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The Fermat rule for multifunctions on Banach spaces*

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Abstract. Using variational analysis, we study vector optimization problems with objectives being closed multifunctions on Banach spaces or in Asplund spaces. In particular, in terms of the coderivatives, we present Fermat's rules as necessary conditions for an optimal solution of the above problems. As applications, we also provide some necessary conditions (in terms of Clarke's normal cones or the limiting normal cones) for Pareto efficient points.

Key words. Multifunction – Normal cone – Coderivative – Pareto efficient point – Pareto solution

1. Introduction

The main objective of this paper is to study the following vector optimization problem

$$C - \min_{x \in X} \Phi(x). \tag{1.1}$$

Here X, Y are Banach spaces, $\Phi : X \to 2^Y$ is a closed multifunction and $C \subset Y$ is a closed convex pointed non-trivial cone, which specifies a partial order $\leq_C$ on Y as follows: for $y_1, y_2 \in Y$,

$$y_1 \leq_C y_2 \text{ if and only if } y_2 - y_1 \in C.$$

Let A be a subset of Y. Recall that $\bar{a} \in A$ is said to be a Pareto efficient point if there does not exist $a \in A$ with $a \neq \bar{a}$ such that $a \leq_C \bar{a}$, that is,

$$A \cap (\bar{a} - C) = \{\bar{a}\}.$$

We use $\mathrm{E}(A, C)$ to denote the set of all Pareto efficient points of A. For $\bar{x} \in X$ and $\bar{y} \in \Phi(\bar{x})$, we say that $(\bar{x}, \bar{y})$ is a local Pareto solution of the vector optimization problem (1.1) if there exists a neighborhood U of $\bar{x}$ such that

$$\bar{y} \in \mathrm{E}(\Phi(U), C).$$

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Under some restricted conditions (e.g., the ordering cone has a nonempty interior, the spaces are finite dimensional, or Φ is single-valued), many authors (see [7, 9, 14, 15, 26, 29] and references therein) have obtained existence results for Pareto solutions or weak Pareto solutions, while there are only a few who address the issue of sufficient/necessary optimality conditions (for $\bar{x} \in X$ to provide a solution). In particular, Minami [16] studied multiobjective program on a Banach space with a single-valued objective function and with finitely many equality/inequality constraints given by numerical functions. His result on Kuhn-Tucker forms is closely related to one of our results in Section 4 and we will make further comments there. Under the convexity assumptions, Gotz and Jahn [10] studied necessary optimality conditions for weak Pareto solutions using the notion of cotangent derivative. Very recently, Ye and Zhu [27] gave some necessary optimality conditions for single-valued vector optimization problems with respect to an abstract order in an Euclidean space setting. Single-valued vector optimization problems with respect to abstract order (regardless to linear structure) have also been discussed in Zhu [29] and Mordukhovich, Traiman and Zhu [20]. Our approach here differs from the earlier studies mainly in two aspects: firstly Φ is a general closed multifunction, and secondly our main results in Section 3 are valid for general Banach spaces.

In the special case when $Y = R$, $C = [0, +\infty)$ and Φ is given by

$$\Phi(x) = [f(x), +\infty) \text{ for all } x \in X \tag{1.2}$$

where $f : X \to R \cup \{+\infty\}$ is a proper lower semicontinuous function, it is easy to verify that $(\bar{x}, f(\bar{x}))$ is a local Pareto solution of (1.1) if and only if $\bar{x}$ is a local minimum point of f. Note (cf. [6]) also that Clarke's subdifferential $\partial_c f(\bar{x})$ and the associated coderivative $D_c^*\Phi(\bar{x}, f(\bar{x})) : Y^* \to 2^{X^*}$ are related by

$$\partial_c f(\bar{x}) = D_c^*\Phi(\bar{x}, f(\bar{x}))(1). \tag{1.3}$$

In view of the following well known result (Fermat's rule)

$$f \text{ attains a local minimum at } x \Longrightarrow 0 \in \partial_c f(x),$$

it is natural to ask whether or not the following Fermat's rule is also valid: if $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$ is a local Pareto solution of (1.1), does it follow that

$$0 \in D_c^*\Phi(\bar{x}, \bar{y})(c^*) \tag{1.4}$$

for some $c^* \in C^+$ with $\|c^*\| = 1$, where $C^+ := \{y^* \in Y^* : \langle c^*, c\rangle \geq 0 \text{ for all } c \in C\}$ and Y^* denotes the dual space of Y (see Section 2 for undefined terms). Though the answer is negative in general (cf. Example 3.1), we show in Section 3 that the following fuzzy version is valid: If $(\bar{x}, \bar{y})$ is a local Pareto solution of the vector optimization problem (1.1) then for any $\varepsilon > 0$ there exist $x_\varepsilon \in \bar{x} + \varepsilon B_X$, $y_\varepsilon \in \Phi(x_\varepsilon) \cap (\bar{y} + \varepsilon B_Y)$ and $c^* \in C^+$ with $\|c^*\| = 1$ such that

$$0 \in D_c^*\Phi(x_\varepsilon, y_\varepsilon)(c^* + \varepsilon B_{Y^*}) + \varepsilon B_{X^*}, \tag{1.5}$$

where B_X and B_{X^*} respectively denote the closed unit balls of X and X^*. Moreover we show that (1.4) holds if $(\bar{x}, \bar{y})$ is a local Pareto solution of (1.1) and if (at least) one of the following conditions is satisfied.

(i) The ordering cone C has a nonempty interior.
(ii) The ordering cone C is dually compact and $N_c(\mathrm{Gr}(\Phi), \cdot)$ is closed at $(\bar{x}, \bar{y})$.
(iii) There exists a vector hypertangent to $\mathrm{Gr}(\Phi)$ at $(\bar{x}, \bar{y})$.

If X and Y are assumed to be Asplund spaces, the results are strengthened in Section 4: D_c^* in (1.4) and (1.5) can be replaced by D_F^*, the Mordukhovich coderivative defined by limiting Frechet normal cones. In the case when the objective is a closed multifunction with the Aubin property, the corresponding results for constrained vector optimization problems (with set-inclusion together with abstract constraints) are also reported.

In vector optimization theory, another interesting issue is to study necessary and/or sufficient conditions for Pareto efficient points of a closed subset of a Banach space. In Section 5, as applications of our study in earlier sections we provide some necessary conditions for Pareto efficient points of a closed set in a Banach space or an Asplund space.

2. Preliminaries

Throughout this section, we assume that Y is a Banach space. Let $f : Y \to R \cup \{+\infty\}$ be a proper lower semicontinuous function, and let $\mathrm{epi}(f)$ denote the epigraph of f, that is,

$$\mathrm{epi}(f) := \{(y, t) \in Y \times R : \ f(y) \leq t\}.$$

Let $y \in \mathrm{dom}(f) := \{x \in X : \ f(x) < +\infty\}$, $h \in Y$, and let $f^\circ(y, h)$ denote the generalized directional derivative given by Rockafellar (cf. [6]), that is,

$$f^\circ(y, h) := \lim_{\varepsilon \downarrow 0} \limsup_{z \xrightarrow{f} y, t \downarrow 0} \inf_{w \in h + \varepsilon B_Y} \frac{f(z + tw) - f(z)}{t},$$

where the expression $z \xrightarrow{f} y$ means that $z \to y$ and $f(z) \to f(y)$. It is known that $f^\circ(y, h)$ reduces to Clarke's directional derivative when f is locally Lipschitz (cf. [6]). Let

$$\partial_c f(y) := \{y^* \in Y^* : \ \langle y^*, h \rangle \leq f^\circ(y, h) \quad \forall h \in Y\}.$$

Let A be a closed subset of Y and let $N_c(A, a)$ denote Clarke's normal cone of A at a, that is,

$$N_c(A, a) := \begin{cases} \partial_c \delta_A(a) & a \in A \\ \emptyset & a \notin A \end{cases}$$

where δ_A denotes the indicator function of A: $\delta_A(y) = 0$ if $y \in A$ and $\delta_A(y) = +\infty$ otherwise. For $a \in A$, let $T_c(A, a)$ denote Clarke's tangent cone, namely

$$T_c(A, a) := \{h \in Y : \ d_A^\circ(a, h) = 0\}$$

where $d_A(\cdot)$ denotes the distance function to A. It is well known that for $a \in A$,

$$N_c(A, a) = \{y^* \in Y^* : \ \langle y^*, h \rangle \leq 0 \ \text{ for all } h \in T_c(A, a)\}.$$

The following result (cf. [6, P.52, Corollary]) presents an important necessary optimality condition in terms of Clarke's subdiferentials and normal cones for a constrained optimization problem.

Proposition 2.1. *Let $f : Y \to R$ be a locally Lipschitz function and A be a closed subset of Y. Suppose that f attains its minimum over A at $a \in A$. Then* $0 \in \partial_c f(a) + N_c(A, a)$.

