\delta) \setminus F^{-1}(b)$ *and* $y \in F(x) \cap B(b,\delta)$.

Then (iii)⇒(ii)⇒(i). Moreover (i)⇔(ii)⇔(iii) if F *is assumed to be prox-regular at* (a,b).

Proof. For any $y \in Y \setminus \{b\}$, noting that $J(y-b) = \big\{(y-b)/\|y-b\|\big\}$, one has $J_\varepsilon(y-b) \subset (y-b)/\|y-b\| + \varepsilon B_Y$. Hence (iii)⇒(ii) is evident. To prove (ii)⇒(i), suppose to the contrary that (GE) is not metrically subregular at a. Then, for each $n \in \mathbb{N}$ there exists $x_n \in B(a, 1/n) \setminus F^{-1}(b)$ such that $d(x_n, F^{-1}(b)) > nd(b, F(x_n))$. Thus

$$d(x_n, F^{-1}(b)) \leq \|x_n - a\| < 1/n \tag{5.3}$$

and

$$\|y_n - b\| < d(x_n, F^{-1}(b))/n \quad \text{with some } y_n \in F(x_n). \tag{5.4}$$

Define $\phi : X \times Y \to \mathbb{R} \cup \{+\infty\}$ by

$$\phi(x,y) := \|y-b\| + \delta_{\mathrm{Gr}(F)}(x,y) \quad \forall (x,y) \in X \times Y.$$

Then ϕ is lower semicontinuous and

$$\phi(x_n, y_n) < \inf_{(x,y)\in X\times Y} \phi(x,y) + d(x_n, F^{-1}(b))/n.$$

By Lemma 2.3 (applied to $\varepsilon = d(x_n, F^{-1}(b))/n$ and $\lambda = d(x_n, F^{-1}(b))/2$), there exist $(\bar{x}_n, \bar{y}_n)$, $(u_n, v_n) \in X \times Y$ such that

$$\|(u_n, v_n) - (x_n, y_n)\| < d(x_n, F^{-1}(b))/2, \quad \|(\bar{x}_n, \bar{y}_n) - (u_n, v_n)\| < d(x_n, F^{-1}(b))/2 \tag{5.5}$$

and $(\bar{x}_n, \bar{y}_n)$ is a minimizer of the function $\phi + \frac{4}{nd(x_n, F^{-1}(b))}\|(\cdot,\cdot) - (u_n, v_n)\|^2$. By the optimality condition and Lemma 2.1, we have

$$(0,0) \in \partial_P \phi(\bar{x}_n, \bar{y}_n) + 8(\bar{x}_n - u_n, \bar{y}_n - v_n)/nd(x_n, F^{-1}(b)). \tag{5.6}$$

This and the definition of ϕ imply that $\bar{y}_n \in F(\bar{x}_n)$. Noting that $\bar{x}_n \notin F^{-1}(b)$ (by the second inequality of (5.5)), one has $\bar{y}_n \neq b$. Hence the function $(x, y) \mapsto \|y - b\|$ is twice continuously differentiable at $(\bar{x}_n, \bar{y}_n)$, and so one can apply Lemma 2.1 to rewrite (5.6) as

$$(0, 0) \in \partial_P \delta_{\mathrm{Gr}(F)}(\bar{x}_n, \bar{y}_n) + \big(0, (\bar{y}_n - b)/\|\bar{y}_n - b\|\big) + (x_n^*, y_n^*),$$

where $(x_n^*, y_n^*) := 8(\bar{x}_n - u_n, \bar{y}_n - v_n)/nd(x_n, F^{-1}(b))$ (which is of the norm less than $4/n$ (by the second inequality of (5.5)). Thus $(-x_n^*, -y_n^* - (\bar{y}_n - b)/\|\bar{y}_n - b\|) \in N_P(\mathrm{Gr}(F), (\bar{x}_n, \bar{y}_n))$ and so

$$d\big(0, D_P^*F(\bar{x}_n, \bar{y}_n)(y_n^* + (y_n - b)/\|y_n - b\|)\big) \leq \| - x_n^*\| < 4/n,$$

which contradicts (ii) when n is large enough because $(\bar{x}_n, \bar{y}_n) \to (a, b)$ by (5.3)—(5.5). Therefore (ii)$\Rightarrow$(i).

