# ON THE STRUCTURES OF GENERATING ITERATED FUNCTION SYSTEMS OF CANTOR SETS

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ABSTRACT. A generating IFS of a Cantor set F is an IFS whose attractor is F. For a given Cantor set such as the middle-3rd Cantor set we consider the set of its generating IFSs. We examine the existence of a minimal generating IFS, i.e. every other generating IFS of F is an iterating of that IFS. We also study the structures of the semi-group of homogeneous generating IFSs of a Cantor set F in  $\mathbb{R}$  under the open set condition (OSC). If dim<sub>H</sub> F < 1 we prove that all generating IFSs of the set must have logarithmically commensurable contraction factors. From this Logarithmic Commensurability Theorem we derive a structure theorem for the semi-group of generating IFSs of F under the OSC. We also examine the impact of geometry on the structures of the semi-groups. Several examples will be given to illustrate the difficulty of the problem we study.

### 1. INTRODUCTION

It is well known that the standard middle-third Cantor set C is the attractor of the iterated function system (IFS)  $\{\phi_0, \phi_1\}$  where

(1.1) 
$$\phi_0(x) = \frac{1}{3}x, \qquad \phi_1(x) = \frac{1}{3}x + \frac{2}{3}$$

A natural question is: Is it possible to express C as the attractor of another IFS?

Surprisingly, the general question whether the attractor of an IFS can be expressed as the attractor of another IFS, which seems a rather fundamental question in fractal geometry, has hardly been studied, even for some of the best known Cantor sets such as the middle-third Cantor set.

A closer look at this question reveals that it is not as straightforward as it may appear. It is easy to see that for any given IFS  $\{\phi_j\}_{j=1}^N$  one can always iterate it to obtain another

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IFS with identical attractor. For example, the middle-third Cantor set C satisfies

$$C = \phi_0(C) \cup \phi_1(C)$$
  
=  $\phi_0 \circ \phi_0(C) \cup \phi_0 \circ \phi_1(C) \cup \phi_1(C)$   
=  $\phi_0 \circ \phi_0(C) \cup \phi_0 \circ \phi_1(C) \cup \phi_1 \circ \phi_0(C) \cup \phi_1 \circ \phi_1(C)$ 

Hence C is also the attractor of the IFS  $\{\phi_0 \circ \phi_0, \phi_0 \circ \phi_1, \phi_1\}$  and the IFS  $\{\phi_0 \circ \phi_0, \phi_0 \circ \phi_1, \phi_1 \circ \phi_0, \phi_1 \circ \phi_1\}$ , as well as infinitely many other iterations of the original IFS  $\{\phi_0, \phi_1\}$ . The complexity doesn't just stop here. Since C is centrally symmetric, C = -C + 1, we also have

$$C = \left(-\frac{1}{3}C + \frac{1}{3}\right) \cup \left(-\frac{1}{3}C + 1\right).$$
  
Thus C is also the attractor of the IFS  $\{-\frac{1}{3}x + \frac{1}{3}, -\frac{1}{3}x + 1\}$ , or even  $\{-\frac{1}{3}x + \frac{1}{3}, \frac{1}{3}x + \frac{2}{3}\}$ .

**Definition 1.1.** Let  $\Phi = \{\phi_i\}_{i=1}^N$  and  $\Psi = \{\psi_j\}_{j=1}^M$  be two IFSs. We say that  $\Psi$  is *derived* from  $\Phi$  if for each  $1 \leq j \leq M$ ,  $\psi_j = \phi_{i_1} \circ \cdots \circ \phi_{i_k}$  for some  $1 \leq i_1, \ldots, i_k \leq N$ . We say that  $\Psi$  is an *iteration* of  $\Phi$  if  $\Psi$  is derived from  $\Phi$  and  $F_{\Phi} = F_{\Psi}$ , where  $F_{\Phi}$  and  $F_{\Psi}$  denote the attractors of  $\Phi$  and  $\Psi$ , respectively.

We point out that the multiplicities in an IFS are not counted in our study. An IFS  $\Phi$ , after an iteration, may contain redundant maps. For example, let  $\Phi = \{\phi_0, \phi_1, \phi_2\}$  where  $\phi_i = \frac{1}{2}(x+i)$ . Then in  $\{\phi_i \circ \phi_j : 0 \le i, j \le 2\}$  both  $\frac{1}{4}(x+2)$  and  $\frac{1}{4}(x+4)$  appear twice. After removing redundancies we have  $\Psi = \{\frac{1}{4}(x+j) : 0 \le j \le 6\}$  as an iteration of  $\Phi$ .