Recall (cf. [18]) that the Frechet subdifferential of f at $y \in \mathrm{dom}(f)$ is defined by

$$\hat{\partial} f(y) := \left\{ y^* \in Y^* : \liminf_{v \to y} \frac{f(v) - f(y) - \langle y^*, v - y\rangle}{\|v - y\|} \geq 0 \right\}.$$

Let $\varepsilon \geq 0$. The set of ε-normals to A at a is defined by

$$\hat{N}_\varepsilon(A, a) := \left\{ y^* \in Y^* : \limsup_{y \overset{A}{\to} a} \frac{\langle y^*, y - a\rangle}{\|y - a\|} \leq \varepsilon \right\},$$

where $y \overset{A}{\to} a$ means that $y \to a$ with $y \in A$. The set $\hat{N}_0(A, a)$ is simply denoted by $\hat{N}(A, a)$ and is called the Frechet normal cone to A at a. The limiting Frechet normal cone to A at a is defined by

$$N_F(A, a) := \{y^* \in Y^* : \ \exists \varepsilon_n \to 0^+, \ y_n \overset{A}{\to} a, \ y_n^* \overset{w^*}{\to} y^* \text{ with } y_n^* \in \hat{N}_{\varepsilon_n}(A, y_n)\}.$$

The limiting Frechet subdifferential of a proper lower semicontinuous function $f : Y \to R \cup \{+\infty\}$ at $y \in \mathrm{dom}(f)$ is defined by

$$\partial_F f(y) := \{y^* \in Y^* : \ (y^*, -1) \in N_F(\mathrm{epi}(f), (y, f(y)))\}.$$

Recall that a Banach space Y is called an Asplund space if every continuous convex function defined on an open convex subset D of Y is Frechet differentiable at each point of a dense G_δ subset of D. It is well known that Y is an Asplund space if and only if every separable subspace of Y has a separable dual. The class of Asplund spaces is well investigated in geometric theory of Banach spaces; see [21, 18] and references therein. When Y is an Asplund space, it is well known that

$$N_F(A, a) := \{y^* \in Y^* : \ \exists y_n \overset{A}{\to} a, \ y_n^* \overset{w^*}{\to} y^* \text{ with } y_n^* \in \hat{N}(A, y_n)\} \tag{2.1}$$

and that $N_c(A, a)$ is the weak* closed convex hull of $N_F(A, a)$ (cf. [18]).

For $\Phi : X \to 2^Y$ a multifunction from another Banach space X to Y, let $\mathrm{Gr}(\Phi)$ denote the graph of Φ, that is,

$$\mathrm{Gr}(\Phi) := \{(x, y) \in X \times Y : \ y \in \Phi(x)\}.$$

We say that Φ is closed if $\mathrm{Gr}(\Phi)$ is a closed subset of $X \times Y$. For $x \in X$ and $y \in \Phi(x)$, let $\hat{D}^*\Phi(x, y)$ and $D_F^*\Phi(x, y) : Y^* \to 2^{X^*}$ respectively denote Frechet and limiting coderivatives of Φ at (x, y) in Mordukhovich's sense, that is,

$$\hat{D}^*\Phi(x, y)(y^*) := \{x^* \in X^* : \ (x^*, -y^*) \in \hat{N}(\mathrm{Gr}(\Phi), (x, y))\} \text{ for all } y^* \in Y^* \tag{2.2}$$

and

$$D_F^*\Phi(x,y)(y^*) := \{x^* \in X^* :\ (x^*, -y^*) \in N_F(\mathrm{Gr}(\Phi), (x,y))\} \text{ for all } y^* \in Y^*. \quad (2.3)$$

Mimicking this definition, we employ Clarke's normal cone to define another kind of coderivative

$$D_c^*\Phi(x,y)(y^*) := \{x^* \in X^* :\ (x^*, -y^*) \in N_c(\mathrm{Gr}(\Phi), (x,y))\} \text{ for all } y^* \in Y^*.$$

When Φ is single-valued, we denote $\hat{D}^*\Phi(x,\Phi(x))$, $D_F^*\Phi(x,\Phi(x))$ and $D_c^*\Phi(x,\Phi(x))$ by $\hat{D}^*\Phi(x)$, $D_F^*\Phi(x)$ and $D_c^*\Phi(x)$, respectively. The following two lemmas dealing with possibly non-convex sets in generalizing the Separation Theorem will be useful for us. As remarked by one of the referees, it is strange that Lemma 2.1 below and the above definition of the Clarke coderivative do not seem available in print before.

Lemma 2.1. *Let A be a closed convex subset of Y with a nonempty interior and let B be a closed (not necessarily convex) subset of Y. Suppose $\mathrm{int}(A) \cap B = \emptyset$ and $a \in A \cap B$. Then there exists $a^* \in Y^*$ with $\|a^*\| = 1$ such that*

$$a^* \in N_c(B,a) \text{ and } \langle a^*, a\rangle = \inf\{\langle a^*, x\rangle :\ x \in A\}.$$

Proof. Let $a_0 \in \mathrm{int}(A)$ and P be the Minkowski functional of $A - a_0$, namely

$$P(y) := \inf\{t > 0 : y \in t(A - a_0)\} \text{ for all } y \in Y.$$

Then by well known results in functional analysis,

$$\mathrm{int}(A) - a_0 = \{y \in Y :\ P(y) < 1\} \text{ and } A - a_0 = \{y \in Y :\ P(y) \le 1\}$$

(cf. [22]). Therefore,

$$1 = P(a - a_0) = \inf\{P(y - a_0) + \delta_B(y) :\ y \in Y\}.$$

Noting that the Minkowski functional P is Lipschitz (because it is positively homogeneous, subadditive and continuous), it follows from Proposition 2.1 that $0 \in \partial P(a - a_0) + N_c(B,a)$. Noting that P is convex and $P(0) < P(a - a_0)$, one has $0 \notin \partial P(a - a_0)$. Hence there exist $r > 0$ and $a^* \in N_c(B,a)$ with $\|a^*\| = 1$ such that $-ra^* \in \partial P(a - a_0)$. Thus,

$$\langle -ra^*, y - a\rangle \le P(y - a_0) - P(a - a_0) \le 0 \text{ for all } y \in A$$

and so $\langle a^*, a\rangle = \inf\{\langle a^*, y\rangle :\ y \in A\}$. This completes the proof. □

Lemma 2.2. *Let A and B be closed subsets of Y with $A \cap B = \emptyset$. Let $a \in A$, $b \in B$ and $\varepsilon > 0$ be such that $\|a - b\| \le d(A,B) + \varepsilon^2$, where $d(A,B) := \inf\{\|x - y\| :\ x \in A \text{ and } y \in B\}$. Then there exist $a_\varepsilon \in A$, $b_\varepsilon \in B$, $a_\varepsilon^* \in N_c(A, a_\varepsilon) + \varepsilon B_{Y^*}$ and $b_\varepsilon^* \in N_c(B, b_\varepsilon) + \varepsilon B_{Y^*}$ with $\|a_\varepsilon^*\| = \|b_\varepsilon^*\| = 1$ such that*

$$a_\varepsilon^* + b_\varepsilon^* = 0 \text{ and } \|a_\varepsilon - a\| + \|b_\varepsilon - b\| \le \varepsilon.$$

Proof. Define $f : Y \times Y \to R \cup \{+\infty\}$ by

$$f(x, y) := \delta_{A\times B}(x, y) + \|x - y\| \text{ for all } (x, y) \in Y \times Y.$$

Then $\inf\{f(x, y) : (x, y) \in Y \times Y\} = d(A, B)$ and so, by assumption

$$f(a, b) \leq \inf\{f(x, y) : (x, y) \in Y \times Y\} + \varepsilon^2.$$

Equipping $Y \times Y$ with the norm $\|(x, y)\| = \|x\| + \|y\|$, by the Ekeland Variational Principle there exists $(a_\varepsilon, b_\varepsilon) \in A \times B$ such that

$$\|a - a_\varepsilon\| + \|b - b_\varepsilon\| \leq \varepsilon \tag{2.4}$$

and

$$f(a_\varepsilon, b_\varepsilon) \leq f(x, y) + \varepsilon(\|x - a_\varepsilon\| + \|y - b_\varepsilon\|) \quad \forall (x, y) \in Y \times Y.$$

Letting

$$g(x, y) := \|x - y\| + \varepsilon(\|x - a_\varepsilon\| + \|y - b_\varepsilon\|) \text{ for all } (x, y) \in Y \times Y,$$

this implies that $g(x, y)$ attains its minimum over $A \times B$ at $(a_\varepsilon, b_\varepsilon)$. It follows from Proposition 2.1 that

$$(0, 0) \in \partial_c g(a_\varepsilon, b_\varepsilon) + N_c(A \times B, (a_\varepsilon, b_\varepsilon)). \tag{2.5}$$

Let $h(x, y) := \|x - y\|$ and $T(x, y) = x - y$ for any $(x, y) \in Y \times Y$. It follows from [6, Theorem 2.3.10] that $\partial h(a_\varepsilon, b_\varepsilon) = T^*[\partial(\|\cdot\|)(a_\varepsilon - b_\varepsilon)]$, where T^* is the conjugate operator of the bounded linear operator T. Noting that $T^*(z^*) = (z^*, -z^*)$ for any $z^* \in Y^*$, $a_\varepsilon - b_\varepsilon \neq 0$ (since $A \cap B = \emptyset$ and $(a_\varepsilon, b_\varepsilon) \in A \times B$) and

$$\partial(\|\cdot\|)(a_\varepsilon - b_\varepsilon) = \{z^* \in X^* : \|z^*\| = 1 \text{ and } \langle z^*, a_\varepsilon - b_\varepsilon\rangle = \|a_\varepsilon - b_\varepsilon\|\},$$

the subdifferential of the convex function $h(x, y)$ at $(a_\varepsilon, b_\varepsilon)$ is equal to the set

$$D := \{(z^*, -z^*) \in Y^* \times Y^* : \|z^*\| = 1 \text{ and } \langle z^*, a_\varepsilon - b_\varepsilon\rangle = \|a_\varepsilon - b_\varepsilon\|\}.$$