It remains to show that if F is prox-regular at (a, b) then (i)$\Rightarrow$(iii) holds. To do this, suppose by (i) that there exist $\tau, r \in (0, +\infty)$ such that

$$d(x, F^{-1}(b)) \leq \tau d(b, F(x)) \quad \forall x \in B(a, r). \tag{5.7}$$

By the prox-regularity of F at (a, b), there exist $\sigma \in (0, +\infty)$ and $r_1 \in (0, r)$ such that

$$\langle x^*, u - x\rangle \leq \langle y^*, b - y\rangle + \sigma(\|u - x\|^2 + \|b - y\|^2) \tag{5.8}$$

whenever $x \in B(a, r_1)$, $u \in F^{-1}(b) \cap B(a, r_1)$, $y \in F(x) \cap B(b, r_1)$, $y^* \in B_Y$ and $x^* \in D_c^*F(x, y)(y^*) \cap B_X$. Let $\varepsilon \in (0, 1)$, $\varepsilon' \in (\varepsilon, 1)$ and take $\delta \in (0, r_1/2)$ such that

$$(1 - \varepsilon - \sigma\delta)/\tau - \sigma\delta > (1 - \varepsilon')/\tau.$$

We will prove that (5.2) holds for $\eta := \min\{(1 - \varepsilon')/\tau, 1\}$. To do this, let $x \in B(a, \delta) \setminus F^{-1}(b)$, $y \in F(x) \cap B(b, \delta)$, $y^* \in (y - b)/\|y - b\| + \varepsilon B_Y$ and $x^* \in D_P^*F(x, y)(y^*)$. We have to show that $\|x^*\| \geq \eta$. Since $\eta \leq 1$, we can assume that $\|x^*\| \leq 1$. Noting that

$$0 < d(x, F^{-1}(b)) \leq \|x - a\| < \delta,$$

there exists a sequence $\{u_n\}$ in $F^{-1}(b))$ such that

$$\|x - u_n\| < \min\big\{(1 + 1/n)d(x, F^{-1}(b)), \delta\big\} \quad \forall n \in \mathbb{N}.$$

Our choice of y^* clearly implies that

$$\begin{aligned}\langle y^*, b-y\rangle &= \langle (y-b)/\|y-b\|, b-y\rangle + \langle y^* - (y-b)/\|y-b\|, b-y\rangle \\ &\leq -(1-\varepsilon)\|y-b\|.\end{aligned}$$

Noting that $\|u_n - a\| \leq \|u_n - x\| + \|x-a\| < 2\delta < r_1$, it follows from (5.8) that

$$\begin{aligned}\langle x^*, u_n - x\rangle &\leq -(1-\varepsilon)\|y-b\| + \sigma(\|x-u_n\|^2 + \|y-b\|^2) \\ &\leq -(1-\varepsilon)\|y-b\| + \sigma(\delta\|x-u_n\| + \delta\|y-b\|) \\ &\leq -(1-\varepsilon-\sigma\delta)\|y-b\| + \sigma\delta\|x-u_n\| \\ &\leq -(1-\varepsilon-\sigma\delta)d(b, F(x)) + \sigma\delta\|x-u_n\|\end{aligned}$$

(because $-1+\varepsilon+\sigma\delta < 0$ and $y \in F(x)$). Since $x \in B(a,r)$, it follows from (5.7) that

$$\begin{aligned}\langle x^*, x-u_n\rangle &\geq (1-\varepsilon-\sigma\delta)/\tau d(x, F^{-1}(b)) - \sigma\delta\|x-u_n\| \\ &\geq n(1-\varepsilon-\sigma\delta)/(n+1)\tau\|x-u_n\| - \sigma\delta\|x-u_n\|\end{aligned}$$