**Definition 1.2.** Let F be a compact set in  $\mathbb{R}^d$ . A generating IFS of F is an IFS  $\Phi$  whose attractor is F. A generating IFS family of F is a set  $\mathcal{I}$  of generating IFSs of F. A generating IFS family  $\mathcal{I}$  of F is said to have a minimal element  $\Phi_0 \in \mathcal{I}$  if every  $\Psi \in \mathcal{I}$  is an iteration of  $\Phi_0$ .

The objective of this paper is to study the existence of a minimal IFS in a generating IFS family of a self-similar set  $F \subset \mathbb{R}$ . We have already pointed out the complexity of this problem even for the middle-third Cantor set. Naturally, one cannot expect the existence of a minimal IFS in a generating IFS family  $\mathcal{I}$  of a set F to be the general rule — not without first imposing restrictions on  $\mathcal{I}$  and F. But what are these restrictions? A basic restriction is the *open set condition* (OSC). Without the OSC either the existence of a minimal IFS is hopeless, or the problem appears rather intractable. But even with the OSC

a compact set may have generating IFSs that superficially seem to bear little relation to one another. One such example is the unit interval F = [0, 1]. For each integer  $N \ge 2$  the IFS  $\Phi_N = \{\frac{1}{N}(x+j): 0 \le j < N\}$  is a generating IFS for F satisfying the OSC, and for  $N_2 > N_1$  that is not a power of  $N_1$ ,  $\Phi_{N_2}$  is not an iteration of  $\Phi_{N_1}$ . It is evident that other restrictions will be needed. We study this issue in this paper.

While the questions we study in the paper appear to be rather fundamental questions of fractal geometry in themselves, our study is also motivated by several questions in related areas. One of the well known questions in tiling is whether there exists a 2-reptile that is also a 3-reptile in the plane ([5]). A compact set T with  $T = \overline{T^o}$  is called a *k*-reptile if there exists a measure disjoint partition  $T = \bigcup_{j=1}^k T_j$  of T such that each  $T_j$  is similar to T and all  $T_j$  are congruent. Suppose that  $T_j = \phi_j(T)$  for some similarity  $\phi_j$ . Then T is the attractor of the IFS  $\{\phi_j\}_{j=1}^k$ . So this question, or more generally whether an *m*-reptile can also be an *n*-reptile, is a special case of the questions we study here.

Another motivation comes from the application of fractal geometry to image compression, see Barnsley [2] or Lu [12]. The basic premise of fractal image compression is that a digital image can be partitioned into pieces in which each piece is the attractor of an affine IFS. So finding a generating IFS of a given set plays the central role in this application. Naturally, better compressions are achieved by choosing a minimal generating IFS for each piece if possible, see also Deliu, Geronimo and Shonkwiler [6].

Although not directly related, there are two other questions that have also motivated our study. One is a question raised by Mattila: Is it true that any self-similar subset F of the middle-third Cantor set C is trivial, in the sense that F has a generating IFS that is derived from the generating IFS { $\phi_0, \phi_1$ } of C given in (1.1)? As we shall see in §5, this is not true. There are indeed self-similar subsets F of C, of which no generating IFS is derived from { $\phi_0, \phi_1$ }. The other question concerns the symmetry of a self-similar set such as the Sierpinski Gasket, see e.g. [4] and [18]. We have already seen from the middle-third Cantor set that symmetry complicates the study of existence of minimal IFSs. How the two questions relate is perhaps a problem worth further exploiting.

For any IFS  $\Phi$  we shall use  $F_{\Phi}$  to denote its attractor. We call an IFS  $\Phi = \{\rho_j x + a_j\}_{j=1}^N$ homogeneous if all contraction factors  $\rho_j$  are identical. In this case we use  $\rho_{\Phi}$  to denote the homogeneous contraction factor. We call  $\Phi$  positive if all  $\rho_j > 0$ . A fundamental result concerning the structures of generating IFSs of a self-similar set is the Logarithmic Commensurability Theorem stated below. It is the foundation of many of our results in this paper.

**Theorem 1.1 (The Logarithmic Commensurability Theorem).** Let F be the attractor of a homogeneous IFS  $\Phi = {\phi_i}_{i=1}^N$  in  $\mathbb{R}$  satisfying the OSC.

- (i) Suppose that  $\dim_H F = s < 1$ . Let  $\psi(x) = \lambda x + d$  such that  $\psi(F) \subseteq F$ . Then  $\log |\lambda| / \log |\rho_{\Phi}| \in \mathbb{Q}$ .
- (ii) Suppose that  $\dim_H F = 1$  and F is not a finite union of intervals. Let  $\psi(x) = \lambda x + d$ such that  $\psi(F) \subseteq F$  and  $\min(F) \in \psi(F)$ . Then  $\log |\lambda| / \log |\rho_{\Phi}| \in \mathbb{Q}$ .