Hence

$$\partial_c g(a_\varepsilon, b_\varepsilon) \subset D + \varepsilon B_{Y^*} \times \varepsilon B_{Y^*}.$$

Since $N_c(A \times B, (a_\varepsilon, b_\varepsilon)) = N_c(A, a_\varepsilon) \times N_c(B, b_\varepsilon)$, it follows from (2.5) that there exists $z^* \in Y^*$ with $\|z^*\| = 1$ such that

$$(0, 0) \in (z^*, -z^*) + \varepsilon B_{Y^*} \times \varepsilon B_{Y^*} + N_c(A, a_\varepsilon) \times N_c(B, b_\varepsilon).$$

Note then that

$$-z^* \in \varepsilon B_{Y^*} + N_c(A, a_\varepsilon) \text{ and } z^* \in \varepsilon B_{Y^*} + N_c(B, b_\varepsilon).$$

Together with (2.4), the lemma is established by letting $a^*_\varepsilon = -z^*$ and $b^*_\varepsilon = z^*$. □

Remark. Suppose that $A \cap B = \emptyset$ and that $(a, b) \in A \times B$ satisfies $d(A, B) = \|a - b\|$. From the proof of Lemma 2.2, one can see that there exist $a^* \in N_c(A, a)$ and $b^* \in N_c(B, b)$ such that

$$a^* + b^* = 0 \text{ and } \|a^*\| = \|b^*\| = 1.$$

In contrast to Proposition 2.1, the following result (valid for Asplund spaces) is given in terms of Frechet normal cones and subdifferentials; see [5] and references therein for the detail.

Proposition 2.2. *Let Y be an Asplund space and $f : Y \to R$ a locally Lipschitz function, and let A be a closed subset of Y. Suppose that f attains its minimum over A at $a \in A$. Then for any $\varepsilon > 0$ there exist $a_\varepsilon \in a + \varepsilon B_Y$ and $u_\varepsilon \in A \cap (a + \varepsilon B_Y)$ such that*

$$0 \in \hat{\partial} f(a_\varepsilon) + \hat{N}(A, u_\varepsilon) + \varepsilon B_{Y^*}.$$

Similar to the proof of Lemma 2.2 but applying Proposition 2.2 in place of Proposition 2.1, we have the following result applicable to the case when Y is an Asplund space.

Lemma 2.2′. *Let Y, A, B, a, b and $\varepsilon > 0$ be as in Lemma 2.2 then there exist $a_\varepsilon \in A$, $b_\varepsilon \in B$, $a^*_\varepsilon \in \hat{N}(A, a_\varepsilon) + 2\varepsilon B_{Y^*}$ and $b^*_\varepsilon \in \hat{N}(B, b_\varepsilon) + 2\varepsilon B_{Y^*}$ with $\|a^*_\varepsilon\| = \|b^*_\varepsilon\| = 1$ such that*

$$a^*_\varepsilon + b^*_\varepsilon = 0 \text{ and } \|a_\varepsilon - a\| + \|b_\varepsilon - b\| < 2\varepsilon.$$

Remark. Similar to Lemma 2.2′, one can establish a result corresponding to Lemma 2.1 in the Asplund space setting. Since this is not needed for our further works here, we omit the details.

Lemma 2.3. *Let X, Y, Z be Asplund spaces, $\Phi : X \to 2^Y$ be a closed multifunction and $\phi : X \to Z$ be a locally Lipschitz single-valued mapping. Let*

$$(\Phi, \phi)(x) := \{(y, \phi(x)) \in Y \times Z : \ y \in \Phi(x)\} \text{ for all } x \in X.$$

Then

$$D^*_F(\Phi, \phi)(x, (y, \phi(x)))(y^*, z^*) \subset D^*_F \Phi(x, y)(y^*) + D^*_F \phi(x)(z^*) \tag{2.6}$$

for any $(x, y) \in \mathrm{Gr}(\Phi)$ and $(y^*, z^*) \in Y^* \times Z^*$.

Proof. Let x^* be any element in the set on the left-hand side of (2.6). Then there exist sequences $\{(x^*_k, y^*_k, z^*_k)\}$ in $X^* \times Y^* \times Z^*$ and $\{(x_k, y_k)\}$ in $\mathrm{Gr}(\Phi)$ such that

$$x_k \to x, \ y_k \to y, \ x^*_k \xrightarrow{w^*} x^*, \ y^*_k \xrightarrow{w^*} y^*, \ z^*_k \xrightarrow{w^*} z^* \tag{2.7}$$

and

$$x^*_k \in \hat{D}^*(\Phi, \phi)(x_k, y_k, \phi(x_k))(y^*_k, z^*_k) \text{ for any natural number } k.$$

Hence for each k there exists $\delta_k > 0$ such that for any $(u, v) \in \mathrm{Gr}(\Phi)$ with $\|u - x_k\| + \|v - y_k\| < \delta_k$,

$$0 \leq -\langle x_k^*, u - x_k\rangle + \langle y_k^*, v - y_k\rangle + \langle z_k^*, \phi(u) - \phi(x_k)\rangle \\ +\frac{1}{k}(\|u - x_k\| + \|v - y_k\| + \|\phi(u) - \phi(x_k)\|).$$

Since ϕ is locally Lipschitz at x, by (2.7) we can assume without loss of generality that there exists a constant $L > 1$ such that for any k and $u \in X$ with $\|u - x_k\| < \delta_k$,

$$\|u - x_k\| + \|\phi(u) - \phi(x_k)\| \leq L\|u - x_k\|.$$

Let us fix k and define $f : X \times Y \to R$ by

$$f(u, v) := -\langle x_k^*, u - x_k\rangle + \langle y_k^*, v - y_k\rangle + \langle z_k^*, \phi(u) - \phi(x_k)\rangle \\ +\frac{L}{k}(\|u - x_k\| + \|v - y_k\|)$$

for any $(u, v) \in X \times Y$. Then $f(x_k, y_k) = 0 \leq f(u, v)$ for any $(u, v) \in \mathrm{Gr}(\Phi)$ with $\|u - x_k\| + \|v - y_k\| < \delta_k$. It follows from [30, Theorem 2.12] that there exist $u_k \in X$ and $(u_k', v_k') \in \mathrm{Gr}(\Phi)$ such that

$$\|u_k - x_k\| + \|u_k' - x_k\| + \|v_k' - y_k\| < \min\{\frac{1}{k}, \delta_k\}$$

and

$$(0, 0) \in (-x_k^*, y_k^*) + \hat{\partial}(z_k^* \circ \phi)(u_k) \times \{0\} + \hat{N}(\mathrm{Gr}(\Phi), (u_k', v_k')) + \frac{L+1}{k}(B_{X^*} \times B_{Y^*}).$$

Noting that $\hat{\partial}(z_k^* \circ \phi)(u_k) \subset \hat{D}^*\phi(u_k)(z_k^*)$, it follows that

$$(0, 0) \in (-x_k^*, y_k^*) + \hat{D}^*\phi(u_k)(z_k^*) \times \{0\} + \hat{N}(\mathrm{Gr}(\Phi), (u_k', v_k')) + \frac{L+1}{k}(B_{X^*} \times B_{Y^*}).$$

Letting $k \to \infty$ and noting that $\sup\{\|u^*\| : u^* \in \hat{D}^*\phi(u_k)(z_k^*)\} \leq L\|z_k^*\|$, by (2.7) one has

$$(0, 0) \in (-x^*, y^*) + D_F^*\phi(x)(z^*) \times \{0\} + N_F(\mathrm{Gr}(\Phi), (x, y)),$$

that is,

$$x^* \in D_F^*\phi(x)(z^*) + D_F^*\Phi(x, y)(y^*).$$

This shows that (2.6) holds. □

3. Fermat Rules for multifunctions in Banach spaces

In this section, we always assume that X and Y are Banach spaces. For convenience we define the norm on $X \times Y$ by $\|(x, y)\| = \|x\| + \|y\|$. First we provide a fuzzy version of Fermat Rule for multifunctions in a general setting.

