ON EXTENSION OF FENCHEL DUALITY AND ITS APPLICATION

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Abstract. By considering the epigraphs of conjugate functions, we extend the Fenchel duality, applicable to a (possibly infinite) family of proper lower semicontinuous convex functions on a Banach space. Applications are given in providing fuzzy KKT conditions for semi-infinite programming.

Key words. Fenchel duality, epigraph, KKT conditions, semi-infinite programming.

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1. Introduction. The famous Fenchel duality theorem can be stated as follows (cf. [30, Corollary 2.8.5]): for any family of finitely many proper lower semicontinuous convex functions $f_0, f_1, \ldots, f_n$ on a Banach space X, if $\mathrm{dom} f_{i_0} \cap \mathrm{int}\big(\bigcap_{i\neq i_0} \mathrm{dom} f_i\big) \neq \emptyset$ for some $i_0 \in \{0, 1, \ldots, n\}$, then their conjugate functions $f_0^*, f_1^*, \ldots, f_n^*$ satisfy the relation

$$\inf_{x\in X}\left(\sum_{i=0}^{n} f_i(x)\right) = \max\{-\sum_{i=0}^{n} f_i^*(x_i^*) : \sum_{i=0}^{n} x_i^* = 0\}, \tag{1.1}$$

and in fact the following stronger relation holds for any $x^* \in X^*$:

$$\inf_{x\in X}\{\sum_{i=0}^{n} f_i(x) - \langle x^*, x\rangle\} = \max\{-\sum_{i=0}^{n} f_i^*(x_i^*) : \sum_{i=0}^{n} x_i^* = x^*\}. \tag{1.2}$$

Background information on the Fenchel duality theory can be found in Rockafellar [28] (see also [2, 27, 30]). This theory is a fundamental tool for establishing penalty results in nonlinear programming (cf. [8]). Moreover, it also plays an important role in the theory of best approximation (cf. [14, 20]), error bound analysis [12] and in the study of monotone operators [25], and also in the KKT theory in connection with the following convex programming

$$\begin{aligned} &\min_{x\in X} \ f_0(x) \\ &\text{s.t.} \ \ f_i(x) \leq 0 \ \ (i = 1, \ldots, n). \end{aligned}$$

The Fenchel duality enables us to transform original problem (primal problem) into an optimization problem on the dual space (dual problem). In some cases, especially in optimal control problems, the dual problems are easier to handle than the original ones (see [13, Example 25.2], [15]). Stimulated by the study of semi-infinite programming problems (see [17, 21] and the references therein), it is both interesting and useful to extend the Fenchel duality applicable to a family $\{f_i\}_{i\in I}$ of proper lower semicontinuous convex functions on a Banach space with the index set I which is allowed to be infinite. In this present paper, much of our study is based on the consideration of epigraphs of the conjugate functions and is motivated by the recent work of Jeyakumar and his collaborators (see [9, 10, 19] for example); we provide characterizations (and sufficient conditions) for the following property: for any $x^* \in X^*$

$$\begin{aligned} &\inf_{x\in X}\{f(x) - \langle x^*, x\rangle\} \\ &= \max\{-\sum_{i\in I} f_i^*(x_i^*) : x_i^* \in X^* \text{ and } \sum_{i\in I}\langle x_i^*, x\rangle = \langle x^*, x\rangle \text{ for any } x \in X\}, \end{aligned}$$

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where f is the sum function of $\{f_i : i \in I\}$, that is $f(x) = \sum_{i\in I} f_i(x)$ for all $x \in X$. As an application, we present a fuzzy KKT conditions in section 5 for the semi-infinite programming problem.

2. Preliminaries. Throughout this paper, X denotes a Banach space and X^* denotes its topological dual. We use $B(x, \epsilon)$ (resp. $\overline{B}(x, \epsilon)$) to denote the open (resp. closed) ball of X with center x and radius ϵ. For a set A in X, the interior (resp. relative interior, closure, convex hull, affine hull, linear span) of A is denoted by $\mathrm{int}A$ (resp. $\mathrm{ri}A$, $\overline{A}$, $\mathrm{co}A$, $\mathrm{aff}A$, $\mathrm{span}A$) (if A is a subset of X^*, its weak* closure is denoted by $\overline{A}^{w^*}$). Let A be a nonempty subset of X. The indicator function $\delta_A : X \to \mathbb{R}\cup\{+\infty\}$ and the support function $\sigma_A : X^* \to \mathbb{R} \cup \{+\infty\}$ of A are respectively defined by

$$\delta_A(x) := \begin{cases} 0, & \text{if} \quad x \in A, \\ +\infty, & \text{otherwise}, \end{cases} \tag{2.1}$$

and $\sigma_A(x^*) = \sup_{x\in A}\langle x^*, x\rangle$ for all $x^* \in X^*$. Let $\Gamma(X)$ denote the class of proper lower semicontinuous convex functions on X, $\Gamma_c(X) := \{f \in \Gamma(X) : f$ is continuous and real-valued on $X\}$ and $\Gamma_+(x) := \{f \in \Gamma(X) : f$ is nonnegative on $X\}$. For a proper function f on X, the effective domain and the epigraph are respectively defined by $\mathrm{dom}f := \{x \in X : f(x) < +\infty\}$ and $\mathrm{epi}f := \{(x, r) \in X \times \mathbb{R} : f(x) \le r\}$. The subdifferential of f at $x \in X$ is defined by

$$\partial f(x) = \begin{cases} \{x^* \in X^* : \langle x^*, y - x\rangle \le f(y) - f(x)\,\forall\, y \in X\}, \text{ if } x \in \mathrm{dom}f, \\ \emptyset, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad , \text{otherwise}. \end{cases} \tag{2.2}$$

More generally, for any $\epsilon \ge 0$, the ϵ-subdifferential of f at $x \in X$ is defined by

$$\partial_\epsilon f(x) = \begin{cases} \{x^* \in X^* : \langle x^*, y - x\rangle \le f(y) - f(x) + \epsilon\,\forall\, y \in X\}, \text{ if } x \in \mathrm{dom}f, \\ \emptyset, \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad , \text{otherwise}. \end{cases} \tag{2.3}$$

As usual, for a proper function f on X, its conjugate function $f^* : X^* \to \mathbb{R} \cup \{+\infty\}$ is defined by $f^*(x^*) = \sup\limits_{x\in X}\{\langle x^*, x\rangle - f(x)\}$ for all $x^* \in X^*$. In particular, one has

$$(\delta_A)^*(x^*) = \sigma_A(x^*), \text{ for all } x^* \in X^*. \tag{2.4}$$

The definition of f^* entails that $\langle x^*, x\rangle \le f^*(x^*) + f(x)$ (Young's inequality) for any $x \in X$ and $x^* \in X^*$. Moreover, for any $\epsilon \ge 0$ and $x \in \mathrm{dom}f$

$$x^* \in \partial_\epsilon f(x) \;\Leftrightarrow\; f^*(x^*) + f(x) \le \langle x^*, x\rangle + \epsilon \;\Leftrightarrow\; (x^*, \epsilon + \langle x^*, x\rangle - f(x)) \in \mathrm{epi}f^*. \tag{2.5}$$

In particular, we have the following Young's equality

$$x^* \in \partial f(x) \;\Leftrightarrow\; \langle x^*, x\rangle = f^*(x^*) + f(x).$$

From the definitions, it is clear that for any proper functions f_1, f_2 on X,

$$f_1 \le f_2 \;\Leftrightarrow\; f_1^* \ge f_2^* \;\Leftrightarrow\; \mathrm{epi}f_1^* \subseteq \mathrm{epi}f_2^*. \tag{2.6}$$

Moreover, it is known that $f^* \in \Gamma(X^*)$ for any $f \in \Gamma(X)$ (cf. [30, Theorem 2.3.3]). As usual, $X^* \times \mathbb{R}$ and $(X \times \mathbb{R})^*$ are identified and, for convenience, we use the norms defined by

$$\|(x, \alpha)\| = \max\{\|x\|, |\alpha|\}, \quad \forall\, (x, \alpha) \in X \times \mathbb{R}$$

and

$$\|(x^*, \alpha)\| = \|x^*\| + |\alpha|, \quad \forall\, (x^*, \alpha) \in X^* \times \mathbb{R}.$$

If H is a subspace of X, the restrictions and the corresponding norms of the restrictions are defined as follows: $x^*|_H \in H^*$, $(x^*|_H, \alpha) \in H^* \times \mathbb{R} = (H \times \mathbb{R})^*$, $\|x^*\|_H := \sup\{\langle x^*, x\rangle : x \in H, \|x\| \leq 1\}$, and

$$\|(x^*|_H, \alpha)\| = \|x^*\|_H + |\alpha|. \tag{2.7}$$

Let I be an index set and let $\mathcal{F}(I)$ denote the collection of all finite subsets of I (thus $\mathcal{F}(I)$ is a directed set ordered under the inclusion relation). Let $\{a_i : i \in I\} \subseteq \mathbb{R} \cup \{+\infty\}$. We define the sum of $\{a_i : i \in I\}$ by

$$\sum_{i\in I} a_i = \lim_{A\in\mathcal{F}(I)} \sum_{i\in A} a_i$$

provided that the (unconditional) limit $\lim\limits_{A\in\mathcal{F}(I)} \sum\limits_{i\in A} a_i$ exists as a member of $\mathbb{R} \cup \{+\infty\}$. In particular, if $a_i \geq 0$ for all $i \in I$, then $\sum_{i\in I} a_i$ exists and

$$\sum_{i\in I} a_i = \sup_{A\in\mathcal{F}(I)} \sum_{i\in A} a_i \leq +\infty. \tag{2.8}$$

Remark 2.1. *Let $\{a_i, b_i, c_i\}_{i\in I} \subseteq \mathbb{R}$ be such that $a_i \leq b_i \leq c_i$ for all $i \in I$. Suppose that $\sum_{i\in I} a_i$ and $\sum_{i\in I} c_i$ exist in $\mathbb{R}$. Then $\sum_{i\in I} b_i$ also exists in $\mathbb{R}$ (because $0 \leq b_i - a_i \leq c_i - a_i$ and $\sum_{i\in I}(c_i - a_i) < +\infty$).*