An immediate corollary of the above theorem is:

**Corollary 1.2.** Let F be the attractor of a homogeneous IFS  $\Phi = \{\phi_i\}_{i=1}^N$  satisfying the OSC. Suppose that  $\Psi = \{\psi_j(x) = \lambda_j x + b_j\}_{j=1}^M$  is another generating IFS of F.

- (i) If  $\dim_H F = s < 1$ , then  $\log |\lambda_j| / \log |\rho_{\Phi}| \in \mathbb{Q}$  for all  $1 \le j \le M$ .
- (ii) If  $\dim_H F = 1$  and F is not a finite union of intervals, and if  $\Psi$  is homogeneous, then  $\log |\rho_{\Psi}| / \log |\rho_{\Phi}| \in \mathbb{Q}$ .

Note that the set of all homogeneous generating IFSs of a self-similar set F forms a semigroup. Let  $\Phi = \{\phi_i\}_{i=1}^N$  and  $\Psi = \{\psi_j\}_{j=1}^M$  be two generating IFSs of F. We may define  $\Phi \circ \Psi$  by  $\Phi \circ \Psi = \{\phi_i \circ \psi_j : 1 \le i \le N, 1 \le j \le M\}$ . Then clearly  $\Phi \circ \Psi$  is also a generating IFS of F.

**Definition 1.3.** Let F be any compact set in  $\mathbb{R}$ . We shall use  $\mathcal{I}_F$  to denote the set of all homogeneous generating IFSs of F satisfying the OSC, augmented by the "identity"  $Id = \{id(x) := x\}$ . We shall use  $\mathcal{I}_F^+$  to denote the set of all positive homogeneous generating IFSs of F satisfying the OSC, augmented by the identity Id.

Clearly both  $\mathcal{I}_F$  and  $\mathcal{I}_F^+$ , equipped with the composition as product, are semi-groups. If F is not the attractor of a homogeneous IFS with OSC then  $\mathcal{I}_F$  is trivial. The Logarithmic Commensurability Theorem leads to the following structure theorem for  $\mathcal{I}_F$  and  $\mathcal{I}_F^+$ :

**Theorem 1.3.** Let F be a compact set in  $\mathbb{R}$  that is not a finite union of intervals. Then  $\mathcal{I}_F$  is Abelian. Let  $\Phi = \{\phi_i\}_{i=1}^N \in \mathcal{I}_F, N > 1$ .

- (i) Both  $\mathcal{I}_F$  and  $\mathcal{I}_F^+$  are finitely generated semi-groups.
- (ii) Suppose that N is not a power of another integer. If ρ<sub>Φ</sub> > 0 then Φ is the minimal element for I<sup>+</sup><sub>F</sub>, namely I<sup>+</sup><sub>F</sub> =< Φ >:= {Φ<sup>m</sup> : m ≥ 0}. If ρ<sub>Φ</sub> < 0 then either I<sup>+</sup><sub>F</sub> has a minimal element, or I<sup>+</sup><sub>F</sub> =< Φ<sup>2</sup>, Ψ > for some Ψ with ρ<sub>Ψ</sub> = ρ<sup>q</sup><sub>Φ</sub> where q ∈ N is odd and Ψ<sup>2</sup> = Φ<sup>2q</sup>.
- (iii) Suppose that N is not a power of another integer. Then either  $\mathcal{I}_F = \langle \Phi \rangle$  or  $\mathcal{I}_F = \langle \Phi, \Psi \rangle$  for some  $\Psi$  with  $\rho_{\Psi} = -\rho_{\Phi}^q$  where  $q \in \mathbb{N}$  and  $\Psi^2 = \Phi^{2q}$ .

Due to the technical nature of the proof of the Logarithmic Commensurability Theorem we shall postpone it until §4. Theorem 1.3 establishes the structures of  $\mathcal{I}_F$  and  $\mathcal{I}_F^+$  purely on algebraic grounds. However, the structures of the semi-groups are also dictated by the geometric structures of F. We shall exploit the impact of geometry on the structures of the semi-groups in §2. In §3 we study the existence of minimal IFSs for IFS families with non-homogeneous contraction factors. Geometry plays a considerably bigger role in this setting. In §4 we prove the Logarithmic Commensurability Theorem, along with other related results. Finally in §5 we present various counterexamples, including counterexamples to Mattila's question.

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### 2. STRUCTURES OF THE SEMI-GROUPS AND THE CONVEX OPEN SET CONDITION

In this section we prove Theorem 1.3, and examine the impact of geometry to the structures of the semi-groups  $\mathcal{I}_F$  and  $\mathcal{I}_F^+$ . Although the proof of Theorem 1.1 and Corollary 1.2 will be given later in §4, their proofs do not depend on the results in this section. Hence we shall assume their validity in this section and use them to prove our results.