Theorem 3.1. *Let $\Phi : X \to 2^Y$ be a closed multifunction and $(\bar{x}, \bar{y})$ be a local Pareto solution of the vector optimization problem (1.1). Then for any $\varepsilon > 0$ there exist $x_\varepsilon \in \bar{x} + \varepsilon B_X$, $y_\varepsilon \in \Phi(x_\varepsilon) \cap (\bar{y} + \varepsilon B_Y)$ and $c^* \in C^+$ with $\|c^*\| = 1$ such that*

$$0 \in D_c^* \Phi(x_\varepsilon, y_\varepsilon)(c^* + \varepsilon B_{Y^*}) + \varepsilon B_{X^*}. \tag{3.1}$$

Proof. We will prove the following equivalent form of the result: there exist a sequence $\{(x_n, y_n)\}$ in $\mathrm{Gr}(\Phi)$ and a sequence $\{c_n^*\}$ in C^+ with $\|c_n^*\| = 1$ (for all n) such that $(x_n, y_n) \to (\bar{x}, \bar{y})$ and

$$d((0, -c_n^*), N_c(\mathrm{Gr}(\Phi), (x_n, y_n))) \to 0. \tag{3.2}$$

By assumption there exists $\delta > 0$ such that $\bar{y} \in E(\Phi(\bar{x} + \delta B_X), C)$. Let

$$A := \{(x, y) \in \mathrm{Gr}(\Phi) : \ x \in \bar{x} + \delta B_X\}$$

and take $c_0 \in C \setminus -C$ with $\|c_0\| = 1$ (such an element exists because the ordering cone C is pointed and non-trivial). For simplicity, let $B_n := \bar{y} - \frac{1}{n^2} c_0 - C$. We claim that for all natural number n large enough,

$$A \cap [X \times B_n] = \emptyset. \tag{3.3}$$

Indeed if this is not the case, then there exists $y' \in \Phi(\bar{x} + \delta B_X)$ such that $y' \leq_C \bar{y} - \frac{1}{n^2} c_0$, contradicting $\bar{y} \in \mathrm{E}(\Phi(\bar{x} + \delta B_X), C)$. Hence (3.3) holds. By Lemma 2.2 (applied to $a = (\bar{x}, \bar{y})$ and $b = (\bar{x}, \bar{y} - \frac{1}{n^2} c_0)$), there exist

$$\begin{aligned} &(x_n, y_n) \in A, (u_n, v_n) \in X \times B_n, \\ &(x_n^*, y_n^*) \in N_c(A, (x_n, y_n)) + \frac{1}{n}(B_{X^*} \times B_{Y^*}) \end{aligned} \tag{3.4}$$

and

$$(u_n^*, v_n^*) \in N_c(X \times B_n, (u_n, v_n)) + \frac{1}{n}(B_{X^*} \times B_{Y^*})$$

with $\|(x_n^*, y_n^*)\| = \|(u_n^*, v_n^*)\| = 1$ such that $(x_n^*, y_n^*) + (u_n^*, v_n^*) = 0$,

$$\|(x_n, y_n) - (\bar{x}, \bar{y})\| \leq \frac{1}{n} \ \text{ and } \ \|(u_n, v_n) - (\bar{x}, \bar{y} - \frac{1}{n^2} c_0)\| \leq \frac{1}{n}.$$

Then by the following well known relation on normal cones

$$N_c(X \times B_n, (u_n, v_n)) = \{0\} \times N_c(B_n, v_n) \subset \{0\} \times C^+,$$

there exist $r_n \in [1 - \frac{1}{n},\ 1 + \frac{1}{n}]$ and $c_n^* \in C^+$ with $\|c_n^*\| = 1$ such that

$$(u_n^*, v_n^*) \in r_n(0, c_n^*) + \frac{1}{n}(B_{X^*} \times B_{Y^*}),$$

namely

$$-(x_n^*, y_n^*) \in r_n(0, c_n^*) + \frac{1}{n}(B_{X^*} \times B_{Y^*}).$$

This and (3.4) imply that

$$\begin{aligned}(0, -c_n^*) &\in \frac{1}{r_n}(x_n^*, y_n^*) + \frac{1}{nr_n}(B_{X^*} \times B_{Y^*})\\ &\subset N_c(A, (x_n, y_n)) + \frac{2}{nr_n}(B_{X^*} \times B_{Y^*}).\\ &= N_c(\mathrm{Gr}(\Phi), (x_n, y_n)) + \frac{2}{nr_n}(B_{X^*} \times B_{Y^*})\end{aligned}$$

where the last equality holds because $A = \mathrm{Gr}(\Phi) \cap ((\bar{x} + \delta B_X) \times Y)$ and $(\bar{x} + \delta B_X) \times Y$ is a neighborhood of (x_n, y_n) (for n large enough). Thus (3.2) holds. The proof is completed. □

The following example shows that $\varepsilon > 0$ in Theorem 3.1 cannot be replaced by $\varepsilon = 0$.

Example 3.1. Let X be an infinite dimensional separable Banach space and $\{x_n\}$ be a countable dense subset of X with each $x_n \neq 0$. Let $D = \{-\frac{x_n}{n\|x_n\|}\}$ and A be the closed convex hull of $D \cup -D$. Then A is a compact subset of X and $A = -A$. Moreover, it is easy to verify that

$$X = \mathrm{cl}(\mathrm{span}(A)) \ \text{ and } \mathrm{span}(A) = \bigcup_{n=1}^{\infty} nA, \tag{3.5}$$

where $\mathrm{span}(A)$ denotes the linear subspace of X generated by A. By Baire Category Theorem, it follows that $X \neq \mathrm{span}(A)$. Let $\Phi : X \to 2^X$ be defined by $\Phi(x) = \{x\}$ if $x \in A$ and $\Phi(x) = \emptyset$ otherwise. Then $\mathrm{Gr}(\Phi)$ is a compact convex subset of $X \times X$. Take $e \in X \setminus \mathrm{span}(A)$ and consider the ordering cone C defined by $C := \{te :\ t \geq 0\}$. By the choice of e, it is easy to verify that $(0, 0)$ is a global solution of the vector optimization $C\text{-}\min_{x \in X} \Phi(x)$. We claim that

$$0 \notin D_c^*\Phi(0, 0)(y^*) \ \text{ for all } y^* \in X^* \setminus \{0\}. \tag{3.6}$$

Indeed let $y^* \in X^*$ satisfy $0 \in D_c^*\Phi(0, 0)(y^*)$. By definition and convexity of Φ, one has that $\langle y^*, y\rangle \leq 0$ for all $y \in A$. It follows from (3.5) that $\langle y^*, x\rangle \leq 0$ for all $x \in X$ and hence $y^* = 0$. This shows that (3.6) holds.

Next we provide results showing that, in many interesting cases, one can indeed take $\varepsilon = 0$ in (3.1). For each of Theorems 3.2, 3.3 and 3.4, we will make the following blanket assumptions:

Assumption 3.1. $\Phi : X \to 2^Y$ is a closed multifunction.

Assumption 3.2. $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$ is a local Pareto solution of the vector optimization problem (1.1).

Theorem 3.2. *Let Assumptions 3.1 and 3.2 hold. Suppose that the ordering cone C in Y has a nonempty interior. Then there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that $0 \in D_c^*\Phi(\bar{x}, \bar{y})(c^*)$.*

Proof. Take $\delta > 0$ such that $\bar{y} \in E(\Phi(\bar{x} + \delta B_X), C)$. Letting

$$A := \mathrm{Gr}(\Phi) \cap ((\bar{x} + \delta B_X) \times Y),$$

it follows that $A \cap \mathrm{int}(X \times (\bar{y} - C)) = \emptyset$. By Lemma 2.1 there exists $(x^*, y^*) \in X^* \times Y^*$ with $\|(x^*, y^*)\| = 1$ such that $-(x^*, y^*) \in N_c(A, (\bar{x}, \bar{y}))$ and

$$\langle x^*, \bar{x}\rangle + \langle y^*, \bar{y}\rangle = \sup\{\langle x^*, x\rangle + \langle y^*, y\rangle : \ (x, y) \in X \times (\bar{y} - C)\}.$$

It follows that $x^* = 0$ and $y^* \in C^+$. Moreover

$$(0, -y^*) \in N_c(A, (\bar{x}, \bar{y})) = N_c(\mathrm{Gr}(\Phi), (\bar{x}, \bar{y})).$$

This shows that $0 \in D_c^*\Phi(\bar{x}, \bar{y})(y^*)$. Thus one can take $c^* := y^*$. □

Remark. In the case when $\mathrm{int}(C) \neq \emptyset$, many authors consider, in addition to Pareto solution, weak Pareto solutions of (1.1). Let A be a subset of Y. Recall that $a \in A$ is called a weak Pareto efficient point of A if $A \cap (a - \mathrm{int}(C)) = \emptyset$. Let $\mathrm{WE}(A, C)$ denote the set of all weak Pareto efficient points of A. We say that $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$ is a local weak Pareto solution of (1.1) if there exists a neighborhoond U of $\bar{x}$ such that $\bar{y} \in \mathrm{WE}(\Phi(U), C)$. From the proof of Theorems 3.1 and 3.2 (taking $c_0 \in \mathrm{int}(C)$ in the proof of Theorem 3.1), one sees that if $\mathrm{int}(C) \neq \emptyset$ and $(\bar{x}, \bar{y})$ is a local weak Pareto solution of (1.1) then there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that $0 \in D_c^*\Phi(\bar{x}, \bar{y})(c^*)$. Thus Theorem 3.2 remains true if Assumption 3.2 is replaced by:

Assumption 3.2*. $\mathrm{int}(C) \neq \emptyset$*, and* $(\bar{x}, \bar{y}) \in \mathrm{Gr}(\Phi)$ *is a local weak Pareto solution of (1.1).*

For a subset K of Y, let

$$\mathcal{W}(K) := \{y^* \in Y^* : \ \|y^*\| \leq \sup\{\langle y^*, y\rangle : \ y \in K\}\}.$$

If $c \in \mathrm{int}(C)$ then $c + \delta B_Y \subset C$ for some $\delta > 0$; thus, for any $c^* \in C^+$,

$$0 \leq \inf\{\langle c^*, x\rangle : \ x \in c + \delta B_Y\} = \langle c^*, c\rangle - \delta\|c^*\|$$

and so $\|c^*\| \leq \langle c^*, \frac{c}{\delta}\rangle$. Therefore,

$$c \in \mathrm{int}(C) \Rightarrow C^+ \subset \mathcal{W}(\{rc\}) \ \text{ for some } r > 0$$

(recalling that C^+ is called a Bishop-Phelps cone if there exists a singleton K such that $C^+ \subset \mathcal{W}(K)$, and so $\mathrm{int}(C) \neq \emptyset \Rightarrow C^+$ is a Bishop-Phelps cone). Thus the following concept extends the condition that $\mathrm{int}(C) \neq \emptyset$.