(by our choice of u_n). This implies that $\|x^*\| \geq n(1-\varepsilon-\sigma\delta)/(n+1)\tau - \sigma\delta$ and so, in limits,

$$\|x^*\| \geq (1-\varepsilon-\sigma\delta)/\tau - \sigma\delta \geq (1-\varepsilon')/\tau$$

as wished to show. The proof is completed.

Remark. Let Y, λ and F be as in (1.2) with a proper lower semicontinuous function f. Then $D_P^*F(x, f(x))(1) = \partial_P f(x)$ and $\mathrm{dom}(D_P^*F(x,y)) = \{0\}$ for any $x \in \mathrm{dom}(f)$ and $y \in (f(x),\ +\infty)$; thus Theorem 5.1 implies that $a \in S$ is a weak sharp minimum of f if and only if there exist $\eta, \delta \in (0,\ +\infty)$ such that

$$d(0, \partial_P f(x)) \geq \eta \quad \forall x \in B(a,\delta) \setminus S \text{ with } |f(x) - \lambda| < \delta.$$

The following example shows that if the prox-regularity assumption is dropped then Theorem 5.1 and the above characterization are not true even if f is assumed to be Lipschitz.

Example. For any $n \in \mathbb{N}$, let $\phi_n : \left[2^{-2n},\ 2^{-2(n-1)}\right) \to \mathbb{R}$ be such that

$$\phi_n(x) = \begin{cases} \frac{3x}{2} - 2^{-2n+1}, & x \in \left[2^{-2n+1},\ 2^{-2(n-1)}\right) \\ 2^{-2n}, & x \in \left[2^{-2n},\ 2^{-2n+1}\right). \end{cases}$$

Let $X = \mathbb{R}$ and $f : X \to \mathbb{R}$ be defined as follows.

$$f(x) = \begin{cases} x, & x \in [1,\ +\infty) \\ \phi_n(x), & x \in \left[2^{-2n},\ 2^{-2(n-1)}\right) \text{ and } n \in \mathbb{N} \\ 0, & x \in (-\infty,\ 0]. \end{cases}$$

Then,

$$f'(x) = \begin{cases} 0, & x \in (1,\ +\infty) \\ \frac{3}{2}, & x \in \left(2^{-2n+1},\ 2^{-2(n-1)}\right) \text{ and } n \in \mathbb{N} \\ 0, & x \in \left(2^{-2n},\ 2^{-2n+1}\right) \text{ and } n \in \mathbb{N} \\ 0, & x \in (-\infty,\ 0). \end{cases}$$

Hence f is a Lipshcitz function and $\inf\limits_{x \in X} f(x) = 0$. Note that $S = \{x \in X : \ f(x) = 0\} = (-\infty,\ 0]$ and $f(x) \geq \frac{1}{2}x = \frac{1}{2}d(x, S)$ for all $x \in [0,\ +\infty)$. Thus 0 is a weak sharp minimum of f. But, $\partial_P f(x) = \{0\}$ for any $x \in \left(2^{-2n},\ 2^{-2n+1}\right)$ and $n \in \mathbb{N}$.