Let $\{f_i : i \in I\}$ be a family of extended-real valued functions on X. We define their sum function f as follows: let $D_f := \{x \in X : \sum_{i\in I} f_i(x)$ exists in $\mathbb{R} \cup \{+\infty\}\}$; we define

$$f(x) = \sum_{i\in I} f_i(x) \text{ for all } x \in D_f.$$

In particular, if $f_i \in \Gamma_+(X)$ for all $i \in I$ then $D_f = X$ and

$$(\sum_{i\in I} f_i)(x) = \sup_{A\in\mathcal{F}(I)} \sum_{i\in A} f_i(x) \text{ for all } x \in X. \tag{2.9}$$

For $x^* \in X^*$ and a family $\{x_i^*\}_{i\in I}$ of elements in X^*, the notation

$$x^* = {\sum_{i\in I}}^* x_i^* \tag{2.10}$$

means that $\langle x^*, h\rangle = \lim\limits_{A\in\mathcal{F}(I)} \sum\limits_{i\in A} \langle x_i^*, h\rangle$, for each $h \in X$. Let $\{A_i\}_{i\in I}$ be a family of subsets of X^*. The set $\{x^* \in X^* : \exists x_i^* \in A_i \,\forall\, i \in I$ such that $x^* = {\sum\limits_{i\in I}}^* x_i^*\}$ will be denoted by ${\sum\limits_{i\in I}}^* A_i$. It is easy to check that ${\sum\limits_{i\in I}}^* A_i$ is convex if each A_i is convex, and that ${\sum\limits_{i\in I}}^* A_i = \sum\limits_{i\in I} A_i$ if I is a finite set. Moreover, $\{A_i\}_{i\in I}$ is said to be weak* summable if ${\sum\limits_{i\in I}}^* x_i^*$ exists in X^* (that is, (2.10) holds for some $x^* \in X^*$) whenever $x_i^* \in A_i$ for each $i \in I$.

Remark 2.2. *The above definition is slightly different from [32]: our notation ${\sum\limits_{i\in I}}^* A_i$ does not require the family $\{A_i\}_{i\in I}$ to be weak* summable.*

A useful relationship between epif^* and $\partial_\epsilon f$ is given in the following formula observed by Jeyakumar et. al. in [9] (we note that, as observed in [3], this formula works even when f is merely a proper function):

$$\text{epi} f^* = \bigcup_{\epsilon\geq 0} \{(x^*, \epsilon + \langle x^*, x\rangle - f(x)) : x^* \in \partial_\epsilon f(x)\} \quad \forall\ f \in \Gamma(X),\ x \in \text{dom} f. \tag{2.11}$$

Throughout this paper, unless explicitly mentioned otherwise, I is an arbitrary index set (that is, the cardinality $|I| \leq +\infty$). For convenience, we list below several known results that will be useful for us.

LEMMA 2.1. *(cf. [30]) Let I be a finite set and let $\{f, f_i : i \in I\} \subseteq \Gamma(X)$ be such that $f(x) = \sum_{i\in I} f_i(x)$ for all $x \in X$. Then $\text{epi} f^* = \overline{\sum_{i\in I} \text{epi} f_i^*}^{w^*}$ and, moreover the result can be strengthened to $\text{epi} f^* = \sum_{i\in I} \text{epi} f_i^*$ if there exists $i_0 \in I$ such that $\text{dom} f_{i_0} \cap \text{int}\big(\bigcap_{i\neq i_0} (\text{dom} f_i)\big) \neq \emptyset$.*

REMARK 2.3. *Let I be a finite set and let C be a closed convex subset of X. Recall that $\text{sqri} C := \{x \in C : \bigcup_{\lambda\geq 0} \lambda(C - x)$ is a closed subspace$\}$. A weaker generalized interior point regularity condition ensuring $\text{epi} f^* = \sum_{i\in I} \text{epi} f_i^*$ is as follows (cf. [5, 23]): there exists $i_0 \in I$ such that*

$$0 \in \text{sqri} \prod_{i\neq i_0} (\text{dom} f_i - \text{dom} f_{i_0}).$$

The following lemma can be found in [20, Lemma 2.3]. We note that it has been also derived in [4, Section 4.3] via a different approach.

LEMMA 2.2. *Let $\{f_i : i \in I\} \subseteq \Gamma(X)$. Suppose that there exists $x_0 \in X$ such that $\sup_{i\in I} f_i(x_0) < \infty$. Then*

$$\text{epi}(\sup_{i\in I} f_i)^* = \overline{\text{co} \bigcup_{i\in I} \text{epi} f_i^*}^{w^*},$$

where $\sup_{i\in I} f_i : X \to \mathbb{R} \cup \{+\infty\}$ is defined by $(\sup_{i\in I} f_i)(x) = \sup_{i\in I} f_i(x)$ for all $x \in X$.

REMARK 2.4. *Let $f \in \Gamma(X)$ and $A := \{x : f(x) \leq 0\} \neq \emptyset$. Then $\delta_A = \sup_{\lambda>0} \lambda f$ and it follows from Lemma 2.2 that,*

$$\text{epi}(\delta_A)^* = \overline{\text{co} \bigcup_{\lambda>0} \text{epi}(\lambda f)^*}^{w^*} = \overline{\bigcup_{\lambda>0} \text{epi}(\lambda f)^*}^{w^*}, \tag{2.12}$$

where the last equality holds because $\bigcup_{\lambda>0} \text{epi}(\lambda f)^*$ is a convex set.

For continuous functions, the following result in [31] will play an important role.

LEMMA 2.3. *Let $\{f, f_i : i \in I\} \subseteq \Gamma_c(X)$ be such that $f(x) = \sum_{i\in I} f_i(x)$ for all $x \in X$. Then $\{\partial f_i(x)\}_{i\in I}$ is weak* summable and the following relation holds*

$$\partial f(x) = \overline{\sum_{i\in I}{}^{*} \partial f_i(x)}^{w^*} \qquad \text{for all } x \in X.$$

Moreover, if I is countable then $\sum_{i\in I}^{*} \partial f_i(x)$ is weak closed and hence*

$$\partial f(x) = \sum_{i\in I}{}^{*} \partial f_i(x) \qquad \text{for all } x \in X.$$

3. Strong Fenchel Duality and its Characterization. In this section, we provide some characterization of the strong Fenchel duality (in the sense that (1.2) holds for all $x^* \in X^*$). To do this, we need the following lemma.

LEMMA 3.1. *Let $\{f, f_i : i \in I\} \subseteq \Gamma(X)$ be such that*

$$f(x) = \sum_{i\in I} f_i(x) \text{ for all } x \in X. \tag{3.1}$$

Then the following inclusion holds:

$$\overline{\sum_{i\in I}^{*}\mathrm{epi} f_i^*}^{w^*} \subseteq \mathrm{epi} f^*. \tag{3.2}$$

Proof. Let $(x^*,\alpha)\in \sum_{i\in I}^{*}\mathrm{epi} f_i^*$, that is, for each $i\in I$ there exists $(x_i^*,\alpha_i)\in X^*\times\mathbb{R}$ with $f_i^*(x_i^*)\le\alpha_i$ such that

$$\sum_{i\in I}\alpha_i=\alpha \text{ and } \sum_{i\in I}\langle x_i^*,x\rangle=\langle x^*,x\rangle\ \forall\ x\in X. \tag{3.3}$$

Since $\mathrm{epi} f^*$ is weak* closed, to prove (3.2), it suffices to show $f^*(x^*)\le\alpha$. Let $x\in\mathrm{dom} f$. Note that

$$\langle x_i^*,x\rangle-f_i(x)\le\sup_{z\in X}\{\langle x_i^*,z\rangle-f_i(z)\}=f_i^*(x_i^*)\le\alpha_i. \tag{3.4}$$

Applying Remark 2.1 and making use of (3.1), (3.3) and (3.4), we note that $\sum_{i\in I}f_i^*(x_i^*)$ exists and

$$\langle x^*,x\rangle-f(x)=\sum_{i\in I}\big(\langle x_i^*,x\rangle-f_i(x)\big)\le\sum_{i\in I}f_i^*(x_i^*)\le\sum_{i\in I}\alpha_i=\alpha.$$

Taking supremum over all x in $\mathrm{dom} f$, this implies that

$$f^*(x^*)=\sup_{x\in\mathrm{dom} f}\big(\langle x^*,x\rangle-f(x)\big)\le\sum_{i\in I}f_i^*(x_i^*)\le\alpha, \tag{3.5}$$

as required to show. This completes the proof. □

The following result is known [9] (see also [11, Corollary 3.4]) for the special case when I is finite.

THEOREM 3.2. *Let $\{f,f_i: i\in I\}$ be as in Lemma 3.1. Then the following statements are equivalent:*

(i) $$\partial_\epsilon f(x)\subseteq\bigcup\{\sum_{i\in I}^{*}\partial_{\epsilon_i}f_i(x):\sum_{i\in I}\epsilon_i=\epsilon,\ \textit{each }\epsilon_i\ge0\}\ \forall\ \epsilon\ge0\ \textit{and}\ x\in X. \tag{3.6}$$

(ii) $$\partial_\epsilon f(x)=\bigcup\{\sum_{i\in I}^{*}\partial_{\epsilon_i}f_i(x):\sum_{i\in I}\epsilon_i=\epsilon,\ \textit{each }\epsilon_i\ge0\}\ \forall\ \epsilon\ge0\ \textit{and}\ x\in\mathrm{dom} f.$$

(iii) $\mathrm{epi} f^*=\sum_{i\in I}^{*}\mathrm{epi} f_i^*$.

(iv) *For any $x^*\in X^*$,*

$$\inf_{x\in X}\{f(x)-\langle x^*,x\rangle\}=\max\{-\sum_{i\in I}f_i^*(x_i^*):\sum_{i\in I}^{*}x_i^*=x^*\},$$

that is, $f^*(x^*)=\min\{\sum_{i\in I}f_i^*(x_i^*):\sum_{i\in I}^{*}x_i^*=x^*\}$.

Any of the statements (i)-(iv) *implies*

(v) $$\inf_{x\in X}f(x)=\max\{-\sum_{i\in I}f_i^*(x_i^*):\sum_{i\in I}^{*}x_i^*=0\}.$$

Proof. First, (v) follows from (iv) by letting $x^*=0$. Thus, we only need to show the equivalence of (i)-(iv).