Definition 3.1. *A closed convex cone $C \subset Y$ is said to be dually compact if there exists a compact subset K of Y such that*

$$C^+ \subset \mathcal{W}(K). \tag{3.7}$$

There are two important types of cones C in Y satisfying this property:

(a) Y is finite dimensional (because one can then take $K = B_Y$).
(b) $\mathrm{int}(C) \neq \emptyset$.

Recall that a set A in Y^* is a weakly (resp. weak*) locally compact if every point of A lies in a weakly (resp. weak*) open set V such that $\overline{V}^w \cap A$ (resp. $\overline{V}^{w^*} \cap A$) is weakly (resp. weak*) compact (cf. [15]), where $\overline{V}^w$ (resp. $\overline{V}^{w^*}$) denotes the closure of V with respect to the weak (resp. weak*) topology of Y^*. Loewen [15] proved that if Y is reflexive and K is a compact subset of Y then $\mathcal{W}(K)$ is weakly locally compact ([15, Proposition 3.5]). Since a set in a reflexive Banach space is weakly compact if and only if it is bounded and weakly closed, the implication (i)⇒(iii) of the following proposition for the reflexive case implies the result of Loewen.

Proposition 3.1. *Let C be a closed convex cone in a Banach space Y. Then the following properties are equivalent.*

(i) C is dually compact.
(ii) There exists a weak* *open set V containing 0 such that $V \cap C^+$ is bounded.*
(iii) C^+ is weak* *locally compact.*

Proof. (i)⇒(iii). By (i) there exists a compact subset K of Y such that (3.7) holds. By compactness of K there exist $y_1, \cdots, y_m \in K$ such that $K \subset \bigcup_{i=1}^{m} (y_i + \frac{1}{2} B_Y)$. Therefore, for any $z^* \in C^+$, (3.7) implies that

$$\begin{aligned}\|z^*\| &\le \max\{\langle z^*, y\rangle : \ y \in \bigcup_{i=1}^{m}(y_i + \frac{1}{2} B_Y)\}\\ &= \max\{\langle z^*, y_i\rangle : \ i = 1, \cdots, m\} + \frac{1}{2}\|z^*\|.\end{aligned}$$

Hence

$$\|z^*\| \le 2\max\{\langle z^*, y_i\rangle : \ i = 1, \cdots, m\} \ \text{ for all } z^* \in C^+. \tag{3.8}$$

Let $V := \{y^* \in Y^* : \ \langle y^*, y_i\rangle < 1, \ i = 1, \cdots, n\}$. Then, for any $c^* \in C^+$, $c^* + V$ is a weak* open set containing c^* and

$$\overline{c^* + V}^{w*} = c^* + \{y^* \in Y^* : \ \langle y^*, y_i\rangle \le 1, \ i = 1, \cdots, n\}.$$

It follows from (3.8) that for any $z^* \in \overline{c^* + V}^{w*} \cap C^+$,

$$\|z^*\| \le 2\max\{\langle c^*, y_i\rangle : \ i = 1, \cdots, n\} + 2.$$

Therefore, $\overline{c^* + V}^{w*} \cap C^+$ is weak* compact (because it is weak* closed and bounded). This shows that (iii) holds.

(iii) $\Rightarrow$(ii) is trivial.

(ii) $\Rightarrow$(i). By (ii) there exists a weak* open set V containing 0 and a constant $M > 0$ such that

$$\|y^*\| \leq M \quad \text{for all } y^* \in V \cap C^+.$$

Take $z_1, \cdots, z_n \in Y$ such that

$$V \supset \{y^* \in Y^* : \langle y^*, z_i \rangle \leq 1, i = 1, \cdots, n\}.$$

Let $z^* \in C^+$ and $r := \max\{\langle z^*, z_i\rangle : i = 1, \cdots, n\}$. In the case when $r \leq 0$, $tz^* \in V \cap C^+$ and hence $t\|z^*\| \leq M$ for any $t > 0$. This implies that $z^* = 0$. In the case when $r > 0$, $\frac{z^*}{r} \in V \cap C^+$ and hence

$$\|z^*\| \leq \max\{\langle z^*, Mz_i\rangle : i = 1, \cdots, n\}.$$

This shows that (3.7) holds with $K = \{Mz_1, \cdots, Mz_n\}$. The proof is completed. □

Remark. It is known (cf. [13, Theorem 3.8.6]) that the ordering cone C has a nonempty interior if and only if C^+ has a weak*-compact base (i.e., there exists a weak*-compact convex set Θ such that $0 \notin \Theta$ and $C = \{t\theta : \theta \in \Theta \text{ and } t \geq 0\}$). Therefore,

$$C^+ \text{ has a weak}^*\text{-compact base } \Rightarrow C^+ \text{ is weak}^* \text{ locally compact.}$$

In general, the converse implication is not true. For example, let $Y = R^2$ and $C = \{0\} \times R_+$. Clearly, C^+ is weak* locally compact, but $C^+ = R \times R_+$ has no weak*-compact base. However, under the condition that C^+ is pointed, the converse implication is true (cf [8, Theorem 3]). Song [23] gave some interesting equivalence results for a number of classes of cones used in vector optimization.

By (3.8), one has that if C is dually compact then

$$y_n^* \xrightarrow{w^*} 0 \Leftrightarrow y_n^* \to 0 \quad \text{for any (generalized) sequence } \{y_n^*\} \text{ in } C^+. \tag{3.9}$$

Let A be a closed subset of Y. Recall that A is said to be epi-Lipschizian at a (cf. [3]) if there exist a neighborhood V of a, a nonempty open set U and $\lambda > 0$ such that

$$A \cap V + (0, \ \lambda)U \subset A.$$

In this case, any non-zero vector in U is said to be hypertangent to A at a. We say that A is epi-Lipschitz-like at a (cf. [3, 14]) if there exist $\lambda > 0$, a neighborhood V of a and a convex set S with its polar S° being weak* locally compact such that

$$A \cap V + (0, \ \lambda)S \subset A.$$

Mimicking Mordukhovich's idea in defining partially sequentially normal compactness (cf. [17–19]) by virtue of the coderivative $\hat{D}^*\Phi$, we employ the Clarke coderivative D_c^* to define that the multifunction Φ is partially sequentially normal compact at $(\bar{x}, \bar{y})$

with respect to $D_c^*\Phi$ if following implication holds for any (generalized) sequence $\{(x_n, y_n, x_n^*, y_n^*)\}$:

$$x_n^* \in D_c^*\Phi(x_n, y_n)(y_n^*),\ (x_n, y_n) \to (\bar{x}, \bar{y}),\ x_n^* \to 0 \ \text{ and } \ y_n^* \xrightarrow{w^*} 0 \Longrightarrow y_n^* \to 0.$$

Using similar arguments as in [17–19] , one can show that the above implication holds if $\mathrm{Gr}(\Phi)$ is epi-Lipschitz-like at $(\bar{x}, \bar{y})$.

We say that $N_c(A, \cdot)$ is closed at $a \in A$ if for (generalized) sequences

$$a_n \to a,\ a_n^* \in N_c(A, a_n),\ a_n^* \xrightarrow{w^*} a^* \Longrightarrow a^* \in N_c(A, a)$$

(cf. [6, P.58, Corollary]). It is well known that $N_c(A, \cdot)$ is closed at every point of A if A is convex. It is easy to verify that $N_c(A, \cdot)$ is also closed at a if a is a smooth boundary point of A in the sense that there exist a neighborhood V of a and a continuously Frechet differentiable function f such that $f'(a) \neq 0$ and $V \cap A = V \cap \{x \in X : \ f(x) \leq 0\}$.

Theorem 3.3. *Let Assumptions 3.1 and 3.2 hold. Suppose that* $N_c(\mathrm{Gr}(\Phi), \cdot)$ *is closed at* $(\bar{x}, \bar{y})$ *(this condition is automatically satisfied if* Φ *is assumed to be a closed convex multifunction). Further suppose that one of the following two conditions holds.*

(i) The ordering cone C *in* Y *is dually compact.*
(ii) Φ *is partially sequentially normal compact at* $(\bar{x}, \bar{y})$ *with respect to* $D_c^*\Phi$*. Then there exists* $c^* \in C^+$ *with* $\|c^*\| = 1$ *such that* $0 \in D_c^*\Phi(\bar{x}, \bar{y})(c^*)$.