Theorem 5.1 are motivated by Ioffe's work [18] and Result I in which the sufficient conditions and characterizations are expressed in terms of the subdifferentials and coderivatives of the concerned function *outside the solution set*. In a different line, in terms of some properties of the concerned function *inside the solution set*, many authors [7, 16, 23, 28, 38-40] studied the weak sharp minima or the metric regularity. In this line and in the Euclidean space case, Burke and Ferris [8] used technique of conjugate functions to study weak sharp minima of a convex function and established several interesting characterizations for weak sharp minima. Using the same technique as [8], Burke and Deng [7] extended Burke and Ferris' results to the Hilbert space case. In particular, they proved the following result:

Result II. *Let X be a Hilbert space and $f : X \to \mathbb{R} \cup \{+\infty\}$ be a proper lower semicontinuous convex function. The following statements are equivalent.*

(i) $a \in S$ is a weak sharp minimum of f:

(ii) There exist $\tau, \delta \in (0, +\infty)$ such that

$$N(S, x) \cap B_{X^*} \subset [0, \tau]\partial f(x) \quad \forall x \in S \cap B(a, \delta).$$

(iii) There exist $\tau, \delta \in (0, +\infty)$ such that

$$\bigcup_{x \in S \cap B(a,\delta)} N(S, x) \cap B_{X^*} \subset \bigcup_{x \in S \cap B(a,\delta)} [0, \tau]\partial f(x).$$

(iv) There exist $\tau, \delta \in (0, +\infty)$ such that

$$d(h, T(S, u)) \leq \tau[d^+ f(u)(h)]_+ \quad \forall u \in S \cap B(a, \delta) \text{ and } h \in X.$$

(v) There exist $\tau, \delta \in (0, +\infty)$ such that

$$[d^+ f(u)(h)]_+ \geq \tau\|h\| \ \ \forall u \in S \cap B(a, \delta) \text{ and } h \in N(S, u).$$

(vi) There exist $\tau, \delta \in (0, +\infty)$ such that

$$[d^+ f(u)(x - u)]_+ \geq d(x, S) \ \ \forall x \in B(a, \delta) \text{ and } u \in P_S(x),$$

where $P_S(x) := \{u \in S : \ \|x - u\| = d(x, S)\}$.

The authors [39,40] extended (i)⇔(ii)⇔(iv) to convex generalized equations and (i)⇔(ii) to subsmooth (not necessarily convex) generalized equations in Banach spaces.

Next, in terms of some properties of the concerned multifunction inside the solution set, we consider sufficient and/or necessary conditions for the metric suregularity of generalized equation (GE) in the prox-regularity case. In particular, relaxing the convexity assumption, we generalize Result II to more general generalized equations in Hilbert spaces. To do this, we need the following inclusions which are immediate from the related definitions.

$$T(F^{-1}(b), u) \subset DF(u, b)^{-1}(0) \quad \forall u \in F^{-1}(b) \tag{5.9}$$

and

$$D_P^* F(u, b)(Y) \subset N_P(F^{-1}(b), u) \quad \forall u \in F^{-1}(b). \tag{5.10}$$

Theorem 5.2. *Let (GE) be prox-regular at* (a, b) *and consider the following statements.*

(i) (GE) is metrically subregular at a*.*

(ii) There exist $\tau, \delta \in (0, +\infty)$ *such that*

$$N_P(F^{-1}(b), u) \cap B_X \subset \tau D_P^* F(u, b)(B_Y) \quad \forall u \in F^{-1}(b) \cap B(a, \delta). \tag{5.11}$$

(iii) There exist $\tau, \delta \in (0, +\infty)$ *such that*

$$\bigcup_{u \in F^{-1}(b) \cap B(a,\delta)} N_P(F^{-1}(b), u) \cap B_X \subset \bigcup_{u \in F^{-1}(b) \cap B(a,\delta)} D_P^* F(u, b)(\tau B_Y). \tag{5.12}$$

(iv) There exist $\tau, \delta \in (0, +\infty)$ *such that*

$$d(x, T(F^{-1}(b), u)) \le \tau d(0, DF(u, b)(x)) \quad \forall u \in F^{-1}(b) \cap B(a, \delta) \text{ and } x \in X. \tag{5.13}$$

(v) There exist $\tau, \delta \in (0, +\infty)$ *such that*

$$d(0, DF(u, b)(h)) \ge \tau \|h\| \ \ \forall u \in F^{-1}(b) \cap B(a, \delta) \text{ and } h \in N_P(F^{-1}(b), u). \tag{5.14}$$

(vi) There exist $\tau, \delta \in (0, +\infty)$ *such that for all* $x \in B(a, \delta)$ *and* $u \in P_{F^{-1}(b)}(x)$*,*

$$d(0, DF(u, b)(x - u)) \ge \tau d(x, F^{-1}(b)). \tag{5.15}$$

Then (i)⇔(ii)⇔(iv), (ii)⇒(iii) and (iv)⇒(v)⇒(vi).