**Definition 2.1.** Let  $\Phi = \{\phi_j\}_{j=1}^N$  be an IFS in  $\mathbb{R}$ . We say  $\Phi$  satisfies the separation condition (SC) if  $\phi_i(F_{\Phi}) \cap \phi_j(F_{\Phi}) = \emptyset$  for all  $i \neq j$ . We say  $\Phi$  satisfies the convex open set condition (COSC) if  $\Phi$  satisfies the OSC with a convex open set.

Among the results we prove in this section, the following is another main theorem in this paper:

**Theorem 2.1.** Let  $F \subset \mathbb{R}$  be a compact set that is not a finite union of intervals such that F is the attractor of a homogeneous IFS satisfying the COSC. Let  $\Phi$  be any generating IFS of F with the OSC. Then  $\Phi$  also satisfies the COSC. Furthermore we have:

- (i) The semi-group  $\mathcal{I}_F^+$  has a minimal element  $\Phi_0$ , namely  $\mathcal{I}_F^+ = \langle \Phi_0 \rangle$ .
- (ii) Suppose that F is not symmetric. Then  $\mathcal{I}_F$  has a minimal element  $\Phi_0$ ,  $\mathcal{I}_F = \langle \Phi_0 \rangle$ .
- (iii) Suppose that F is symmetric. Then there exist  $\Phi_+$  and  $\Phi_-$  in  $\mathcal{I}_F$  with  $\rho_{\Phi_+} = -\rho_{\Phi_-} > 0$  such that every  $\Psi \in \mathcal{I}_F$  can be expressed as  $\Psi = \Phi^m_+$  if  $\rho_{\Psi} > 0$  and  $\Psi = \Phi^m_+ \circ \Phi_-$  if  $\rho_{\Psi} < 0$  for some  $m \in \mathbb{N}$ .

We shall first prove several results leading up to our main theorem.

**Lemma 2.2.** Let  $\Phi = \{\phi_j\}$  be an IFS in  $\mathbb{R}$ . Then  $\Phi$  satisfies the COSC if and only if for all  $i \neq j$  we have  $\phi_i(x) \leq \phi_j(y)$  for all  $x, y \in F_{\Phi}$  or  $\phi_i(x) \geq \phi_j(y)$  for all  $x, y \in F_{\Phi}$ .

**Proof.** Suppose that  $\Phi$  satisfies the COSC. Then the convex open set for the OSC must be an interval U = (a, b). Since  $\phi_i(U) \cap \phi_j(U) = \emptyset$  for all  $i \neq j$ , and noting that  $F_{\Phi} \subseteq \overline{U}$ , we immediately know that  $\phi_i(F)$  must lie entirely on one side of  $\phi_j(F)$ .

Conversely, suppose that  $\phi_i(F)$  lies entirely on one side of  $\phi_j(F)$ ,  $i \neq j$ . Let U be the interior of the convex hull of F, which is an interval. Then  $\phi_i(U) \cap \phi_j(U) = \emptyset$ , and clearly  $\phi_i(F) \subset F$  implies that  $\phi_i(U) \subset U$ . Hence  $\Phi$  satisfies the COSC.

**Proposition 2.3.** Let  $\Phi$  and  $\Psi$  be two homogeneous IFSs in  $\mathbb{R}$  satisfying the OSC. If  $\rho_{\Phi} = \rho_{\Psi}$  and  $F_{\Phi} = F_{\Psi}$ , then  $\Phi = \Psi$ .

**Proof.** Let  $\Phi = \{\phi_i(x) := \rho x + a_i\}_{i=1}^N$  and  $\Psi = \{\psi_j(x) := \rho x + b_j\}_{j=1}^M$ . Denote  $F = F_{\Phi} = F_{\Psi}$ . To see N = M observe that by the OSC of  $\Phi$  and  $\Psi$  we have  $\dim_H F = \frac{\log N}{-\log |\rho|} = \frac{\log M}{-\log |\rho|}$ . It follows that N = M.

Let  $\nu_F$  be the normalized s-dimensional Hausdorff measure restricted to F, where  $s = \dim_H F$ , i.e.  $\nu_F = \frac{1}{\mathcal{H}^s(F)} \mathcal{H}^s$ . We assert that  $\nu_F$  is the self-similar measure defined by  $\Phi$  with equal weights, i.e.

$$\nu_F = \frac{1}{N} \sum_{j=1}^N \nu_F \circ \phi_j^{-1}.$$

For any  $E \subset F$  let  $E_i = \phi_i(F) \cap E$ . Note that  $\phi_j^{-1}(E_i) \cap F \subseteq \phi_j^{-1}(\phi_i(F) \cap \phi_j(F))$ . The OSC now implies  $\nu_F(\phi_j^{-1}(E_i)) = 0$ , as well as  $\nu_F(E_i \cap E_j) = 0$  for any  $i \neq j$ . Therefore

$$\nu_F \circ \phi_j^{-1}(E) = \nu_F(\phi_j^{-1}(E_j)) = \rho^{-s} \nu_F(E_j) = N \nu_F(E_j).$$

It follows that

$$\frac{1}{N}\sum_{j=1}^{N}\nu_F \circ \phi_j^{-1}(E) = \sum_{j=1}^{N}\nu_F(E_j) = \nu_F(E).$$

This proves the assertion. Similarly,  $\nu_F$  is also the self-similar measure defined by the IFS  $\Psi$  with equal weights.