Proof. By Theorem 3.1 there exists a sequence $(x_n, y_n, x_n^*, y_n^*, c_n^*)$ with each $(x_n, y_n) \in \mathrm{Gr}(\Phi)$, $c_n^* \in C^+$, $\|c_n^*\| = 1$ and $x_n^* \in D_c^*\Phi(x_n, y_n)(y_n^*)$ such that

$$(x_n, y_n) \to (\bar{x}, \bar{y}),\ x_n^* \to 0 \ \text{ and } \ \|y_n^* - c_n^*\| \to 0.$$

Since the unit ball of Y^* is weak* compact, without loss of generality we can assume that $c_n^* \xrightarrow{w^*} c_0^* \in C^+$ (and hence $y_n^* \xrightarrow{w^*} c_0^*$). Since $N_c(\mathrm{Gr}(\Phi), \cdot)$ is closed at $(\bar{x}, \bar{y})$,

$$0 \in D_c^*\Phi(\bar{x}, \bar{y})(c_0^*). \tag{3.10}$$

Thus the proof will be completed provided that $c_0^* \neq 0$. This is certainly the case if (ii) holds because $\|y_n^*\| \to 1$ and $y_n^* \xrightarrow{w^*} c_0^*$. Next suppose that (i) holds. Then, we must also have $c_0^* \neq 0$, in view of (3.9). The proof is completed. □

Recall (cf. [6, P.58, Corollary]) that if a closed set A is epi-Lipschitzian at a then $N_c(A, \cdot)$ is closed at $a \in A$. The following corollary is a consequence of Theorem 3.3 ((ii) is automatically satisfied thanks to the epi-Lipschitz assumption).

Corollary 3.1. *Let Assumptions 3.1 and 3.2 hold. Suppose that* $\mathrm{Gr}(\Phi)$ *is epi-Lipschitzian at* $(\bar{x}, \bar{y})$*. Then there exists* $c^* \in C^+$ *with* $\|c^*\| = 1$ *such that* $0 \in D_c^*\Phi(\bar{x}, \bar{y})(c^*)$.

4. Asplund space setting

Throughout this section, let X and Y denote Asplund spaces (thus $X \times Y$ is also an Asplund space). In this setting Theorems 3.1, 3.2 and 3.3 can be strengthened to following Theorems 4.1 and 4.2 in which D_c^* is replaced by the Mordukhovich derivative D_F^* (recall that $\hat{N}(A, a) \subset N_F(A, a)$ and $N_c(A, a)$ is the weak*-closed convex hull of $N_F(A, a)$). The proofs are the same as before but use Lemma 2.2′ in place of Lemma 2.2.

Theorem 4.1. *Let X and Y be Asplund spaces and $\Phi : X \to 2^Y$ be a closed multifunction. Suppose that $(\bar{x}, \bar{y})$ is a local Pareto solution of (1.1). Then for any $\varepsilon > 0$ there exist $x_\varepsilon \in \bar{x} + \varepsilon B_X$, $y_\varepsilon \in \Phi(x_\varepsilon) \cap (\bar{y} + \varepsilon B_Y)$ and $c^* \in C^+$ with $\|c^*\| = 1$ such that*

$$0 \in \hat{D}^*\Phi(x_\varepsilon, y_\varepsilon)(c^* + \varepsilon B_{Y^*}) + \varepsilon B_{X^*}$$

(where the notion $\hat{D}^*$ is defined by (2.2)).

Remark. From the proof of Theorem 3.1, one sees that if $(\bar{x}, \bar{y})$ is a local Pareto solution of (1.1) then it is a local extremal point of the system $\{\mathrm{Gr}(\Phi),\ \bar{y} - C\}$ (cf. [18]). Thus one can also prove Theorem 4.1 by using the extremal principle (cf. [14]) instead of Lemma 2.2′. Lemma 2.2′ in general implies the extremal principle but clearly its converse is not true.

Following Mordukhovich and Shao [18, 19], we say that the multifunction Φ is partially sequentially normally compact with respect to Y at $(x, y) \in \mathrm{Gr}(\Phi)$ if any sequence (x_n, y_n, x_n^*, y_n^*) satisfying $x_n^* \in D_F^*\Phi(x_n, y_n)(y_n^*)$, $(x_n, y_n) \to (x, y)$, $\|x_n^*\| \to 0$ and $y_n^* \xrightarrow{w^*} 0$ as $n \to \infty$ contains a subsequence with $\|y_{n_k}^*\| \to 0$ as $k \to \infty$.

Theorem 4.2. *Let X and Y be Asplund spaces. Suppose that Assumptions 3.1 and 3.2 hold. Then there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that $0 \in D_F^*\Phi(\bar{x}, \bar{y})(c^*)$ provided that one of the following conditions is satisfied.*

(a) Φ is partially sequentially normally compact with respect to Y at $(\bar{x}, \bar{y})$.
(b) The ordering cone C is dually compact.
(c) $\mathrm{int}(C) \neq \emptyset$ or Y is finite dimensional.

Proof. Since (c)$\Longrightarrow$(b), we need only to deal with (a) and (b). By Theorem 4.1 there exists a sequence (x_n, y_n, x_n^*, y_n^*) such that

$$x_n^* \in \hat{D}^*\Phi(x_n, y_n)(y_n^*),\ \|y_n^*\| = 1,\ (x_n, y_n) \to (\bar{x}, \bar{y}),\ \|x_n^*\| \to 0 \text{ and } d(y_n^*, C^+) \to 0.$$

Without loss of generality we can assume that $y_n^* \xrightarrow{w^*} c_0^* \in C^+$. It follows from (2.1) that $0 \in D_F^*\Phi(\bar{x}, \bar{y})(c_0^*)$. It remains to show that $c_0^* \neq 0$. However this can be done exactly as in the proof of Theorem 3.3. □

Let Ω be a closed subset of X and consider the following constrained vector optimization problem.

$$C - \min_{x \in \Omega} \Phi(x). \tag{4.1}$$

We say that $(\bar{x}, \bar{y})$ is a local solution of (4.1) if there exists a neighborhood U of $\bar{x}$ such that $\bar{y} \in E(\Phi(U \cap \Omega), C)$. With Φ defined by (1.2), we note that $\bar{x}$ is a local minimum point of f on Ω if and only if $(\bar{x}, f(\bar{x}))$ is a local solution of (4.1). Recall [5] that if $f : X \to R$ is assumed to be locally Lipschitz then the following Fermat's rule is valid:

$$f \text{ attains a local mimimum at } \bar{x} \text{ over } \Omega \Longrightarrow 0 \in \partial_F f(\bar{x}) + N_F(\Omega, \bar{x}).$$

Thus it is reasonable for us to make a similar provision (of local Lipschitz property) in our multifunction setting. Recall [1] that a multifunction $\Phi : X \to 2^Y$ is said to have the Aubin property (or pseudo-Lipschitzian property) at $\bar{x}$ for $\bar{y} \in \Phi(\bar{x})$ if there exist a constant $l > 0$, neighborhoods U of $\bar{x}$ and V of $\bar{y}$ such that

$$\Phi(x) \cap V \subset \Phi(u) + l\|x - u\| B_Y \quad \text{for any } x, u \in U.$$

We shall need the following known result (cf. [17, Theorem 3.2]).

Proposition 4.1. *Let $\Phi : X \to 2^Y$ be a closed multifunction with the Aubin property at $\bar{x} \in X$ for $\bar{y} \in \Phi(\bar{x})$. Then there exist $L, \delta > 0$ such that*

$$\sup\{\|x^*\| : x^* \in \hat{D}^*\Phi(x, y)(y^*)\} \le L\|y^*\|$$

for any $(x, y) \in \mathrm{Gr}(\Phi) \cap (B(\bar{x}, \delta) \times B(\bar{y}, \delta))$ and any $y^* \in Y^*$.

In the remainder of this section, we always assume that X, Y, Z are Asplund spaces, Ω is a closed subset of X, $\Phi : X \to 2^Y$ is a closed multifunction with the Aubin property, and that $\phi : X \to Z$ is a locally Lipschitz single-valued mapping. Let C_Z be a closed convex cone in Z and let $\le_{C_Z}$ denote the preorder induced by C_Z. Next consider the following vector optimization problem with more general constraint:

$$\begin{aligned} C - \min \Phi(x) \\ \phi(x) \le_{C_Z} 0 \\ x \in \Omega. \end{aligned} \tag{4.2}$$

We say that $(\bar{x}, \bar{y})$ is a local Pareto solution of (4.2) if $\bar{x} \in \Omega$, $\phi(\bar{x}) \le_{C_Z} 0$ and there exists a neighborhood U of $\bar{x}$ such that $\bar{y} \in E(\Phi(U \cap \Omega \cap \phi^{-1}(-C_Z)), C)$.