Proof. Since (GE) is prox-regular at a, there exist $\sigma, r \in (0, +\infty)$ such that

$$\langle u^*, x - u\rangle \le \langle v^*, y - b\rangle + \sigma(\|x - u\|^2 + \|y - b\|^2) \tag{5.16}$$

whenever $x \in B(a, r)$, $u \in F^{-1}(b) \cap B(a, r)$, $y \in F(x) \cap B(b, r)$, $v^* \in B_Y$ and $u^* \in D_c^* F(u, b)(v^*) \cap B_X$. Hence $N_P(\mathrm{Gr}(F), (u, b)) = N_c(\mathrm{Gr}(F), (u, b))$ and so

$$T(\mathrm{Gr}(F), (u, b)) = T_c(\mathrm{Gr}(F), (u, b)) \quad \forall u \in F^{-1}(b) \cap B(a, r).$$

This means that

$$D_P^* F(u, b) = D_c^* F(u, b) \text{ and } DF(u, b) = D_c F(u, b) \ \ \forall u \in F^{-1}(b) \cap B(a, r). \tag{5.17}$$

Hence $DF(u,b)$ is convex and

$$D^*(DF(u,b))(0,0) = D_P^*F(u,b) \quad \forall u \in F^{-1}(b) \cap B(a,r). \tag{5.18}$$

(i)⇒(ii). By [40, Theorem 4.2], (i) implies that there exist $\tau, \delta \in (0,\ +\infty)$ such that

$$\hat{N}(F^{-1}(b),u) \cap B_X \subset \tau D_c^*F(u,b)(B_Y) \quad \forall u \in F^{-1}(b) \cap B(a,\delta)$$

and so (5.11) holds because $N_p(F^{-1}(b),u) \subset \hat{N}(F^{-1}(b),u)$ for all $u \in F^{-1}(b)$. Thus (ii) holds.

The proof of (ii)⇒(i) is similar to that of [40, Theorem 4.4] (but with Proposition 2.1 replacing [42, Theorem 3.1]).

(i)⇒(iv). By (i) there exist $\tau \in (0,\ +\infty)$ and $\delta \in (0,\ r)$ such that

$$d(x, F^{-1}(b)) \leq \tau d(b, F(x)) \quad \forall x \in B(a,\delta).$$

By the already established implication (i)⇒(ii), we may assume that (5.11) holds (take a smaller δ and a larger τ if necessary). This together with (5.10) and (5.17) implies that

$$N_P(F^{-1}(b),u) = D_P^*F(u,b)(Y) = D_c^*F(u,b)(Y) \ \ \forall u \in F^{-1}(b) \cap B(a,\delta). \tag{5.19}$$

By (5.18), we can also rewrite (5.11) as

$$N_P(F^{-1}(b),u) \cap B_X \subset \tau D^*(DF(u,b))(0,0)(B_Y) \quad \forall u \in F^{-1}(b) \cap B(a,\delta). \tag{5.20}$$