Now taking the Fourier transform of  $\nu_F$  and applying the self-similarity yield

$$\widehat{\nu}_F(\xi) = A(\rho\xi)\widehat{\nu}_F(\rho\xi) = B(\rho\xi)\widehat{\nu}_F(\rho\xi)$$

where  $A(\xi) := \frac{1}{N} \sum_{j=1}^{N} e^{2\pi i a_j \xi}$  and  $B(\xi) := \frac{1}{M} \sum_{j=1}^{M} e^{2\pi i b_j \xi}$ . Therefore  $A(\xi) = B(\xi)$  and  $\{a_j\} = \{b_j\}$ , proving the lemma.

**Proposition 2.4.** Let  $\Phi$  and  $\Psi$  be two homogeneous IFSs satisfying the OSC. Then  $F_{\Phi} = F_{\Psi}$  if and only if  $\Phi \circ \Psi = \Psi \circ \Phi$ .

**Proof.** Suppose that  $F_{\Phi} = F_{\Psi}$  then both  $\Phi \circ \Psi$  and  $\Psi \circ \Phi$  are generating IFSs of F with identical contraction factors, and both satisfy the OSC. Hence  $\Phi \circ \Psi = \Psi \circ \Phi$ .

Conversely, if  $\Phi \circ \Psi = \Psi \circ \Phi$  then  $\Phi \circ \Psi^m = \Psi^m \circ \Phi$  for any  $m \in \mathbb{N}$ . Therefore  $\Phi \circ \Psi^m(F_\Psi) = \Psi^m \circ \Phi(F_\Psi)$ . But  $\Psi^m(E) \longrightarrow F_\Psi$  as  $m \longrightarrow \infty$  in the Hausdorff metric for any compact set E. Taking limit we obtain  $\Phi(F_\Psi) = F_\Psi$ . Therefore  $F_\Phi = F_\Psi$ .

**Lemma 2.5.** Let  $\Phi$  and  $\Psi$  be two homogeneous IFSs such that  $\rho_{\Phi} = -\rho_{\Psi}$  and  $F_{\Phi} = F_{\Psi}$ . Assume that  $\Phi$  satisfies the COSC. Then  $F_{\Phi}$  must be symmetric.

**Proof.** Let  $\Phi = \{\phi_i(x) := \rho x + a_i\}_{i=1}^N$  and  $\Psi = \{\psi_j(x) := -\rho x + b_j\}_{j=1}^M$ .  $\Psi$  also satisfies the COSC by Theorem 2.1. (See the proof below; the proof of that part does not depend on this lemma.) Without loss generality we assume that  $\rho > 0$  and  $a_1 < a_2 < \cdots < a_N$ ,  $b_1 < b_2 < \cdots < b_M$ . Denote  $\mathcal{A} = \{a_i\}$  and  $\mathcal{B} = \{b_j\}$ . The OSC for  $\Phi$  and  $\Psi$  now implies  $\Phi^2 = \Psi^2$ . Observe that

$$\Phi^{2} = \{\rho^{2}x + a_{i} + \rho a_{j}\}_{i,j=1}^{N}, \qquad \Psi^{2} = \{\rho^{2}x + b_{i} - \rho b_{j}\}_{i,j=1}^{M}$$

It follows from the COSC for  $\Phi^2$  that the lexicographical order for  $\{a_i + \rho a_j\}_{i,j=1}^N$  also yields a strictly increasing order for the set. Similarly, the lexicographical order for  $\{b_i - \rho b_{M+1-j}\}_{i,j=1}^M$  also yields a strictly increasing order for the set. Therefore M = N and  $a_i + \rho a_j = b_i - \rho b_{N+1-j}$  for all i, j. Fix j = 1 yields  $a_i = b_i + c$  for some constant c. Fix i = 1 yields  $a_j = -b_{N+1-j} + c'$  for some constant c'. Thus  $a_j = a_{N+1-j} + c''$  for some constant c''. Hence  $\mathcal{A}$  is symmetric, which implies that  $F_{\Phi}$  is symmetric.