Theorem 4.3. *Let $(\bar{x}, \bar{y})$ be a local Pareto solution of the constrained vector optimization problem (4.2). Suppose that both C and C_Z are dually compact. Then there exist $c^* \in C^+$ and $c_Z^* \in C_Z^+$ with $\|c^*\| + \|c_Z^*\| = 1$ such that*

$$0 \in D_F^*\Phi(\bar{x}, \bar{y})(c^*) + D_F^*\phi(\bar{x})(c_Z^*) + N_F(\Omega, \bar{x}). \tag{4.3}$$

Proof. By assumption there exists $\delta > 0$ such that

$$\bar{y} \in E(\Phi[(\bar{x} + \delta B_X) \cap \Omega \cap \phi^{-1}(-C_Z)], C). \tag{4.4}$$

Let

$$A := \{(x, y, \phi(x)) \in X \times Y \times Z : \ y \in \Phi(x) \ \text{ and } \ x \in \bar{x} + \delta B_X\}$$

and take $c_0 \in C$ with $\|c_0\| = 1$. For all natural number n large enough, let

$$B_n := \Omega \times (\bar{y} - \frac{1}{n^2}c_0 - C) \times (\phi(\bar{x}) - C_Z).$$

Then $A \cap B_n = \emptyset$. Indeed, if this is not the case then there exist $x' \in \bar{x} + \delta B_X$ and $y' \in \Phi(x')$ such that $x' \in \Omega$, $y' \leq \bar{y} - \frac{1}{n^2}c_0$ and $\phi(x') \leq_{C_Z} \phi(\bar{x}) \leq_{C_Z} 0$. This contradicts (4.4). By Lemma 2.2$'$ (applied to $a = (\bar{x}, \bar{y}, \phi(\bar{x}))$ and $b = (\bar{x}, \bar{y} - \frac{1}{n^2}c_0, \phi(\bar{x}))$), there exists a sequence $(x_n, y_n, x_n^*, y_n^*, z_n^*, u_n, v_n, w_n, u_n^*, v_n^*, w_n^*)$ with each

$$\begin{aligned} &(x_n, y_n, \phi(x_n)) \in A, \quad (u_n, v_n, w_n) \in B_n,\\ &(x_n^*, y_n^*, z_n^*) \in \hat{N}(A, (x_n, y_n, \phi(x_n))) \end{aligned} \tag{4.5}$$

and

$$(u_n^*, v_n^*, w_n^*) \in \hat{N}(B_n, (u_n, v_n, w_n)) \tag{4.6}$$

such that

$$\begin{aligned} &\lim_{n\to\infty}(\|(x_n, y_n, \phi(x_n)) - (\bar{x}, \bar{y}, \phi(\bar{x}))\| + \|(u_n, v_n, w_n) - (\bar{x}, \bar{y} - \frac{1}{n^2}c_0, \phi(\bar{x}))\|) = 0,\\ &\lim_{n\to\infty}\|(x_n^*, y_n^*, z_n^*)\| = \lim_{n\to\infty}\|(u_n^*, v_n^*, w_n^*)\| = 1 \end{aligned} \tag{4.7}$$

and

$$\lim_{n\to\infty}\|(x_n^*, y_n^*, z_n^*) + (u_n^*, v_n^*, w_n^*)\| = 0. \tag{4.8}$$

By (4.6) and (4.7), and making use of the following well-known relation

$$\begin{aligned} \hat{N}(B_n, (u_n, v_n, w_n)) &= \hat{N}(\Omega, u_n) \times \hat{N}(\bar{y} - \frac{1}{n^2}c_0 - C, v_n) \times \hat{N}(\phi(\bar{x}) - C_Z, w_n)\\ &\subset \hat{N}(\Omega, u_n) \times C^+ \times C_Z^+, \end{aligned}$$

we can assume without loss of generality that

$$(u_n^*, v_n^*, w_n^*) \xrightarrow{w^*} (u^*, \tilde{c}^*, \tilde{c}_Z^*) \in N_F(\Omega, \bar{x}) \times C^+ \times C_Z^+. \tag{4.9}$$

Noting that $A = \mathrm{Gr}(\Phi, \phi) \cap [(\bar{x} + \delta B_X) \times Y \times Z]$ and since $(\bar{x} + \delta B_X) \times Y \times Z$ is a neighborhood of $(x_n, y_n, \phi(x_n))$ (for all large enough n), (4.5) can be rewritten as

$$(x_n^*, y_n^*, z_n^*) \in \hat{N}(\mathrm{Gr}(\Phi, \phi), (x_n, y_n, \phi(x_n))),$$

that is,

$$x_n^* \in \hat{D}^*(\Phi, \phi)(x_n, y_n, \phi(x_n))(-y_n^*, -z_n^*). \tag{4.10}$$

It follows from (4.8) and (4.9) that that

$$x_n^* \xrightarrow{w^*} -u^*, \ y_n^* \xrightarrow{w^*} -\tilde{c}^*, \ z_n^* \xrightarrow{w^*} -\tilde{c}_Z^*$$

and

$$-u^* \in D_F^*(\Phi, \phi)(\bar{x}, \bar{y}, \phi(\bar{x}))(\tilde{c}^*, \tilde{c}_Z^*).$$

This and Lemma 2.3 imply that

$$-u^* \in D_F^*\Phi(\bar{x}, \bar{y})(\tilde{c}^*) + D_F^*\phi(\bar{x})(\tilde{c}_Z^*).$$

Thus (4.3) holds with $c^* = \frac{\tilde{c}^*}{\|\tilde{c}^*\|+\|\tilde{c}_Z^*\|}$ and $c_Z^* = \frac{\tilde{c}_Z^*}{\|\tilde{c}^*\|+\|\tilde{c}_Z^*\|}$ provided that $(\tilde{c}^*, \tilde{c}_Z^*) \neq (0, 0)$. Suppose for contradiction that $\tilde{c}^* = 0$ and $\tilde{c}_Z^* = 0$. Then $v_n^* \xrightarrow{w^*} 0$ and $w_n^* \xrightarrow{w^*} 0$. It follows from the dual compactness of C and C_Z that

$$\|v_n^*\| \to 0 \;\text{ and }\; \|w_n^*\| \to 0. \tag{4.11}$$

But on the other hand, since Φ and ϕ have respectively the Aubin property and local Lipschitz property at $(\bar{x}, \bar{y})$ and $\bar{x}$, one can apply Proposition 4.1 and (4.10) to conclude that there exists a constant $L > 0$ such that $\|x_n^*\| \le L(\|y_n^*\| + \|z_n^*\|)$ for all large enough n. It follows from (4.7) that there exists $r > 0$ such that $2r \le \|y_n^*\| + \|z_n^*\|$ for all large enough n. Therefore, by (4.8), $r \le \|v_n^*\| + \|w_n^*\|$ for all large enough n. This contradicts (4.11). The proof is completed. □

Setting $\phi(x) := 0$ for all $x \in X$, the following corollary is an immediate consequence of Theorem 4.3.

Corollary 4.1. *Let $(\bar{x}, \bar{y})$ be a local Pareto solution of the constrained vector optimization problem (4.1). Suppose that C is dually compact. Then there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that*

$$0 \in D_F^*\Phi(\bar{x}, \bar{y})(c^*) + N_F(\Omega, \bar{x}). \tag{4.12}$$

Remark. In the case when Φ is a Lipschitz single-valued mapping and Y is finite dimensional, by [18, Theorem 5.7] one has that

$$D_F^*\Phi(\bar{x})(y^*) = \partial_F(y^* \circ \Phi)(\bar{x}) \;\text{ for any } y^* \in Y^*, \tag{4.13}$$

and hence (4.12) is reduced to

$$0 \in \partial_F(c^* \circ \Phi)(\bar{x}) + N_F(\Omega, \bar{x}).$$

Recall that a Lipschitz single-valued mapping ϕ is strictly differentiable at x with a strict derivative $\phi'(x)$, a bounded linear operator from X to Y, provided that for each $h \in X$,

$$\lim_{z \to x\; t \downarrow 0} \frac{\phi(z + th) - \phi(z)}{t} = \phi'(x)(h).$$

It is known that $D_F^*\phi(x)(y^*) = (\phi'(x))^*(y^*)$ for all $y^* \in Y^*$ if ϕ is strictly differentiable at x, where $(\phi'(x))^*$ denotes the conjugate operator of $\phi'(x)$ (cf. [19, Theorem 3.5]).

Thus, in the the case when the objective function Φ in (4.1) is a Lipschitz single-valued mapping ϕ which is strictly differentiable at $\bar{x}$, (4.12) is the same as

$$0 \in (\phi'(\bar{x}))^*(c^*) + N_F(\Omega, \bar{x}).$$

Let $Z := R^{n+m}$, $C_Z := R^n_+ \times \{\mathbf{0}_m\}$ and $\phi : X \to Z$ be defined by

$$\phi(x) := (g_1(x), \cdots, g_n(x), h_1(x), \cdots, h_m(x)) \quad \text{for all } x \in X,$$

where $\mathbf{0}_m$ is the zero element of R^m and $g_i, h_j : X \to R$ are locally Lipschitz functions ($i = 1, \cdots, n$ and $j = 1, \cdots, m$). Thus (4.2) is reduced to the following problem:

$$\begin{aligned} C - \min \Phi(x) \\ g_i(x) \le 0, \quad i = 1, \cdots, n \\ h_j(x) = 0, \quad j = 1, \cdots, m \\ x \in \Omega \qquad . \end{aligned} \tag{4.14}$$

Corollary 4.2. *Let $(\bar{x}, \bar{y})$ be a local Pareto solution of (4.14). Suppose that C is dually compact. Then there exist $c^* \in C^+$, $\lambda_i \in R_+$ $(i = 1, \cdots, n)$ and $\mu_j \in R$ $(j = 1, \cdots, m)$ such that*

(i) $0 \in D_F^*\Phi(\bar{x}, \bar{y})(c^*) + \sum_{i=1}^{n} \lambda_i \partial_F g_i(\bar{x}) + \sum_{j=1}^{m} \partial_F(\mu_j h_j)(\bar{x}) + N_F(\Omega, \bar{x})$,

(ii) $\lambda_i g_i(\bar{x}) = 0$ $(i = 1, \cdots, n)$,

(iii) $\|c^*\| + \sum_{i=1}^{n} \lambda_i + \sum_{j=1}^{m} |\mu_j| = 1$.