Let $u \in F^{-1}(b) \cap B(a,\delta)$. We claim that

$$T(F^{-1}(b),u) = DF(u,b)^{-1}(0) \tag{5.21}$$

and

$$N(T(F^{-1}(b),u),0) \subset N_P(F^{-1}(b),u). \tag{5.22}$$

To show (5.21), let $h \in DF(u,b)^{-1}(0)$; we need only show that $h \in DF(u,b)^{-1}(0)$ (thanks to (5.9)). Noting that $(h,0) \in T(\mathrm{Gr}(F),(u,b))$, there exists a sequence $\{(t_n,h_n,y_n)\}$ in $\mathbb{R} \times X \times Y$ such that

$$t_n \to 0^+,\ (h_n,y_n) \to (h,0),\ b + t_n y_n \in F(u + t_n h_n) \ \text{ and } \ u + t_n h_n \in B(a,\delta).$$

Then, by our choice of τ and δ,

$$d(u + t_n h_n, F^{-1}(b)) \leq \tau d(b, F(u + t_n h_n)) \leq \tau t_n \|y_n\|.$$

It follows that there exists $u_n \in F^{-1}(b)$ such that $\|u + t_n h_n - u_n\| \leq 2\tau t_n \|y_n\|$ and so $u_n = u + t_n(h_n + \alpha_n)$ for some $\alpha_n \in 2\tau\|y_n\| B_X$. Noting that $h_n + \alpha_n \to h$, this implies that $h \in T(F^{-1}(b), u)$, as wished to show. To prove (5.22), suppose to the contrary that there exists

$$v \in N(T(F^{-1}(b), u), 0) \setminus N_P(F^{-1}(b), u). \tag{5.23}$$

We claim that $N_P(F^{-1}(b), u)$ is closed. Let $\{x_n^*\} \subset N_P(F^{-1}(b), u)$ converge to x^*. By (5.11) there exists a sequence $\{y_n^*\}$ in τB_Y such that $\frac{x_n^*}{M} \in D_P^* F(u, b)(y_n^*)$ for all $n \in \mathbb{N}$, where $M := 1 + \sup\limits_{n \in \mathbb{N}} \|x_n^*\|$. It follows from (5.16) that

$$\langle x_n^*/M, x - u\rangle \leq (1 + \tau)\sigma \|x - u\|^2 \quad \forall x \in F^{-1}(b) \cap B(a, r).$$

Letting $n \to \infty$, one has

$$\langle x^*, x - u\rangle \leq M(1 + \tau)\sigma \|x - u\|^2 \quad \forall x \in F^{-1}(b) \cap B(a, r)$$

and so $x^* \in N_P(F^{-1}(b), u)$. This shows that $N_P(F^{-1}(b), u)$ is closed. Since a proximal normal cone is always convex, it follows from (5.23) and the separation theorem that there exists $h \in X$ such that

$$\langle x^*, h\rangle \leq 0 < \langle v, h\rangle \quad \forall x^* \in N_P(F^{-1}(b), u). \tag{5.24}$$

This and (5.19) imply that

$$\langle (x^*, y^*), (h, 0)\rangle \leq 0 \quad \forall (x^*, y^*) \in N_c(\mathrm{Gr}(F), (u, b)).$$

This means that $h \in D_c F(u, b)^{-1}(0)$, and so $h \in T(F^{-1}(b), u)$ (by (5.17) and (5.21)). It follows from (5.23) that $\langle v, h\rangle \leq 0$, contradicting (5.24). This shows that (5.22) holds. By (5.20)—(5.22), one has

$$N(DF(u, b)^{-1}(0), 0) \cap B_X \subset D^*(DF(u, b))(0, 0)(\tau B_Y).$$

It follows from [39, Theorem 3.1] that

$$d(x, DF(u,b)^{-1}(0)) \leq \tau d(0, DF(u,b)(x)) \quad \forall x \in X.$$

This and (5.21) imply that (5.13) holds because u is arbitrary in $F^{-1}(b) \cap B(a,\delta)$. This completes the proof of (i)⇒(iv).