[(i) $\Rightarrow$ (ii)] Let $x \in \mathrm{dom} f$, $\epsilon \geq 0$ and $\epsilon_i \geq 0$ be such that $\sum_{i\in I} \epsilon_i = \epsilon$. To prove (i) $\Rightarrow$ (ii), it suffices to show that

$$\sum_{i\in I}^{*} \partial_{\epsilon_i} f_i(x) \subseteq \partial_\epsilon f(x). \tag{3.7}$$

To do this, let $x^* = \sum_{i\in I}^{*} x_i^* \in X^*$, where each $x_i^* \in \partial_{\epsilon_i} f_i(x)$. Then from Young's inequality and (2.5) we have

$$\langle x_i^*, x\rangle - \epsilon_i \leq f_i^*(x_i^*) + f_i(x) - \epsilon_i \leq \langle x_i^*, x\rangle.$$

Therefore, by Remark 2.1, $\sum_{i\in I} (f_i^*(x_i^*) + f_i(x) - \epsilon_i)$ exists in $\mathbb{R}$ and

$$\sum_{i\in I} f_i^*(x_i^*) + f(x) - \epsilon = \sum_{i\in I} \big(f_i^*(x_i^*) + f_i(x) - \epsilon_i\big) \leq \sum_{i\in I} \langle x_i^*, x\rangle = \langle x^*, x\rangle. \tag{3.8}$$

On the other hand, note that $f^*(x^*) \leq \sum_{i\in I} f_i^*(x_i^*)$ because for each $z \in \mathrm{dom} f$ one has

$$\langle x^*, z\rangle - f(z) = \sum_{i\in I} \big(\langle x_i^*, z\rangle - f_i(z)\big) \leq \sum_{i\in I} f_i^*(x_i^*). \tag{3.9}$$

Thus, by (3.8),

$$f^*(x^*) + f(x) - \epsilon \leq \sum_{i\in I} f_i^*(x_i^*) + f(x) - \epsilon \leq \langle x^*, x\rangle.$$

Therefore $x^* \in \partial_\epsilon f(x)$ and (3.7) holds.
[(ii) $\Rightarrow$ (iii)] In view of Lemma 3.1, it suffices to show that $\mathrm{epi} f^* \subseteq \sum_{i\in I}^{*} \mathrm{epi} f_i^*$. To do this, let $(x^*, \alpha) \in \mathrm{epi} f^*$. We have to show that $(x^*, \alpha) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*$. Take an arbitrary $x \in \mathrm{dom} f$; from (2.11), there exists $\epsilon \geq 0$ such that $x^* \in \partial_\epsilon f(x)$ and $\alpha = \epsilon + \langle x^*, x\rangle - f(x)$. It follows from (ii) that there exist $\epsilon_i \geq 0$ and $x_i^* \in \partial_{\epsilon_i} f_i(x)$ $\big($so $(x_i^*, \alpha_i) \in \mathrm{epi} f_i^*$ where $\alpha_i := \epsilon_i + \langle x_i^*, x\rangle - f_i(x)\big)$ such that $\epsilon = \sum_{i\in I} \epsilon_i$ and $x^* = \sum_{i\in I}^{*} x_i^*$. Thus

$$(x^*, \alpha) = \sum_{i\in I}^{*} (x_i^*, \alpha_i) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*,$$

as required to show.
[(iii) $\Rightarrow$ (iv)] Let $x^* \in X^*$. Note first that, by (3.9),

$$-f^*(x^*) = \inf_{z\in \mathrm{dom} f} \{f(z) - \langle x^*, z\rangle\} \geq -\sum_{i\in I} f_i^*(x_i^*) \tag{3.10}$$

whenever $\{x_i^* : i \in I\} \subseteq X^*$ with $x^* = \sum_{i\in I}^{*} x_i^*$. Thus to prove (iv), it remains to show that there exists $x_i^* \in X^*$ $(i \in I)$ such that $x^* = \sum_{i\in I} x_i^*$ and

$$\inf_{z\in \mathrm{dom} f} \{f(z) - \langle x^*, z\rangle\} \leq -\sum_{i\in I} f_i^*(x_i^*). \tag{3.11}$$

To do this, we can suppose that $\inf_{z\in \mathrm{dom} f} \{f(z) - \langle x^*, z\rangle\} > -\infty$, that is $f^*(x^*) < +\infty$. Then $(x^*, f^*(x^*)) \in \mathrm{epi} f^*$. It follows from (iii) that $(x^*, f^*(x^*)) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*$, that is, there exist $(x_i^*, \alpha_i) \in \mathrm{epi} f_i^*$ $(i \in I)$ such that

$$\sum_{i\in I}^{*} x_i^* = x^* \quad \text{and} \quad \sum_{i\in I} \alpha_i = f^*(x^*). \tag{3.12}$$

We claim that $\{x_i^* : i \in I\}$ satisfies (3.11). In fact, since $(x_i^*, \alpha_i) \in \text{epi} f_i^*$ $(i \in I)$, Young's inequality implies that for any $z \in X$

$$\langle x_i^*, z\rangle - f_i(z) \le f_i^*(x_i^*) \le \alpha_i \ (i \in I). \tag{3.13}$$

Since $\sum_{i\in I} f_i(z) = f(z) \in \mathbb{R}$ if $z \in \text{dom} f$, it follows from (3.12) and Remark 2.1 that $\sum_{i\in I} f_i^*(x_i^*)$ exists and for any $z \in \text{dom} f$

$$\langle x^*, z\rangle - f(z) = \sum_{i\in I}\big(\langle x_i^*, z\rangle - f_i(z)\big) \le \sum_{i\in I} f_i^*(x_i^*) \le \sum_{i\in I}\alpha_i = f^*(x^*).$$

Taking supremum over all $z \in \text{dom} f$, this implies that $f^*(x^*) \le \sum_{i\in I} f_i^*(x_i^*) \le \sum_{i\in I}\alpha_i = f^*(x^*)$. In view of (3.13), this forces that $f_i^*(x_i^*) = \alpha_i$ for all $i \in I$. Therefore, we obtain that

$$\inf_{z\in\text{dom} f}\{f(z) - \langle x^*, z\rangle\} = -f^*(x^*) = -\sum_{i\in I}\alpha_i = -\sum_{i\in I} f_i^*(x_i^*).$$

Thus (3.11) holds as claimed.
[(iv) $\Rightarrow$ (i)] Let $\epsilon \ge 0$, $x \in X$ and $x^* \in \partial_\epsilon f(x)$. By definition of $f^*(x^*)$, (iv) means that

$$f^*(x^*) = \min\{\sum_{i\in I} f_i^*(x_i^*) : {\sum_{i\in I}}^* x_i^* = x^*\}.$$

Thus, there exist $x_i^* \in X^*$ with ${\sum_{i\in I}}^* x_i^* = x^*$ such that $f^*(x^*) = \sum_{i\in I} f_i^*(x_i^*)$. Hence

$$f^*(x^*) + f(x) - \langle x^*, x\rangle = \sum_{i\in I}\big(f_i^*(x_i^*) + f_i(x) - \langle x_i^*, x\rangle\big).$$

where $0 \le f_i^*(x_i^*) + f_i(x) - \langle x_i^*, x\rangle$ for all $i \in I$ (by Young's inequality). Since $x^* \in \partial_\epsilon f(x)$ (that is $f^*(x^*) + f(x) - \langle x^*, x\rangle \le \epsilon$), it follows that there exist $\epsilon_i \ge 0$ $(i \in I)$ such that $\sum_{i\in I}\epsilon_i = \epsilon$ and

$$f_i^*(x_i^*) + f_i(x) - \langle x_i^*, x\rangle \le \epsilon_i \quad \forall\, i \in I.$$

Then $x_i^* \in \partial_{\epsilon_i} f_i(x)$ $(i \in I)$ and $x^* \in {\sum_{i\in I}}^* \partial_{\epsilon_i} f_i(x)$ (as $x^* = {\sum_{i\in I}}^* x_i^*$). Therefore x^* belongs to the set on the right hand side of (i). This completes the proof. $\square$

Note 3.1. *The property* (v)*, sometimes referred as the Fenchel duality, is strictly weaker (even when* $|I| = 2$*) than the properties* (i)-(iv) *listed in Theorem 3.2. Examples can be found in [5, page 2798-2799] and [26, Example 11.1 and Example 11.3] .*

Corollary 3.3. *(Extension of Fenchel duality) Let* $\{f_i, h, f : i \in I \cup J\} \subseteq \Gamma(X)$ *with* $I \cap J = \emptyset$, $|J| < +\infty$ *and*

$$h(x) = \sum_{i\in I} f_i(x) \text{ and } f(x) = \sum_{i\in I} f_i(x) + \sum_{j\in J} f_j(x) \text{ for all } x \in X.$$

Suppose that

$$\text{epi} h^* = {\sum_{i\in I}}^* \text{epi} f_i^* \tag{3.14}$$

and (at least) one of the following conditions holds:

(i) $\text{dom}\, h \cap \text{int}\Big(\bigcap_{j\in J} \text{dom} f_j\Big) \ne \emptyset.$ (3.15)

(ii) *There exists $j_0 \in J$ such that*

$$\mathrm{int}(\mathrm{dom}\, h) \cap \mathrm{dom} f_{j_0} \cap \mathrm{int}\Big(\bigcap_{j \in J \setminus \{j_0\}} \mathrm{dom} f_j \Big) \neq \emptyset. \tag{3.16}$$

Then

$$\mathrm{epi} f^* = \sum_{i \in I}^{*} \mathrm{epi} f_i^* + \sum_{j \in J} \mathrm{epi} f_j^*, \tag{3.17}$$

and in particular, one has

$$\inf_{x \in X} f(x) = \max\{- \sum_{i \in I} f_i^*(x_i^*) - \sum_{j \in J} f_j^*(y_j^*) : \sum_{i \in I}^{*} x_i^* + \sum_{j \in J} y_j^* = 0\}. \tag{3.18}$$

Proof. First, from the implication (iii) $\Rightarrow$ (v) in Theorem 3.2, we need only to show (3.17). Since $\{f_j, h, f : j \in J\} \subseteq \Gamma(X)$ and $f = h + \sum_{j \in J} f_j$, Lemma 2.1 implies that

$$\mathrm{epi} f^* = \mathrm{epi} h^* + \sum_{j \in J} \mathrm{epi} f_j^*,$$

provided that (i) or (ii) holds. Consequently (3.17) holds by (3.14). □

4. Sufficient Conditions. This section is devoted to provide sufficient conditions ensuring that for $\{f_i, f : i \in I\} \subseteq \Gamma(X)$, $\mathrm{epi} f^* = \sum_{i \in I}^{*} \mathrm{epi} f_i^*$ (see Theorem 3.2 (iii)), where

$$f(x) = \sum_{i \in I} f_i(x) \text{ for all } x \in X. \tag{4.1}$$

4.1. Continuous type. Throughout this subsection, we assume f and each f_i are continuous, that is,

$$\{f_i, f : i \in I\} \subseteq \Gamma_c(X). \tag{4.2}$$

THEOREM 4.1. *Assume (4.1) and (4.2). Then*

$$\mathrm{epi} f^* = \overline{\sum_{i \in I}^{*} \mathrm{epi} f_i^*}^{w^*}. \tag{4.3}$$