Proof. Let $I := \{1 \le i \le n : \; g_i(\bar{x}) = 0\}$, $Z := R^{|I|+m}$ and $C_Z := R_+^{|I|} \times \{\mathbf{0}_m\}$. Let $\phi(x) := ((g_i(x))_{i \in I}, h_1(x), \cdots, h_m(x))$ for all $x \in X$. By assumption, it is clear that $(\bar{x}, \bar{y})$ is a local Pareto solution of the following problem:

$$\begin{aligned} C - \min \Phi(x) \\ \phi(x) \le_{C_Z} 0 \\ x \in \Omega \qquad . \end{aligned}$$

By Theorem 4.3 there exist $c^* \in C^+$, $\lambda_i \in R_+$ $(i \in I)$ and $\mu_j \in R$ $(j = 1, \cdots, m)$ such that $\|c^*\| + \sum_{i \in I} \lambda_i + \sum_{j=1}^{m} |\mu_j| = 1$ and

$$0 \in D^*\Phi(\bar{x}, \bar{y})(c^*) + D_F^*\phi(\bar{x})((\lambda_i)_{i \in I}, \mu_1, \cdots, \mu_m)) + N_F(\Omega, \bar{x}).$$

It follows from (4.13) and [18, Corollary 4.3] that (i), (ii) and (iii) hold with $\lambda_i = 0$ if $i \notin I$. □

Remark. In the special case when $Y = R^k$, $C = R^k_+$ and Φ is a Lipschitz single-valued mapping, (4.14) is reduced to the multiobjective program problem studied by Minami in [16]; noting that $\partial_F(\mu_j h_j)(\bar{x}) \subset \partial_c(\mu_j h_j)(\bar{x}) = \mu_j \partial_c h_j(\bar{x})$, (i), (ii) and (iii) in Corollary 4.2 respectively imply (a), (b) and (c) in [16, Theorem 3.1] (but, on the other hand, the said result in [16] is applicable to a general Banach space).

In the case when X is a reflexive Banach space, $Y = R$, $C = R_+$ and Φ is a single-valued Lipschitz function, Corollary 4.2 implies [4, Corollary 2.5]; if, in addition, $\Omega = X$ then Corollary 4.1 implies [4, Corollary 2.3].

5. Necessary conditions for Pareto efficient points

Throughout this section, we assume that A is a closed subset of a Banach space Y. We shall consider necessary conditions for $a \in A$ to be a Pareto efficient point of A with respect to the ordering cone C. Let $\Phi_A : Y \to 2^Y$ be defined by $\Phi_A(x) = \{x\}$ if $x \in A$ and $\Phi_A(x) = \emptyset$ otherwise. Thus $\mathrm{Gr}(\Phi_A) = \{(x, x) : \ x \in A\}$. It is also clear that

$$a \in E(A, C) \Leftrightarrow (a, a) \text{ is a solution of vector optimization problem } C - \min_{x\in Y} \Phi_A(x). \tag{5.1}$$

Lemma 5.1. *Let Y be a Banach space and $a \in A$. Then*

$$D_c^*\Phi_A(a, a)(y^*) = y^* + N_c(A, a) \ \text{ for all } y^* \in Y^*. \tag{5.2}$$

Proof. Let $T_c(\mathrm{Gr}(\Phi_A), (a, a))$ and $T_c(A, a)$ denote respectively Clarke's tangent cones of $\mathrm{Gr}(\Phi_A)$ at (a, a) and of A at a. Recall [6, Theorem 2.4.5] that $(u, v) \in T_c(\mathrm{Gr}(\Phi_A), (a, a))$ if and only if for every sequence $\{(x_n, y_n)\}$ in $\mathrm{Gr}(\Phi_A)$ converging to (a, a) and sequence $\{t_n\}$ in $(0, +\infty)$ decreasing to 0 there exists a sequence $\{(u_n, v_n)\}$ in $Y \times Y$ converging to (u, v) such that $(x_n, y_n) + t_n(u_n, v_n) \in \mathrm{Gr}(\Phi_A)$ for every natural number n. This and the definition of Φ_A imply that $(u, v) \in T_c(\mathrm{Gr}(\Phi_A), (a, a))$ if and only if $u = v$ and for every sequence $\{a_n\}$ in A converging to a and sequence $\{t_n\}$ in $(0, +\infty)$ decreasing to 0 there exists a sequence $\{v_n\}$ in Y converging to v such that $a_n + t_n v_n \in A$. Thus $T_c(\mathrm{Gr}(\Phi_A), (a, a)) = \{(v, v) : \ v \in T_c(A, a)\}$. Noting that

$$x^* \in D_c^*\Phi_A(a, a)(y^*) \Leftrightarrow \langle x^*, u\rangle - \langle y^*, v\rangle \le 0 \quad \forall (u, v) \in T_c(\mathrm{Gr}(\Phi_A), (a, a)),$$

it follows that

$$\begin{aligned} x^* \in D_c^*\Phi_A(a, a)(y^*) &\Leftrightarrow \langle x^*, v\rangle - \langle y^*, v\rangle \le 0 \quad \forall v \in T_c(A, a) \\ &\Leftrightarrow x^* - y^* \in N_c(A, a). \end{aligned}$$

This shows that (5.2) holds. □

Lemma 5.2. *Let Y be an Asplund space and $a \in A$. Then*

$$D_F^*\Phi_A(a, a)(y^*) = y^* + N_F(A, a) \ \text{ for all } y^* \in Y^*. \tag{5.3}$$

Proof. Let x be an arbitrary point in A. Note that

$$\begin{aligned} (x^*, y^*) \in \hat{N}(\mathrm{Gr}(\Phi_A), (x, x)) &\Leftrightarrow \limsup_{(u,v) \xrightarrow{\mathrm{Gr}(\Phi_A)} (x,x)} \frac{\langle x^*, u - x\rangle + \langle y^*, v - x\rangle}{\|u - x\| + \|v - x\|} \le 0 \\ &\Leftrightarrow \limsup_{v \xrightarrow{A} x} \frac{\langle x^*, v - x\rangle + \langle y^*, v - x\rangle}{\|v - x\|} \le 0 \\ &\Leftrightarrow x^* + y^* \in \hat{N}(A, x), \end{aligned}$$

that is,

$$\hat{N}(\mathrm{Gr}(\Phi_A), (x, x)) = \{(x^*, y^*) : \ x^* + y^* \in \hat{N}(A, x)\}.$$

Since Y is an Asplund space, it follows from (2.1) that (5.3) holds. □

We shall apply results in Section 3 to the multifunction Φ_A and thereby provide necessary conditions for a to be a Pareto efficient point of a set A.

Theorem 5.1. *Let Y be a Banach space and $a \in E(A, C)$. Then for any $\varepsilon > 0$ there exist $a_\varepsilon \in A \cap B(a, \varepsilon)$ and $a_\varepsilon^* \in C^+$ with $\|a_\varepsilon^*\| = 1$ such that*

$$-a_\varepsilon^* \in N_c(A, a_\varepsilon) + \varepsilon B_{Y^*}.$$

Proof. By (5.1) and Theorem 3.1 there exist $a_\varepsilon \in B(a, \varepsilon)$ and $c^* \in C^+$ with $\|c^*\| = 1$ such that

$$0 \in D_c^* \Phi_A(a_\varepsilon, a_\varepsilon)(c^* + \frac{\varepsilon}{2} B_{Y^*}) + \frac{\varepsilon}{2} B_{Y^*}.$$

It follows from (5.2) that $0 \in N_c(A, a_\varepsilon) + c^* + \varepsilon B_{Y^*}$. Thus, the theorem is established by setting $a_\varepsilon^* = c^*$. □

Using Theorems 3.2-3.4 instead of Theorem 3.1 in the above proof, we can show similarly the following results.

Theorem 5.2. *Let Y be a Banach space and $a \in \mathrm{E}(A, C)$. Suppose that one of the following conditions is satisfied.*

(a) C has a nonempty interior.
(b) C is dually compact and $N_c(A, \cdot)$ is closed at a.
(c) There exists a vector in Y hypertangent to A at $a \in A$. Then there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that $-c^* \in N_c(A, a)$.

If Y is assumed to be an Asplund space, then the preceding two theorems can be strengthened to following theorems 5.3 and 5.4 where $\hat{N}(A, \cdot)$ or $N_F(A, \cdot)$ is used in place of $N_c(A, \cdot)$. The proofs are similar as before but one applies (5.3) and results in Section 4 in place of (5.2) and results in Section 3.

Theorem 5.3. *Let Y be an Asplund space and $a \in \mathrm{E}(A, C)$. Then for any $\varepsilon > 0$ there exist $a_\varepsilon \in A \cap B(a, \varepsilon)$ and $a_\varepsilon^* \in C^+$ with $\|a_\varepsilon^*\| = 1$ such that*

$$-a_\varepsilon^* \in \hat{N}(A, a_\varepsilon) + \varepsilon B_{Y^*}.$$

Recall [17, 18] that A is said to be sequentially normally compact at $a \in A$ if any sequence (x_n, x_n^*) satisfying $x_n^* \in N_F(A, x_n)$, $x_n \to a$ and $x_n^* \xrightarrow{w^*} 0$ contains a subsequence with $\|x_{n_k}^*\| \to 0$. It is easy to verify that Φ_A is partially sequentially normally compact at (a, a) with respect to Y if A is sequentially normally compact at a.

Theorem 5.4. *Let Y be an Asplund space and $a \in \mathrm{E}(A, C)$. Suppose that one of the following conditions is satisfied.*

(a) C be dually compact.
(b) A is sequentially normally compact at a. Then there exists $c^* \in C^+$ with $\|c^*\| = 1$ such that $-c^* \in N_c(A, a)$.

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