**Proof of Theorem 2.1.** We prove that  $\Psi$  satisfies the COSC if  $\Phi$  does. By Corollary 1.2 there exist integers m, n such that  $\rho_{\Phi}^n = \rho_{\Psi}^m$ . It follows from Lemma 2.3 that  $\Phi^n = \Psi^m$ . Assume that  $\Psi$  does not satisfy the COSC. Then there exist  $\psi_i, \psi_j \in \Psi$  so that  $\psi_i(x) < \psi_j(y)$ and  $\psi_i(z) > \psi_j(w)$  for some  $x, y, z, w \in F$ . The same inequalities will hold if we replace  $\psi_i$ ,  $\psi_j$  by  $\psi_1^{m-1} \circ \psi_i$  and  $\psi_1^{m-1} \circ \psi_j$ , respectively. But this is impossible because both  $\psi_1^{m-1} \circ \psi_i$ and  $\psi_1^{m-1} \circ \psi_i$  are in  $\Psi^m$ , and hence in  $\Phi^n$ , which satisfies the COSC.

To prove the rest of the theorem we first prove the following claim.

**Claim.** Let  $\Phi, \Psi$  be any two elements in  $\mathcal{I}_F$  with  $|\rho_{\Phi}| > |\rho_{\Psi}|$ . Then there exists a  $\Gamma \in \mathcal{I}_F$  such that  $\Psi = \Phi \circ \Gamma$ , where  $\Phi \circ \Gamma := \{\phi \circ \gamma : \phi \in \Phi, \gamma \in \Gamma\}$ .

Proof of Claim: Let  $\Phi = \{\phi_i(x)\}_{i=1}^N$  and  $\Psi = \{\psi_j(x)\}_{j=1}^M$ . Since both  $\Phi$  and  $\Psi$  satisfy the COSC, we may without loss of generality assume that  $\phi_1(F) \leq \cdots \leq \phi_M(F)$  and  $\psi_1(F) \leq \cdots \leq \psi_N(F)$ , where  $X \leq Y$  for two sets X and Y means  $x \leq y$  for all  $x \in X$  and  $y \in Y$ . Set  $a = \min F$ ,  $b = \max F$  and  $F_0 = [a, b]$ . Clearly each  $\phi_i(F_0)$  (resp.  $\Psi_j(F_0)$ ) is a sub-interval of  $F_0$ , with end points  $\phi_i(a)$  and  $\phi_i(b)$  (resp.  $\psi_i(a)$  and  $\psi_i(b)$ ). The COSC for  $\Phi$  and  $\Psi$  now imply that  $\phi_1(F_0) \leq \cdots \leq \phi_M(F_0)$  and  $\psi_1(F_0) \leq \cdots \leq \psi_N(F_0)$ .

It follows from Corollary 1.2 that  $\frac{\log |\rho_{\Phi}|}{\log |\rho_{\Psi}|} = \frac{n}{m}$  for some positive integers m and n with gcd(m,n) = 1. Thus  $N^m = M^n$ , or  $N = M^{\frac{n}{m}}$ . This forces  $K = M^{\frac{1}{m}}$  to be an integer, for otherwise the co-primeness of n, m makes  $N = M^{\frac{n}{m}}$  an irrational number. Therefore  $M = K^m$  and  $N = K^n$ . In particular,  $\frac{M}{N} = L \in \mathbb{Z}$ .

Now  $\Phi^q = \Psi^r$  by Lemma 2.3, where q = 2m and r = 2n. For each  $\mathbf{i} = i_1 i_2 \cdots i_q \in \{1, \ldots, N\}^q$  denote  $\phi_{\mathbf{i}} := \phi_{i_1} \circ \cdots \circ \phi_{i_q}$ , and similarly define  $\psi_{\mathbf{j}}$  for  $\mathbf{j} \in \{1, \ldots, M\}^r$ . Then  $\Phi^q = \{\phi_{\mathbf{i}} : \mathbf{i} \in \{1, \ldots, N\}^q\}$  and  $\Psi^q = \{\psi_{\mathbf{i}} : \mathbf{i} \in \{1, \ldots, M\}^r\}$ . It follows from  $\phi_1(F_0) \leq \cdots \leq \phi_M(F_0)$  and the COSC that  $\phi_{\mathbf{i}_1}(F_0) \leq \phi_{\mathbf{i}_2}(F_0)$  if and only if  $\mathbf{i}_1 < \mathbf{i}_2$  in the lexicographical order for  $\{1, \ldots, N\}^q$ . Similarly,  $\psi_{\mathbf{j}_1}(F_0) \leq \psi_{\mathbf{j}_2}(F_0)$  if and only if  $\mathbf{j}_1 < \mathbf{j}_2$  in the lexicographical order for  $\{1, \ldots, M\}^r$ . Therefore the sequence of maps  $(\phi_{\mathbf{i}} : \mathbf{i} \in \{1, \ldots, N\}^q)$  in the lexicographical order equals the sequence of maps  $(\psi_{\mathbf{i}} : \mathbf{i} \in \{1, \ldots, M\}^q)$  in the lexicographical order. The first  $N^{q-1}$  maps in  $(\phi_{\mathbf{i}} : \mathbf{i} \in \{1, \ldots, N\}^q)$  are  $\mathcal{J}_1 = \{\phi_{1\mathbf{i}'} : \mathbf{i}' \in \{1, \ldots, N\}^{q-1}\}$ , while the first  $N^{q-1}$  maps in  $(\psi_{\mathbf{j}} : \mathbf{j} \in \{1, \ldots, M\}^r)$  are  $\mathcal{J}_2 = \{\psi_{j_1\mathbf{j}'} : 1 \leq j_1 \leq L, \mathbf{j}' \in \{1, \ldots, M\}^{r-1}\}$ . Therefore  $\mathcal{J}_1 = \mathcal{J}_2$ . Note that