Proof. Let $x \in X$. By continuity, each $\partial f_i(x) \neq \emptyset$; take $x_i^* \in \partial f_i(x)$. By Lemma 2.3, there exists $x^* \in \partial f(x)$ such that $x^* = \sum_{i \in I}^{*} x_i^*$. Denote $r := \langle x^*, x \rangle - f(x)$ and $r_i = \langle x_i^*, x \rangle - f_i(x)$. It follows from (4.1) that $r = \sum_{i \in I} r_i$. Moreover, by (2.5), each $(x_i^*, r_i) \in \mathrm{epi} f_i^*$ and so $(x^*, r) \in \sum_{i \in I}^{*} \mathrm{epi} f_i^*$. Therefore, by Lemma 3.1, $\emptyset \neq \sum_{i \in I}^{*} \mathrm{epi} f_i^* \subseteq \mathrm{epi} f^*$. Thus, since $\mathrm{epi} f^*$ is weak* closed, if (4.3) is not true then there exists $(x^*, \alpha) \in \mathrm{epi} f^* \setminus \overline{\sum_{i \in I}^{*} \mathrm{epi} f_i^*}^{w^*}$. Recalling that a linear functional h on X^* is the form $h(x^*) = \langle a, x^* \rangle$ for some $a \in X$ if and only if h is continuous in the weak* topology of X^* (cf. [29, Page 112, Theorem 1], it follows from the separation theorem that there exists $(x_0, r_0) \in X \times \mathbb{R}$ such that

$$\sup\{\langle y^*, x_0 \rangle + \beta r_0 : \ (y^*, \beta) \in \sum_{i \in I}^{*} \mathrm{epi} f_i^*\} < \langle x^*, x_0 \rangle + \alpha r_0. \tag{4.4}$$

Considering $\beta > 0$ large, it follows that $r_0 \le 0$. We claim that $r_0 < 0$. Indeed, if $r_0 = 0$ then (4.4) means $\sup\{\langle y^*, x_0\rangle : (y^*,\beta) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*\} < \langle x^*, x_0\rangle$. Since $x^* \in \mathrm{dom} f^*$ and $\mathrm{Im}\,\partial f$ is norm dense in $\mathrm{dom} f^*$ (cf. [24, Theorem 3.18]), there exist $a^* \in \mathrm{Im}\,\partial f$ (so $a^* \in \partial f(a)$ for some $a \in X$) such that $\sup\{\langle y^*, x_0\rangle : (y^*,\beta) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*\} < \langle a^*, x_0\rangle$. By Lemma 2.3, this implies that

$$\sup\{\langle y^*, x_0\rangle : (y^*,\beta) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*\} < \langle a_0^*, x_0\rangle \tag{4.5}$$

for some $a_0^* \in \sum_{i\in I}^{*} \partial f_i(a)$. Note that a_0^* can be expressed in the form $a_0^* = \sum_{i\in I}^{*} a_i^*$ with each $a_i^* \in \partial f_i(a)$. Since each $\langle a_i^*, a\rangle = f_i(a) + f_i^*(a_i^*)$ (Young's equality) it follows from (4.1) that $\langle a_0^*, a\rangle = f(a) + \sum_{i\in I} f_i^*(a_i^*)$ and hence that $(a_0^*, \beta_0) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*$ where $\beta_0 := \langle a_0^*, a\rangle - f(a) \in \mathbb{R}$. But then $\sup\{\langle y^*, x_0\rangle : (y^*,\beta) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*\} \ge \langle a_0^*, x_0\rangle$, contradicting (4.5). Henceforth, without loss of generality, we may assume that $r_0 = -1$. Then (4.4) becomes

$$\sup\{\langle y^*, x_0\rangle - \beta : (y^*,\beta) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*\} < \langle x^*, x_0\rangle - \alpha. \tag{4.6}$$

Note that $\langle x^*, x_0\rangle - \alpha \le f(x_0)$ by Young's inequality and the fact that $(x^*, \alpha) \in \mathrm{epi} f^*$ and it follows from (4.6) that

$$\sup\{\langle y^*, x_0\rangle - \beta : (y^*,\beta) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*\} < f(x_0). \tag{4.7}$$

Moreover, for each $i \in I$, pick $x_i^* \in \partial f_i(x_0)$. Define $x_0^* := \sum_{i\in I}^{*} x_i^*$ (this is well-defined by Lemma 2.3). Let $\alpha_0 := \langle x_0^*, x_0\rangle - f(x_0)$. Note from Young's equality that $\langle x_i^*, x_0\rangle = f_i(x_0) + f_i^*(x_0^*)$ for each $i \in I$ and it follows from (4.1) that $\alpha_0 = \sum_{i\in I} f_i^*(x_i^*)$ and hence that $(x_0^*, \alpha_0) = \sum_{i\in I}^{*} (x_i^*, f_i^*(x_i^*)) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*$. Consequently, by (4.7), $\langle x_0^*, x_0\rangle - \alpha_0 < f(x_0)$ contradicting the definition of α_0.□

If I is countable and if another assumption, namely

$$\mathrm{dom}\, f^* = \mathrm{Im}\,\partial f \tag{4.8}$$

is added, the following result shows that the set $\sum_{i\in I}^{*} \mathrm{epi} f_i^*$ is weak* closed.

THEOREM 4.2. *Assume (4.8) in addition to (4.1) and (4.2), and suppose that I is countable. Then* $\mathrm{epi} f^* = \sum_{i\in I}^{*} \mathrm{epi} f_i^*$.

Proof. Noting $\mathrm{epi} f^* = \mathrm{gph} f^* + \{0\} \times [0, +\infty)$ and

$$\sum_{i\in I}^{*} \mathrm{epi} f_i^* + \{0\} \times [0,\infty) \subseteq \sum_{i\in I}^{*} \mathrm{epi} f_i^*. \tag{4.9}$$

(because $\mathrm{epi} f_i^* + \{0\} \times [0,\infty) = \mathrm{epi} f_i^*$ for each i), and making use of Theorem 4.1 we need only to show

$$\mathrm{gph} f^* \subseteq \sum_{i\in I}^{*} \mathrm{epi} f_i^*, \tag{4.10}$$

where $\mathrm{gph} f^*$ denotes the graph of f^*. To see (4.10), let $(x^*, \alpha) \in \mathrm{gph} f^*$. Then $x^* \in \mathrm{dom} f^* = \mathrm{Im}\,\partial f$ thanks to (4.8). Hence there exists $x \in X$ such that $x^* \in \partial f(x)$. By Lemma 2.3, x^* can be expressed in the form

$$x^* = \sum_{i\in I}^{*} x_i^*,$$

where each $x_i^* \in \partial f_i(x)$. By Young's equality, $f^*(x^*) = \langle x^*, x\rangle - f(x)$ and each $f_i^*(x_i^*) = \langle x_i^*, x\rangle - f_i(x)$, and it follows from (4.1) that $\sum_{i\in I} f_i^*(x_i^*) = \langle x^*, x\rangle - f(x)$, that is, $\sum_{i\in I} f_i^*(x_i^*) = f^*(x^*) = \alpha$. Therefore $(x^*, \alpha) = \sum_{i\in I}^{*}(x_i^*, f_i^*(x_i^*)) \in \sum_{i\in I}^{*} \mathrm{epi} f_i^*$. This completes the proof. □

4.2. Nonnegative type. Throughout this subsection, we assume that f and each f_i are nonnegative-valued, that is,

$$\{f_i, f : i \in I\} \subseteq \Gamma_+(X). \tag{4.11}$$

THEOREM 4.3. *Assume (4.1) and (4.11). Then*

$$\mathrm{epi} f^* = \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{i\in J} \mathrm{epi} f_i^*}^{w^*} = \overline{\sum_{i\in I}^{*} \mathrm{epi} f_i^*}^{w^*}. \tag{4.12}$$

Proof. Since each $\mathrm{epi} f_i^*$ is a convex set containing the origin (because $f_i \geq 0$), one has from (2.6) and Lemma 3.1 that

$$\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{i\in J} \mathrm{epi} f_i^* \subseteq \sum_{i\in I}^{*} \mathrm{epi} f_i^* \subseteq \mathrm{epi} f^*$$

and hence

$$\overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{i\in J} \mathrm{epi} f_i^*}^{w^*} \subseteq \overline{\sum_{i\in I}^{*} \mathrm{epi} f_i^*}^{w^*} \subseteq \mathrm{epi} f^*. \tag{4.13}$$

For each $J \subseteq I$ with $|J| < \infty$, let g_J denote the sum function of $\{f_i : i \in J\}$, namely $g_J(x) = \sum_{i\in J} f_i(x)$ for all $x \in X$. Since each f_i is nonnegative-valued, we have that, by (4.1) and (2.9),

$$f = \sum_{i\in I} f_i = \sup_{\substack{J\subseteq I,\\ |J|<\infty}} g_J. \tag{4.14}$$

Hence, by Lemma 2.2 (applied to $\{g_J : J \subseteq I, |J| < +\infty\}$) and Lemma 2.1, we have

$$\mathrm{epi} f^* = \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \mathrm{epi} g_J^*}^{w^*} = \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \overline{\sum_{i\in J} \mathrm{epi} f_i^*}^{w^*}}^{w^*} \tag{4.15}$$

(note that $\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \mathrm{epi} g_J^*$ is a convex set since $\mathrm{epi} g_{J_1}^* \subseteq \mathrm{epi} g_{J_2}^*$ if $J_1 \subseteq J_2$). Combining this with (4.13) and (4.15) we see that (4.12) holds because the set on the right-hand side of (4.15) is equal to that on the left-hand side of (4.13) $\big($to see the latter fact, note that, for any $J \subseteq I$ with $|J| < +\infty$ one has

$$\overline{\sum_{i\in J} \mathrm{epi} f_i^*}^{w^*} \subseteq \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{i\in J} \mathrm{epi} f_i^*}^{w^*},$$

and so

$$\overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \overline{\sum_{i\in J}\text{epi}f_i^*}^{w^*}}^{w^*} \subseteq \overline{\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{i\in J}\text{epi}f_i^*}^{w^*}).$$