(iv)⇒(ii). Suppose that there exist $\tau > 0$ and $\delta \in (0,\ r)$ such that (5.13) holds. Since a contingent cone is always closed, this implies that

$$DF(u,b)^{-1}(0) \subset T(F^{-1}(b), u) \quad \forall u \in F^{-1}(b) \cap B(a,\delta).$$

It follows from (5.9) that (5.21) holds for all $u \in F^{-1}(b) \cap B(a,\delta)$. Therefore, (5.13) means that for each $u \in F^{-1}(b) \cap B(a,\delta)$, the sublinear generalized equation

$$0 \in DF(u,b)(x)$$

is metrically subregular at 0 with the modulus τ. This and [39, Theorem 3.1] imply that

$$N(DF(u,b)^{-1}(0), 0) \cap B_X \subset D^*(DF(u,b))(0,0)(\tau B_Y) \quad \forall u \in F^{-1}(b) \cap B(a,\delta).$$

Thus, by (5.17), (5.18) and (5.21), to prove (ii) it suffices to show that

$$N_P(F^{-1}(b), u) \subset N(T(F^{-1}(b), u), 0) \quad \forall u \in F^{-1}(b) \cap B(a,\delta). \tag{5.25}$$

Let $u \in F^{-1}(b) \cap B(a,\delta)$ and $u^* \in N_P(F^{-1}(b), u)$. Then there exist $\eta, \delta_0 \in (0,\ +\infty)$ such that

$$\langle u^*, x - u\rangle \leq \eta \|x - u\|^2 \quad \forall x \in F^{-1}(b) \cap B(u, \delta_0).$$

For any $h \in T(F^{-1}(b), u)$, there exists a sequence $\{(t_n, h_n)\}$ in $\mathbb{R} \times X$ such that

$$t_n \to 0^+,\ h_n \to h \ \text{ and } \ u + t_n h_n \in F^{-1}(b) \quad \forall n \in \mathbb{N}.$$

Hence $\langle u^*, t_n h_n\rangle \leq \eta \|t_n h_n\|^2$ for all sufficiently large n. This implies that $\langle u^*, h\rangle \leq 0$ for all $h \in T(F^{-1}(b), u)$. Noting (by (5.17) and (5.21)) that $T(F^{-1}(b), u)$ is a closed convex cone, it follows that $u^* \in N(T(F^{-1}(b), u), 0)$. Therefore (5.25) holds, and (iv)⇒(ii) is shown.

(ii)$\Rightarrow$(iii) is trivially true.

To prove (iv)$\Rightarrow$(v), suppose that there exist $\tau, \delta \in (0,\ +\infty)$ such that (5.13) holds. Let $u \in F^{-1}(b) \cap B(a, \delta)$ and $h \in N_P(F^{-1}(b), u)$. Then, there exist $\sigma, r \in (0,\ +\infty)$ such that $\langle h, x - a\rangle \leq \sigma^2\|x - a\|$ for all $x \in F^{-1}(b) \cap B(a, r)$. From the definition of the contingent cone, it is easy to verify that $\langle h, v\rangle \leq 0$ for all $v \in T(F^{-1}(b), u)$. This implies that $d(h, T(F^{-1}(b), u)) = \|h\|$. It follows from (5.13) that (v) holds.

(v)$\Rightarrow$(vi). Suppose that there exist $\tau, \delta \in (0,\ +\infty)$ such that (5.14) holds. Let $x \in B(a, \frac{\delta}{2})$ and $u \in P_{F^{-1}(b)}(x)$. Then $x - u \in N_P(F^{-1}(b), u)$. Noting that

$$\|u - a\| \leq \|u - x\| + \|x - a\| \leq 2\|x - a\| < \delta,$$

it follows from (5.14) that

$$d(0, DF(u, b)(x - u)) \geq \tau\|x - u\| = \tau d(x, F^{-1}(b)).$$

Hence (vi) holds. The proof is completed.

In Theorem 5.2, if F is assumed to be locally convex at (a, b) then the following proposition shows that each of (iii) and (vi) implies (i).