$$\bigcup_{\varphi \in \mathcal{J}_1} \varphi(F) = \phi_1(F), \qquad \bigcup_{\varphi \in \mathcal{J}_2} \varphi(F) = \bigcup_{j=1}^L \psi_j(F).$$

It follows that  $F = \bigcup_{j=1}^{L} \phi_1^{-1} \circ \psi_j(F)$ , so  $\Gamma_1 := \{\phi_1^{-1} \circ \psi_j\}_{j=1}^{L}$  is a generating IFS for F. It clearly satisfies the COSC.

We can continue the same argument by counting the next  $N^{q-1}$  elements in the two sequences. This yields  $F = \bigcup_{j=L+1}^{2L} \phi_2^{-1} \circ \psi_j(F)$ , so  $\Gamma_2 := \{\phi_2^{-1} \circ \psi_j\}_{j=L+1}^{2L}$  is a generating IFS for F. Continue to the end yields  $\Gamma_1, \ldots, \Gamma_N$  in  $\mathcal{I}_F$ , with the property that

(2.1) 
$$\{\psi_j : (k-1)L + 1 \le j \le kL\} = \{\phi_k \circ \varphi : \varphi \in \Gamma_k\}.$$

But all  $\Gamma_k$  are equal because they have the same contraction factor. It follows from (2.1) that  $\Psi = \Phi \circ \Gamma$ , with  $\Gamma := \Gamma_k$ . This proves the Claim.

To prove part (i) of the theorem, let  $\Phi_0 \in \mathcal{I}_F^+$  have the largest contraction factor. Such a  $\Phi_0$  exists because for any  $\Phi \in \mathcal{I}_F^+$  we must have  $\rho_{\Phi} = N^{-\dim_H(F)}$  for some positive integer N. Now any  $\Phi \neq \Phi_0$  in  $\mathcal{I}_F^+$  we have  $\rho_{\Phi} < \rho_{\Phi_0}$ . By the Claim  $\Phi = \Phi_0 \circ \Gamma_1$  for some  $\Gamma_1 \in \mathcal{I}_F^+$ . If  $\Gamma_1 = \Phi_0$  then  $\Phi = \Phi_0^2$ , and we finish the proof. If not then  $\rho_{\Gamma} < \rho_{\Phi_0}$ , yielding  $\Gamma_1 = \Phi_0 \circ \Gamma_2$  for some  $\Gamma_2 \in \mathcal{I}_F^+$ . Apply the Claim recursively, and the process will eventually terminate. Hence  $\Phi = \Phi_0^k$  for some k. The proof of part (i) is now compelte.

To prove part (ii) of the theorem, if  $\mathcal{I}_F = \mathcal{I}_F^+$  then there is nothing we need to prove. Assume that  $\mathcal{I}_F \neq \mathcal{I}_F^+$ . Let  $\mathcal{I}_F^- \subset \mathcal{I}_F$  consisting of all homogeneous IFSs with negative contraction factors, and  $\Phi_- \in \mathcal{I}_F^-$  have the largest contraction factor in absolute value. Let  $\Phi_+ \in \mathcal{I}_F^+$  have the largest contraction factor in  $\mathcal{I}_F^+$ . If  $|\rho_{\Phi_-}| = \rho_{\Phi_+}$  then F is symmetric by Lemma 2.5, a contradiction. So  $|\rho_{\Phi_-}| \neq \rho_{\Phi_+}$ . Note that  $\Phi_-^2 = \Phi_+^m$  for some m by part (i). Thus m = 1 or m > 2. If m > 2 then  $\rho_{\Phi_+} > |\rho_{\Phi_-}|$ . Following the Claim we have  $\Phi_- = \Phi_+ \circ \Gamma$  for some  $\Gamma \in \mathcal{I}_F$ . But  $\rho_{\Gamma} < 0$  and  $|\rho_{\Gamma}| > |\rho_{\Phi_-}|$ . This is a contradiction. Therefore m = 1 and  $\Phi_-^2 = \Phi_+$ . Part (ii) of the theorem follows from part (i) and the Claim. Finally we prove (iii). If F is symmetric, then for any IFS  $\Psi \in \mathcal{I}_F$  there is another  $\Psi' \in \mathcal{I}_F$  such that  $\rho_{\Psi} = -\rho_{\Psi'}$  because F = -F + c for some c. Let  $\Phi_+$  and  $\Phi_-$  be the elements in  $\mathcal{I}_F$  whose contraction factors have the largest absolute values,  $\rho_{\Phi_+} = -\rho_{\Phi_-} > 0$ . Lemma 2.3 and the same argument to prove part (i) now easily apply to prove that for any  $\Psi \in \mathcal{I}_F$ ,  $\Psi = \Phi^m_+$  if  $\rho_{\Psi} > 0$  and  $\Psi = \Phi^m_+ \circ \Phi_-$  if  $\rho_{\Psi} < 0$  for some  $m \in \mathbb{N}$ .