This completes the proof. □

Next, we seek some sufficient conditions to ensure that the set $\sum_{i\in I}^{*}\text{epi}f_i^*$ in Theorem 4.3 is weak* closed. It would be convenient for us to introduce some new notation first. Let Y be a Banach space and let J be a finite set. Let $\{K_i\}_{i\in J}$ be closed convex cones of Y. Following [28], we define $\gamma(K_i;J)$ by

$$\gamma(K_i;J) = \inf\{\|\sum_{i\in J} y_i\| : \sum_{i\in J}\|y_i\| = 1,\ \text{each } y_i\in K_i\}. \tag{4.16}$$

When $J=\{1,2\}$ and Y is a Hilbert space, the corresponding value of $\cos^{-1}\gamma(K_i;J)$ is termed as the angle between the closed convex cones K_1 and K_2 (see [7] for a detailed discussion). Given $y^*\in Y^*$ and any subspace Z of Y, $y^*|_Z$ denotes the restriction of y^* to Z and $\|y^*\|_Z$ denotes the corresponding norm of $y^*|_Z$ in Z^*. Furthermore, let $D\subseteq Y^*$, we define $D|_Z := \{y^*|_Z : y^*\in D\}$. Let K be a subset of Y (resp. Y^*), the (negative) polar of K is denoted by K° and defined by $K^\circ = \{y^*\in Y^* : \langle y^*, y\rangle \le 0$ for all $y\in K\}$) (resp. $K^\circ = \{y\in Y : \langle y^*, y\rangle\le 0$ for all $y^*\in K\}$). From the definition, it is clear that if K_1 and K_2 are two subsets of Y (resp. Y^*) and $K_1\subseteq K_2$, then $K_2^\circ\subseteq K_1^\circ$.

When H is a subspace of X, $Y = X^*\times\mathbb{R}$, $Z = H\times\mathbb{R}$ and each K_i $(i\in J)$ is a weak* closed convex cone of Y, $K_i|_Z$ $(i\in J)$ and $\gamma(K_i|_Z;J)$ are respectively defined by

$$K_i|_Z = \{(x^*|_H,\alpha) : (x^*,\alpha)\in K_i\} \tag{4.17}$$

and

$$\gamma(K_i|_Z;J) = \inf\{\|\sum_{i\in J}(x_i^*|_H,\alpha_i)\| : \sum_{i\in J}\|(x_i^*|_H,\alpha_i)\| = 1,\ \text{each } (x_i^*,\alpha_i)\in K_i\} \tag{4.18}$$

(see (4.16)). If H is finite dimensional, the infimum in (4.18) is attained and hence can be replaced by minimum.

An important special case (that we shall consider in the next theorem) is: each f_i is given in the form

$$f_i(x) = \max\{\langle a_i^*, x\rangle + r_i, 0\} + \delta_{C_i}(x) \tag{4.19}$$

where C_i are closed convex subsets of X with $\bigcap_{i\in I} C_i \ne\emptyset$ and $a_i^*\in X^*$ and $r_i\in\mathbb{R}$. Let D_i denote the convex hull of the set $(a_i^*,-r_i)\cup(0,0)$, and let K_i denote the set epiσ_{C_i}. Then D_i is a weak* compact set in $X^*\times\mathbb{R}$ containing the origin, and K_i is a weak* closed convex cone in $X^*\times\mathbb{R}$. We observe that

$$\text{co}\{(\{a_i^*\}\times[-r_i,\infty))\cup(\{0\}\times[0,\infty))\} = \text{co}\{(a_i^*,-r_i)\cup(0,0)\} + \{0\}\times[0,\infty) \tag{4.20}$$

Indeed, let $(x^*,r)\in \text{co}\{(\{a_i^*\}\times[-r_i,\infty))\cup(\{0\}\times[0,\infty))\}$. There exist $t\in[0,1]$, $\epsilon,\delta\ge 0$ such that $(x^*,r) = t(a_i^*,-r_i+\epsilon)+(1-t)(0,\delta) = t(a_i^*,-r_i)+(0,t\epsilon+(1-t)\delta)$. Note that $t\epsilon+(1-t)\delta\ge 0$. It follows that $(x^*,r)\in\text{co}\{(a_i^*,-r_i)\cup(0,0)\}+\{0\}\times[0,\infty)$ and hence $\text{co}\{(\{a_i^*\}\times[-r_i,\infty))\cup(\{0\}\times[0,\infty))\}\subseteq \text{co}\{(a_i^*,-r_i)\cup(0,0)\}+\{0\}\times[0,\infty)$. As the converse inclusion can be verified similarly, (4.20) is seen to hold. Consequently, we have

$$\begin{aligned}\text{epi}f_i^* &= \text{epi}(\max\{\langle a_i^*,\cdot\rangle + r_i, 0\})^* + \text{epi}(\delta_{C_i})^*\\ &= \text{co}\{\text{epi}(\langle a_i^*,\cdot\rangle + r_i)^*\cup(\{0\}\times[0,\infty))\} + \text{epi}(\delta_{C_i})^*\\ &= \text{co}\{(\{a_i^*\}\times[-r_i,\infty))\cup(\{0\}\times[0,\infty))\} + \text{epi}(\delta_{C_i})^*\\ &= \text{co}\{(a_i^*,-r_i)\cup(0,0)\} + \text{epi}(\delta_{C_i})^*\\ &= D_i + K_i, \end{aligned} \tag{4.21}$$

where the first equality follows from (4.19) and Lemma 2.1, the second equality follows from Lemma 2.3 and the fourth equality holds by (4.20) and the fact $\text{epi}(\delta_{C_i})^* = \text{epi}(\delta_{C_i})^* + \{0\} \times [0, +\infty)$. Therefore, the condition (C1) in the following theorem is satisfied if the functions f_i are given in the form (4.19).

THEOREM 4.4. *Let I be a compact metric space. Assume (4.1), (4.11) and the following assumptions:*
(C1) *For each $i \in I$, there exist a weak* compact convex set D_i in $X^* \times \mathbb{R}$ containing the origin, and a weak* closed convex cone K_i in $X^* \times \mathbb{R}$ such that*

$$\text{epi} f_i^* = D_i + K_i. \tag{4.22}$$

(C2) $\sum_{i\in I} \text{diam}(D_i) < \infty$, *where* $\text{diam}(D_i)$ *denotes the diameter of* D_i $(i \in I)$, *i.e.,* $\text{diam}(D_i) := \sup\{\|x - y\| : x, y \in D_i\}$.
(C3) *There exist $i_0 \in I$ and a finite dimensional subspace H of X such that $K_{i_0}^\circ \subseteq Z := H \times \mathbb{R}$ (denote the corresponding dimension of Z by m).*
(C4) *For any $J \subseteq I$ with $|J| = m$, $\gamma(K_i|_Z; J) > 0$.*
(C5) *The set-valued mapping $i \mapsto K_i|_Z$ is upper semicontinuous, i.e., for any $\bar{i} \in I$,*

$$\limsup_{i\to\bar{i}} (K_i|_Z) \subseteq K_{\bar{i}}|_Z,$$

where $\limsup_{i\to\bar{i}}(K_i|_Z) := \{x^* \in Z^* : \exists\, x_i^* \in K_i|_Z$ *such that* $x^* = \lim_{i\to\bar{i}} x_i^*$ *(in the norm of* Z^**)*$\}$.
Then $\sum_{i\in I}^{*} \text{epi} f_i^*$ *is weak* closed and*

$$\text{epi} f^* = \sum_{i\in I}^{*} \text{epi} f_i^*. \tag{4.23}$$

Proof. By Theorem 4.3, we need only prove the weak*-closedness assertion. Denote $Y := X \times \mathbb{R}$. and so Y^* is identified with $X^* \times \mathbb{R}$. Denote $A_i := \text{epi} f_i^* \subseteq Y^*$ and $A := \sum_{i\in I}^{*} A_i$. Let $a^* \in \overline{A}^{w*}$. We have to show that $a^* \in A$. To do this, we take a sequence $\{a_k^*\} \subseteq A$ such that $a_k^* \to a^*$ on $Z := H \times \mathbb{R}$ (thanks to the assumption that H is finite dimensional and the weak* topology coincides with the norm topology on a finite dimensional space). For each $k \in \mathbb{N}$, noting $a_k^* \in A = \sum_{i\in I}^{*} A_i$, there exists a sequence in $\bigcup_{\substack{J\subseteq I,\\ |J|<\infty}} \sum_{i\in J} A_i$ weak* converging (and hence in norm $\|\cdot\|_Z$) to a^*. Thus, there exist a finite subset I_k of I and $a_{i,k}^* \in A_i$ $(i \in I_k)$ such that

$$\|a_k^* - \sum_{i\in I_k} a_{i,k}^*\|_Z \le \frac{1}{k}. \tag{4.24}$$

Hence

$$\lim_{k\to\infty} \|u_k^* - a^*\|_Z \to 0, \tag{4.25}$$

where $u_k^* := \sum_{i\in I_k} a_{i,k}^*$. Note that $u_k^* \in \sum_{i\in I_k} D_i + \sum_{i\in I_k} K_i$ (by (4.22)). Since Z is of dimension m and each K_i is a (convex) cone it follows from the Carathédory theorem [28, Corollary 17.1.2] that for each $k \in \mathbb{N}$ that there exist $\{i_{1,k}, i_{2,k}, \ldots, i_{m,k}\} \subseteq I_k$, such that

$$u_k^* = \sum_{i\in I_k} \overline{y}_{i,k}^* + \sum_{j=1}^{m} \overline{z}_{i_{j,k}}^* \text{ on } Z \tag{4.26}$$

for some $\overline{y}^*_{i,k} \in D_i$ $(i \in I_k)$ and $\overline{z}^*_{i_{j,k}} \in K_{i_{j,k}}$ $(1 \leq j \leq m)$. Let $I' := \bigcup_{k\in\mathbb{N}} I_k$ and set $\overline{y}^*_{i,k} := 0$ for any $i \in I' \backslash I_k$. For each fixed $k \in \mathbb{N}$, it follows from (4.26) that

$$u^*_k = \sum_{i\in I'} \overline{y}^*_{i,k} + \sum_{j=1}^{m} \overline{z}^*_{i_{j,k}} \text{ on } Z, \tag{4.27}$$

where $\overline{y}^*_{i,k} \in D_i$ for all $i \in I'$ $\big($thanks to the assumption that each D_i contains the origin$\big)$. Next, we show that

$$\{\overline{z}^*_{i_{j,k}}|_Z\}_{k\in\mathbb{N}} \text{ are bounded sequences for all } 1 \leq j \leq m. \tag{4.28}$$