Proposition 5.1. *Suppose that F is locally convex at (a, b), namely there exists $r > 0$ such that $\mathrm{Gr}(F) \cap (B(a, r) \times B(b, r))$ is convex. Further suppose that one of the following properties is satisfied.*
1) There exist $\tau > 0$ and $\delta \in (0,\ r)$ such that (5.12) holds.
2) There exist $\tau, \delta \in (0,\ +\infty)$ such that (5.15) holds for all $x \in B(a, \delta)$ and $u \in P_{F^{-1}(b)}(x)$.
Then (GE) is metrically subregular at a.

Proof. First suppose that 1) holds. By Theorem 5.2, we need only show that (ii) of Theorem 5.2. Let $u \in F^{-1}(b) \cap B(a, \delta)$ and $u^* \in N_P(F^{-1}(b), u) \cap B_X$. It follows from 1) that there exist $z \in F^{-1}(b) \cap B(a, \delta)$ and $v \in B_Y$ such that $u^* \in D_P^*F(z, b)(\tau v)$. From the local convexity assumption, it is easy to verify that

$$\langle u^*, x - z\rangle - \langle \tau v, y - b\rangle \leq 0 \quad \forall x \in B(a, r) \text{ and } y \in F(x) \cap B(b, r). \tag{5.26}$$

Setting $y = b$ in (5.26), it follows that

$$\langle u^*, x - z\rangle \leq 0 \quad \forall x \in F^{-1}(b) \cap B(a, r). \tag{5.27}$$

On the other hand, noting that $u^* \in N_P(F^{-1}(b), u)$ and $F^{-1}(b) \cap B(a, r)$ is convex, one has

$$\langle u^*, x - u\rangle \leq 0 \quad \forall x \in F^{-1}(b) \cap B(a, r).$$

Since $\delta < r$, it follows from (5.26) that $\langle u^*, u\rangle = \langle u^*, z\rangle$. By (5.25), one has $u^* \in D^*_P F(u, b)(\tau v)$. This shows that (ii) of Theorem 5.2 holds.

Next suppose that 2) holds. Without loss of generality, we assume that $\delta < \frac{r}{2}$. Let $x \in B(a, \delta)$ be such that $y \in F(x) \cap B(b, \delta)$. Since $\mathrm{Gr}(F) \cap (B(a, r) \times B(b, r))$ is convex, $F^{-1}(b) \cap \bar{B}(a, 2\delta)$ is a closed convex set. By the projection theorem, there exists $u \in F^{-1}(b) \cap \bar{B}(a, 2\delta)$ such that $u \in P_{F^{-1}(b) \cap \bar{B}(a,2\delta)}(x)$. Since $\|x - a\| \leq \delta$, $d(x, F^{-1}(b)) = d(x, F^{-1}(b) \cap \bar{B}(a, 2\delta))$. Hence $u \in P_{F^{-1}(b)}(x)$. This and 2) imply that (5.15) holds. By the convexity of $\mathrm{Gr}(F) \cap (B(a, r) \times B(b, r))$, one has

$$(x - u, y - b) \in T(\mathrm{Gr}(F) \cap (B(a, r) \times B(b, r)), (u, b)) = T(\mathrm{Gr}(T), (u, b))$$

for all $y \in F(x) \cap B(b, r)$. It follows from (5.15) that

$$\|y - b\| \geq \tau d(x, F^{-1}(b)) \quad \forall y \in F(x) \cap B(a, r).$$

Since $d(x, F^{-1}(b)) < \frac{r}{2}$, one has

$$d(b, F(x)) \geq \min\{\tau, 2\} d(x, F^{-1}(b)).$$

This shows that (GE) is metrically regular at a. The proof is completed.

Remark. In the special case when $F(x) = [f(x), +\infty)$ and $f : X \to \mathbb{R} \cup \{+\infty\}$, Theorem 5.2 together with Proposition 5.1 extends Result II.

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