To prove this, we suppose on the contrary that $\{\overline{z}^*_{i_{j,k}}|_Z\}_{k\in\mathbb{N}}$ is an unbounded sequence for some $j \in \{1, 2, \ldots, m\}$. By passing to a subsequence if necessary, we may assume that

$$\lim_{k\to\infty} \sum_{j=1}^{m} \|\overline{z}^*_{i_{j,k}}\|_Z \to \infty. \tag{4.29}$$

Dividing by $\displaystyle\sum_{j=1}^{m} \|\overline{z}^*_{i_{j,k}}\|_Z$ on both sides of (4.27), we obtain

$$\frac{u^*_k}{\sum_{j=1}^m \|\overline{z}^*_{i_{j,k}}\|_Z} = \frac{\sum_{i\in I'} \overline{y}^*_{i,k}}{\sum_{j=1}^m \|\overline{z}^*_{i_{j,k}}\|_Z} + \sum_{j=1}^{m} \frac{\overline{z}^*_{i_{j,k}}}{\sum_{j=1}^m \|\overline{z}^*_{i_{j,k}}\|_Z} \text{ on } Z. \tag{4.30}$$

Note that $\{\|u^*_k\|_Z\}_{k\in\mathbb{N}}$ is a bounded numerical sequence (since $\|u^*_k - a^*\|_Z \to 0$) and

$$\|\sum_{i\in I'} \overline{y}^*_{i,k}\|_Z \leq \sum_{i\in I'} \|\overline{y}^*_{i,k}\| \leq \sum_{i\in I} \text{diam}(D_i) < \infty. \tag{4.31}$$

Moreover, since I is compact, we may assume without loss of generality that $i_{j,k} \to \overline{i}_j$ for some $\overline{i}_j \in I$ $(1 \leq j \leq m)$ as $k \to \infty$. Considering subsequences if necessary, we may assume that there exists $z^*_j \in X^*$ such that the bounded sequence

$$\frac{\overline{z}^*_{i_{j,k}}}{\sum_{j=1}^m \|\overline{z}^*_{i_{j,k}}\|_Z} \to z^*_j \text{ on } Z \quad (1 \leq j \leq m), \tag{4.32}$$

(thanks to the fact that Z is finite-dimensional). By (4.32), it is clear that $\sum_{j=1}^m \|z^*_j\|_Z = 1$. Moreover, assumption (C5) entails that each $z^*_j|_Z \in K_{\overline{i}_j}|_Z$. Finally, by passing to the limits in (4.30) and making use of (4.31) and (4.29), we have $\sum_{j=1}^m z^*_j = 0$ on Z. Then $\gamma(K_i|_Z, \{\overline{i}_1, \overline{i}_2, \ldots, \overline{i}_m\}) = 0$, contradicting the assumption (C4). Therefore (4.28) is proved.

By the compactness of I again and by passing to subsequences if necessary, we may assume that $i_{j,k} \to \hat{i}_j$ for some $\hat{i}_j \in I$ as $k \to \infty$ $(1 \leq j \leq m)$. For each j, since Z is finite dimensional and by (4.28), we may assume that $\overline{z}^*_{i_{j,k}} \to \overline{z}^*_j$ on Z for some $\overline{z}^*_j \in X^*$. By (C5), $\overline{z}^*_j|_Z \in K_{\hat{i}_j}|_Z$ and so there exists $\omega^*_j \in K_{\hat{i}_j}$ such that $\overline{z}^*_j = \omega^*_j$ on Z. Hence, replacing $\overline{z}^*_j$ by ω^*_j if necessary, we may assume without loss of generality that

$$\overline{z}^*_j \in K_{\hat{i}_j} \quad (1 \leq j \leq m). \tag{4.33}$$

Since $u^*_k \to a^*$ on Z $\big($by (4.25)$\big)$, (4.27) implies that

$$\sum_{i\in I'} \overline{y}^*_{i,k} = u^*_k - \sum_{j=1}^{m} \overline{z}^*_{i_{j,k}} \to a^* - \sum_{j=1}^{m} \overline{z}^*_j \text{ on } Z \quad \text{as } k \to \infty. \tag{4.34}$$

Since I' is countable, we may represent I' in the form that $I' = \{i_1, \dots, i_n, \dots\}$ and hence

$$\sum_{n \in \mathbb{N}} \overline{y}^*_{i_n,k} \to a^* - \sum_{j=1}^{m} \overline{z}^*_j \ \text{ on } Z \quad \text{as } k \to \infty.$$

Since $\overline{y}^*_{i_1,k} \in D_{i_1}$ and $D_{i_1}|_Z$ is compact, there exists an infinite subset $N_1 \subseteq \mathbb{N}$ such that $\{\overline{y}^*_{i_1,k}\}_{k \in N_1}$ converges to $\overline{y}^*_{i_1}$ on Z for some $\overline{y}^*_{i_1} \in D_{i_1}$. Inductively, we can find a sequence of infinite subsets $N_n \subseteq \mathbb{N}$ such that $N_{n+1} \subseteq N_n$ and for each $n \in \mathbb{N}$

$$\{\overline{y}^*_{i_n,k}\}_{k \in N_n} \text{ converges to } \overline{y}^*_{i_n} \text{ on } Z \text{ for some } \overline{y}^*_{i_n} \in D_{i_n}. \tag{4.35}$$

Since $\sum_{n \in \mathbb{N}} \|\overline{y}^*_{i_n}\| \le \sum_{i \in I} \|\overline{y}^*_i\| \le \sum_{i \in I} \mathrm{diam} D_i < +\infty$ (by the assumption (C2)) $\sum_{n \in \mathbb{N}} \overline{y}^*_{i_n}$ exists as an element in X^* and in particular

$$\sum_{n \in \mathbb{N}} \overline{y}^*_{i_n} \in {\sum_{n \in \mathbb{N}}}^* D_{i_n}. \tag{4.36}$$

Similarly, for all $k \in \mathbb{N}$

$$\sum_{n \in \mathbb{N}} \|\overline{y}^*_{i_n,k}\| \le \sum_{i \in I} \mathrm{diam} D_i < +\infty,$$

and thus, for any $\epsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n' \ge n_0$

$$\sum_{n > n'} \|\overline{y}^*_{i_n,k}\| \le \sum_{n > n'} \mathrm{diam}(D_{i_n}) \le \epsilon/2 \text{ for all } k \in \mathbb{N}. \tag{4.37}$$

Note that, for any fixed $n' \ge n_0$, (by (4.35) and (4.34)) there exists $k_0 \in \mathbb{N}$ (depending on n') such that

$$\|\overline{y}^*_{i_n,k_0} - \overline{y}^*_{i_n}\|_Z \le \frac{\epsilon}{4n'} \text{ for all } n \in \{1, \dots, n'\} \quad \text{ and}$$

$$\|a^* - \sum_{j=1}^{m} \overline{z}^*_j - \sum_{n \in \mathbb{N}} \overline{y}^*_{i_n,k_0}\|_Z \le \epsilon/4.$$

It follows from (4.37) that for any $n' \ge n_0$

$$\begin{aligned} \|a^* - \sum_{j=1}^{m} \overline{z}^*_j - \sum_{n=1}^{n'} \overline{y}^*_{i_n}\|_Z &\le \|a^* - \sum_{j=1}^{m} \overline{z}^*_j - \sum_{n \in \mathbb{N}} \overline{y}^*_{i_n,k_0}\|_Z \\ &\quad + \sum_{n=1}^{n'} \|\overline{y}^*_{i_n,k_0} - \overline{y}^*_{i_n}\|_Z + \sum_{n > n'} \|\overline{y}^*_{i_n,k_0}\|_Z \\ &\le \epsilon. \end{aligned}$$

Since ϵ is arbitrary, one has

$$a^* = \sum_{j=1}^{m} \overline{z}^*_j + \sum_{n \in \mathbb{N}} \overline{y}^*_{i_n} \text{ on } Z. \tag{4.38}$$

Then, one has

$$a^* - \sum_{j=1}^{m} \overline{z}^*_j - \sum_{n \in \mathbb{N}} \overline{y}^*_{i_n} \in Z^{\perp}. \tag{4.39}$$

From (4.33) and (4.35) we know that each $\overline{z}_j^* \in K_{\hat{i}_j}$ and $\overline{y}_{i_n} \in D_{i_n}$, and it follows from (4.36) that $\sum_{j=1}^{m} \overline{z}_j^* + \sum_{n\in\mathbb{N}} \overline{y}_{i_n}^* \in \sum_{j=1}^{m} K_{\hat{i}_j} + {\sum_{n\in\mathbb{N}}}^* D_{i_n}$. Therefore, (4.39) entails that

$$a^* \in \sum_{j=1}^{m} K_{\hat{i}_j} + {\sum_{n\in\mathbb{N}}}^* D_{i_n} + Z^{\perp}. \tag{4.40}$$

Note from the bipolar theorem (cf. [30, Theorem 1.1.9]) and the assumption (C3) that $Z^{\perp} \subseteq K_{i_0}$. It follows that $a^* \in \sum_{j=1}^{m} K_{\hat{i}_j} + {\sum_{n\in\mathbb{N}}}^* D_{i_n} + K_{i_0}$. Since $0 \in D_i \cap K_i$, one has

$$a^* \in \sum_{j=1}^{m} K_{\hat{i}_j} + {\sum_{n\in\mathbb{N}}}^* D_{i_n} + K_{i_0} = {\sum_{n\in\mathbb{N}}}^* D_{i_n} + \big(\sum_{j=1}^{m} K_{\hat{i}_j} + K_{i_0}\big) \subseteq {\sum_{i\in I}}^* (D_i + K_i) = A,$$

as required to show. This completes the proof. □

Corollary 4.5. *Let I be a compact metric space and let f_i be given by (4.19). Assume (4.1), (4.11) and the following assumptions:*

(B1) $\sum_{i\in I} (\|a_i^*\| + |r_i|) < \infty$.

(B2) *There exists $i_0 \in I$ such that $H := \operatorname{span}(C_{i_0})$ is of finite dimension. (Denote the corresponding dimension by n).*

(B3) *For any $J \subseteq I\backslash\{i_0\}$ with $|J| \leq n+1$, it holds that*

$$C_{i_0} \cap (\bigcap_{i\in J} \operatorname{int}_H C_i) \neq \emptyset.$$

where $\operatorname{int}_H C_i := \{x \in C_i : \exists \epsilon > 0 \text{ s.t. } B(x,\epsilon) \cap H \subseteq C_i\}$.

(B4) *The mapping $i \mapsto C_i \cap H$ is lower semicontinuous, i.e., for any $\bar{i} \in I$*

$$C_{\bar{i}} \cap H \subseteq \liminf_{i\to\bar{i}} (C_i \cap H).$$

Then the conclusion of Theorem 4.4 holds.

Proof. Define $D_i := \operatorname{co}\{(a_i^*, -r_i) \cup (0,0)\}$, and $K_i := \operatorname{epi}\sigma_{C_i}$. Then, as we have mentioned before, assumption (C1) in Theorem 4.4 holds (see (4.21)). Therefore, to finish the proof, it suffices to show the assumptions (C2)-(C5) in Theorem 4.4 are satisfied. First of all, from the condition (B1), we see that $\sum_{i\in I} \operatorname{diam} D_i < \infty$ and hence the assumption (C2) holds. To see (C3), noting that $Z^{\perp} = (\operatorname{span} C_{i_0})^{\perp} \times \{0\} \subseteq \operatorname{epi}\sigma_{C_{i_0}} = K_{i_0}$, one has $K_{i_0}^{\circ} \subseteq Z$. Therefore assumption (C3) holds with $m := n+1$. To see that assumption (C4) holds with $m := n+1$, we proceed by contradiction and suppose that there exists $J_0 \subseteq I$ with $|J_0| = n+1$ such that $\gamma(\operatorname{epi}\sigma_{C_i}|_{H\times\mathbb{R}}; J_0) = 0$. Noting that $H \times \mathbb{R}$ is finite dimensional, it follows from (4.18) that there exist $(x_i^*, \alpha_i) \in \operatorname{epi}\sigma_{C_i}$ $(i \in J_0)$ such that

$$\sum_{i\in J_0} \big(\|x_i^*\|_H + |\alpha_i|\big) = 1, \tag{4.41}$$

$$\sum_{i\in J_0} x_i^* = 0 \text{ on } H \quad \text{and} \quad \sum_{i\in J_0} \alpha_i = 0. \tag{4.42}$$

We claim that

$$x_i^* = 0 \text{ on } H \text{ for all } i \in J_0. \tag{4.43}$$

Granting this, we have $0 = \sigma_{C_i\cap H}(x_i^*) \leq \sigma_{C_i}(x_i^*) \leq \alpha_i$ for all $i \in J_0$. This together with the second equality of (4.42) implies that $\alpha_i = 0$ for all $i \in J_0$. However, this and (4.43) contradict to (4.41). To see (4.43), we divide our proof into two cases, namely Case 1: $i_0 \notin J_0$ and Case 2: $i_0 \in J_0$. For Case 1, we note that $J_0 \subseteq I \backslash \{i_0\}$ and it follows from (B3) that $\bigcap_{j\in J_0} \mathrm{int}_H C_i \neq \emptyset$. Thus there exist $x_0 \in X$ and $\epsilon > 0$ such that $x_0 \in \overline{\mathbb{B}}(x_0, \epsilon) \cap H \subseteq C_i$ for all $i \in J_0$. Recalling the fact $(x_i^*, \alpha_i) \in \mathrm{epi}\sigma_{C_i}$ $(i \in J_0)$ and the definition of $\|\cdot\|_H$, it follows from (4.42) that

$$\begin{aligned}\epsilon \sum_{i\in J_0} \|x_i^*\|_H = \sum_{i\in J_0} \big(\langle x_i^*, x_0\rangle + \epsilon\|x_i^*\|_H\big) &= \sum_{i\in J_0} \sup_{x\in\overline{\mathbb{B}}(x_0,\epsilon)\cap H} \langle x_i^*, x\rangle \\ &\leq \sum_{i\in J_0} \sup_{x\in C_i} \langle x_i^*, x\rangle \\ &\leq \sum_{i\in J_0} \alpha_i = 0. \end{aligned} \tag{4.44}$$

Thus (4.43) holds in this case. For Case 2, one applies (B3) again, and there exist $x_0 \in C_{i_0}$ and $\epsilon > 0$ such that $x_0 \in \overline{\mathbb{B}}(x_0, \epsilon) \cap H \subseteq C_i$ for all $i \in J_0 \backslash \{i_0\}$. Hence $\sup_{x\in\overline{B}(x_0,\epsilon)\cap H} \langle x_i^*, x\rangle \leq \sup_{x\in C_i} \langle x_i^*, x\rangle$, that is

$$\langle x_i^*, x_0\rangle + \epsilon\|x_i^*\|_H = \sup_{x\in\overline{\mathbb{B}}(x_0,\epsilon)\cap H} \langle x_i^*, x\rangle \leq \sigma_{C_i}(x_i^*) \text{ for all } i \in J_0\backslash\{i_0\}. \tag{4.45}$$

Since $(x_i^*, \alpha_i) \in \mathrm{epi}\sigma_{C_i}$ for all $i \in J_0$ and $x_0 \in C_{i_0}$ (so $\langle x_{i_0}^*, x_0\rangle \leq \alpha_{i_0}$), it follows from (4.42) that

$$\epsilon \sum_{i\in J_0\backslash\{i_0\}} \|x_i^*\|_H = \sum_{i\in J_0} \langle x_i^*, x_0\rangle + \epsilon \sum_{i\in J_0\backslash\{i_0\}} \|x_i^*\|_H \leq \sum_{i\in J_0} \alpha_i = 0.$$

This together with the first equality in (4.42) gives that $x_i^* = 0$ on H for all $i \in J_0$. Thus (4.43) also holds in this case. Finally, for (C5), fix an $\bar{i} \in I$. Consider $i \to \bar{i}$ and $(x_i^*, \alpha_i) \in H^* \times \mathbb{R}$ $(i \in I)$ be such that $(x_i^*, \alpha_i) \in \mathrm{epi}\sigma_{C_i}|_{H\times\mathbb{R}}$ with $(x_i^*, \alpha_i) \to (x^*, \alpha)$ for some $(x^*, \alpha) \in H^* \times \mathbb{R}$. Let $x \in C_{\bar{i}} \cap H$. By (B4) and since H is of finite dimension, there exists a sequence $\{x_i\} \subseteq C_i \cap H$ such that $x_i \to x$. It follows that

$$\langle x^*, x\rangle = \lim_{i\to\bar{i}} \langle x_i^*, x_i\rangle \leq \limsup_{i\to\bar{i}} \sigma_{C_i\cap H}(x_i^*) \leq \lim_{i\to\bar{i}} \alpha_i = \alpha.$$

This implies that $(x^*, \alpha) \in \mathrm{epi}\sigma_{C_{\bar{i}}}|_{H\times\mathbb{R}}$ and hence (C5) in Theorem 4.4 holds. Therefore the assumptions (C2)-(C5) in Theorem 4.4 hold. This finishes the proof. □

5. Application to the KKT theory. We first establish an ϵ-sum rule involving possibly infinitely many convex functions. In the special case that $I = \emptyset$, the following result has been presented in [18] (see also [30, Corollary 2.6.7]).

THEOREM 5.1. *Let I, J be two index sets with $I \cap J = \emptyset$ and $|J| < \infty$. Let $\{f_i\}_{i\in I} \subseteq \Gamma_+(X)$ and $\{f_j\}_{j\in J} \subseteq \Gamma(X)$. Let $f \in \Gamma(X)$ be such that $f(x) = \sum_{i\in I} f_i(x) + \sum_{j\in J} f_j(x)$ for each $x \in X$. Let $\epsilon \geq 0$ and $x \in X$. Then we have*

$$\partial_\epsilon f(x) = \bigcap_{\eta>0} \overline{\bigcup_{\substack{I'\subseteq I,\\ |I'|<\infty}} \{\sum_{i\in I'} \partial_{\epsilon_i} f_i(x) + \sum_{j\in J} \partial_{\epsilon_j} f_j(x) : \sum_{i\in I'} \epsilon_i + \sum_{j\in J} \epsilon_j \leq \epsilon + \eta\}}^{w^*}. \tag{5.1}$$

Proof. First of all, since

$$\bigcup_{\substack{I'\subseteq I,\\ |I'|<\infty}} \{\sum_{i\in I'} \partial_{\epsilon_i} f_i(x) + \sum_{j\in J} \partial_{\epsilon_j} f_j(x) : \sum_{i\in I'} \epsilon_i + \sum_{j\in J} \epsilon_j \leq \epsilon + \eta\} \subseteq \partial_{\epsilon+\eta} f(x),$$

and $\partial_{\epsilon+\eta} f(x)$ is weak* closed for each $\eta > 0$, one has

$$\bigcap_{\eta>0} \overline{\bigcup_{\substack{I'\subseteq I,\\ |I'|<\infty}} \{\sum_{i\in I'} \partial_{\epsilon_i} f_i(x) + \sum_{j\in J} \partial_{\epsilon_j} f_j(x) : \sum_{i\in I'} \epsilon_i + \sum_{j\in J} \epsilon_j \leq \epsilon + \eta\}}^{w^*} \subseteq \bigcap_{\eta>0} \partial_{\epsilon+\eta} f(x) = \partial_\epsilon f(x).$$

To see the reverse inclusion, let $x^* \in \partial_\epsilon f(x)$, $\eta > 0$ and let V be a weak* neighbourhood of 0. It suffices to show that there exist I' with $|I'| < +\infty$, $\epsilon_i, \epsilon_j \geq 0$ $(i \in I', j \in J)$ such that

$$\sum_{i\in I'} \epsilon_i + \sum_{j\in J} \epsilon_j \leq \epsilon + \eta \tag{5.2}$$

and

$$x^* \in \sum_{i\in I'} \partial_{\epsilon_i} f_i(x) + \sum_{j\in J} \partial_{\epsilon_j} f_j(x) + V. \tag{5.3}$$

Let $h := \sum_{i\in I} f_i$. Note that $h \in \Gamma_+(X)$ (since $f_i \in \Gamma_+(X)$ and $h(x_0) < +\infty$ for all $x_0 \in \mathrm{dom} f$). From Lemma 2.1 (applied to $\{h, f_j : j \in J\}$) and Theorem 4.3, we have

$$\begin{aligned}
\mathrm{epi} f^* = \mathrm{epi}\big(h + \sum_{j\in J} f_j\big)^* &= \overline{\mathrm{epi} h^* + \sum_{j\in J} \mathrm{epi} f_j^*}^{w^*} \\
&= \overline{\overline{\bigcup_{\substack{I'\subseteq I,\\ |I'|<\infty}} \sum_{i\in I'} \mathrm{epi} f_i^*}^{w^*} + \sum_{j\in J} \mathrm{epi} f_j^*}^{w^*} \\
&= \overline{\bigcup_{\substack{I'\subseteq I,\\ |I'|<\infty}} \sum_{i\in I'} \mathrm{epi} f_i^* + \sum_{j\in J} \mathrm{epi} f_j^*}^{w^*}.
\end{aligned}$$

Since $x^* \in \partial_\epsilon f(x)$, it follows from (2.5) and the above expression that

$$(x^*, \epsilon + \langle x^*, x\rangle - f(x)) \in \overline{\bigcup_{\substack{I'\subseteq I,\\ |I'|<\infty}} \sum_{i\in I'} \mathrm{epi} f_i^* + \sum_{j\in J} \mathrm{epi} f_j^*}^{w^*}, \tag{5.4}$$

and hence there exist $I' \subseteq I$ with $|I'| < \infty$, $(x_i^*, r_i) \in \mathrm{epi} f_i^*$ $(i \in I')$ and $(x_j^*, r_j) \in \mathrm{epi} f_j^*$ $(j \in J)$ such that

$$x^* \in \sum_{i\in I'} x_i^* + \sum_{j\in J} x_j^* + V \tag{5.5}$$

and

$$\sum_{i\in I'} r_i + \sum_{j\in J} r_j \leq (\epsilon + \eta/2) + \langle x^*, x\rangle - f(x). \tag{5.6}$$

By shrinking V if necessary, we may assume without loss of generality that

$$|\langle \sum_{i\in I'} x_i^* + \sum_{j\in J} x_j^* - x^*,\, x\rangle| \leq \eta/2. \tag{5.7}$$

For each $k \in I' \cup J$, let $\epsilon_k := f_k^*(x_k^*) + f_k(x) - \langle x_k^*, x\rangle$; then $\epsilon_k \geq 0$ (Young's inequality), and $x_k^* \in \partial_{\epsilon_k} f_k(x)$ $\big($see (2.5)$\big)$. Thus (5.3) holds by (5.5). It remains to show (5.2). To see this, note that $(x_k^*, r_k) \in \mathrm{epi} f_k^*$ and so $\epsilon_k \leq r_k + f_k(x) - \langle x_k^*, x\rangle$ for each $k \in I' \cup J$. Further, $\sum_{i\in I'} f_i(x) + \sum_{j\in J} f_j(x) \leq f(x)$ (since f_i are nonnegative for all $i \in I$). It follows from (5.6) that

$$\sum_{k\in I'\cup J} \epsilon_k \leq \sum_{k\in I'\cup J} r_k + f(x) - \langle \sum_{i\in I'\cup J} x_k^*, x\rangle \leq \langle x^* - \sum_{k\in I'\cup J} x_k^*, x\rangle + (\epsilon + \eta/2)$$

and so (5.2) holds by (5.7). This completes the proof. □

Let $|I| \leq +\infty$. Consider the following semi-infinite programming

$$\begin{aligned} &\min_{x\in X} \ f_0(x) \\ &\text{s.t.} \ \ f_i(x) \leq 0 \quad (i \in I) \end{aligned} \tag{5.8}$$

where $\{f_0, f_i : i \in I\} \subseteq \Gamma(X)$. We say that x is a feasible point of (5.8) if $f_0(x) < +\infty$ and $f_i(x) \leq 0$ for all $i \in I$. For any $\epsilon \geq 0$, a feasible point $\overline{x}$ of (5.8) is called an ϵ-solution if $f_0(\overline{x}) \leq f_0(x) + \epsilon$ for all feasible points x of (5.8). As an application of the preceding theorem, we have the following fuzzy KKT result for (5.8). In the special case when $\epsilon = 0$, X is reflexive and $\{f_0, f_i : i \in I\} \subseteq \Gamma_c(X)$ the fuzzy KKT condition was first derived by Jeyakumar et. al. in [19]. In the special case when $\epsilon = 0$, and each f_i $(i \in I)$ is epi-closed, this result was also established by Boţ et. al. in [3, 6] via the perturbation approach (indeed, [3, 6] gave the corresponding result for a more general problem: the cone constraint problem).

Theorem 5.2. *Let $\epsilon \geq 0$ and let $\overline{x}$ be an ϵ-solution of (5.8). Let U be a weak*-neighbourhood of* 0 *and $\eta > 0$. Then there exist a finite subset I' of I and $\{\epsilon_i : i \in \{0\} \cup I'\} \cup \{\lambda_i : i \in I'\} \subseteq [0, +\infty)$ such that*

$$0 \leq \sum_{i\in I'\cup\{0\}} \epsilon_i \leq \epsilon + \eta, \quad -(\epsilon + \eta) \leq \sum_{i\in I'} \lambda_i f_i(\overline{x}) \leq 0 \tag{5.9}$$

and

$$0 \in x_0^* + \sum_{i\in I'} x_i^* + U, \tag{5.10}$$

for some

$$x_0^* \in \partial_{\epsilon_0} f_0(\overline{x}) \ \text{and} \ x_i^* \in \partial_{\epsilon_i} (\lambda_i f_i)(\overline{x}) \ (i \in I'). \tag{5.11}$$

Proof. Define $f : X \to \mathbb{R} \cup \{+\infty\}$ by

$$f(x) = f_0(x) + \sum_{i\in I} g_i(x),$$

where $g_i = \delta_{A_i}$ and $A_i := \{x \in X : f_i(x) \leq 0\}$. Then $f(\overline{x}) = f_0(\overline{x}) < +\infty$ and $f \in \Gamma(X)$. Moreover, since $\overline{x}$ is an ϵ-solution of (5.8), one has $f(\overline{x}) \leq f(x) + \epsilon$ for all $x \in X$, so $0 \in \partial_\epsilon f(\overline{x})$. We assume without loss of generality that the given weak* neighborhood U is convex. Note that each g_i is a nonnegative function $(i \in I)$. From Theorem 5.1 (applied to $\{I, \{0\}, \{g_i\}_{i\in I}, f_0, \epsilon\}$ in place of $\{I, J, \{f_i\}_{i\in I}, \{f_j\}_{j\in J}, \epsilon\}$), there exist a finite subset I' of I and $\{\overline{\epsilon}_i : i \in \{0\} \cup I'\} \subseteq [0, \infty)$ such that

$$\overline{\epsilon}_0 + \sum_{i\in I'} \overline{\epsilon}_i \leq \epsilon + \frac{\eta}{2} \ \text{and} \ 0 \in x_0^* + \sum_{i\in I'} z_i^* + \frac{U}{2}. \tag{5.12}$$

for some $x_0^* \in \partial_{\overline{\epsilon}_0} f_0(\overline{x})$ and $z_i^* \in \partial_{\overline{\epsilon}_i} g_i(\overline{x})$ $(i \in I')$. Since $g_i = \delta_{A_i}$, it follows from (2.4) that $g_i^*(z_i^*) = \sup_{a\in A_i} \langle z_i^*, a\rangle \leq \langle z_i^*, \overline{x}\rangle + \overline{\epsilon}_i$. By (2.12) (applied to f_i in place of f), it follows that

$$(z_i^*, \langle z_i^*, \overline{x}\rangle + \overline{\epsilon}_i) \in \overline{\bigcup_{\lambda>0} \operatorname{epi}(\lambda f_i)^*}^{w^*}.$$

Hence, for each $i \in I'$, there exist $\lambda_i > 0$ and $(x_i^*, s_i) \in \operatorname{epi}(\lambda_i f_i)^*$ such that

$$z_i^* \in x_i^* + \frac{U}{2|I'|}, \tag{5.13}$$

$$\langle z_i^* - x_i^*, \overline{x}\rangle < \eta/(4|I'|) \tag{5.14}$$

and

$$|\langle z_i^*, \overline{x}\rangle + \overline{\epsilon}_i - s_i| < \eta/(4|I'|). \tag{5.15}$$

By (2.11) $\big($applied to $\{\lambda_i f_i, \overline{x}\}$ in place of $\{f, x\}\big)$, for each $i \in I'$ there exists $\epsilon_i \geq 0$ such that

$$x_i^* \in \partial_{\epsilon_i}(\lambda_i f_i)(\overline{x}) \text{ and } s_i = \epsilon_i + \langle x_i^*, \overline{x}\rangle - \lambda_i f_i(\overline{x}). \tag{5.16}$$

Thus, letting $\epsilon_0 := \overline{\epsilon}_0$, (5.11) holds. By (5.12) and (5.13), (5.10) also holds. Since each $\epsilon_i \geq 0$, and $f_i(\overline{x}) \leq 0$ $(i \in I')$, we note that

$$\max\{\sum_{i\in\{0\}\cup I'} \epsilon_i \ , \ -\sum_{i\in I'} \lambda_i f_i(\overline{x})\} \leq \sum_{i\in\{0\}\cup I'} \epsilon_i - \sum_{i\in I'} \lambda_i f_i(\overline{x});$$

thus to prove (5.9), it suffices to show that

$$\sum_{i\in\{0\}\cup I'} \epsilon_i - \sum_{i\in I'} \lambda_i f_i(\overline{x}) \leq \epsilon + \eta. \tag{5.17}$$

To do this, note that, for each $i \in I'$

$$\epsilon_i - \lambda_i f_i(\overline{x}) = s_i - \langle x_i^*, \overline{x}\rangle = (s_i - \langle z_i^*, \overline{x}\rangle) + \langle z_i^* - x_i^*, \overline{x}\rangle < (\overline{\epsilon}_i + \frac{\eta}{4|I'|}) + \frac{\eta}{4|I'|}$$

$\big($see (5.14), (5.15) and (5.16)$\big)$. Hence it follows from (5.12) that

$$\epsilon_0 + \sum_{i\in I'} \big(\epsilon_i - (\lambda_i f_i)(\overline{x})\big) < \overline{\epsilon}_0 + \sum_{i\in I} \overline{\epsilon}_i + \eta/2 \leq \epsilon + \eta.$$

Thus (5.17) is true and the proof is completed. □

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