The COSC in Theorem 2.1 cannot be replaced by the condition SC. We give a counterexample in §5.

**Proof of Theorem 1.3.** Let k be the largest integer such that  $N = L^k$  for some  $L \in \mathbb{N}$ . Suppose that  $\Psi = \{\psi_j\}_{j=1}^M \in \mathcal{I}_F$  and  $\Psi \neq Id$ . Then the dimension formula  $N^{-s} = |\rho_{\Phi}|$  and  $M^{-s} = |\rho_{\Psi}|$  where  $s = \dim_H F$  implies that  $\log M / \log N \in \mathbb{Q}$ . It follows that  $M = L^m$  and  $\rho_{\Psi} = \pm |\rho_{\Phi}|^{\frac{m}{k}}$  for some  $m \in \mathbb{N}$ .

We first prove (ii). By assumption N = L. Let  $\Psi = \{\psi_j\}_{j=1}^M \in \mathcal{I}_F^+$ . Then  $M = N^m$  for some m, which implies that  $\rho_{\Psi} = |\rho_{\Phi}|^m$ . If  $\rho_{\Phi} > 0$  then  $\Psi = \Phi^m$  via Lemma 2.3 because they have the same contraction factor. Thus  $\mathcal{I}_F^+ = \langle \Phi \rangle$ . Suppose that  $\rho_{\Phi} < 0$ . We have two cases: Either every  $\Psi \in \mathcal{I}_F^+$  has  $\rho_{\Psi} = |\rho_{\Phi}|^{2m'}$  for some m', or there exists a  $\Psi \in \mathcal{I}_F^+$ with  $\rho_{\Psi} = |\rho_{\Phi}|^m$  for some odd m. In the first case every  $\Psi \in \mathcal{I}_F^+$  has  $\Psi = (\Phi^2)^m$  again by Lemma 2.3. Hence  $\mathcal{I}_F^+ = \langle \Phi^2 \rangle$ . In the second case, let q be the smallest odd integer such that  $\rho_{\Psi_0} = |\rho_{\Phi}|^q$  for some  $\Psi_0 \in \mathcal{I}_F^+$ . For any  $\Psi \in \mathcal{I}_F^+$  we have  $\rho_{\Psi} = |\rho_{\Phi}|^m$ . If m = 2m' then  $\Psi = (\Phi^2)^{m'}$ . If m is odd then  $m \geq q$  and m - q = 2m'. Thus  $\rho_{\Psi} = \rho_{\Phi^{2m'}\circ\Psi_0}$ , and hence  $\Psi = \Phi^{2m'} \circ \Psi_0$ . It follows that  $\mathcal{I}_F^+ = \langle \Phi^2, \Psi_0 \rangle$  with  $\Psi_0^2 = (\Phi^2)^q$ . This proves (ii).

We next prove (iii), which is rather similar to (ii). Again, any  $\Psi \in \mathcal{I}_F$  must have  $\rho_{\Psi} = \pm |\rho_{\Phi}|^m$  for some m. If  $\mathcal{I}_F = \langle \Phi \rangle$  we are done. Otherwise there exists a  $\Psi_0 \in \mathcal{I}_F$  such that  $\Psi_0 \notin \langle \Phi \rangle$  and it has the largest contraction factor in absolute value. Since  $\rho_{\Psi_0} = \pm |\rho_{\Phi}|^q$  for some q, and  $\Psi_0 \neq \Phi^q$ , we must have  $\rho_{\Psi_0} = -\rho_{\Phi}^q$ . We show that  $\mathcal{I}_F = \langle \Phi^2, \Psi_0 \rangle$ . For any  $\Psi \in \mathcal{I}_F$  either  $\Psi = \Phi^m$  for some m or  $\rho_{\Psi} = -\rho_{\Phi}^m$ . In the latter case  $m \geq q$ . So  $\rho_{\Psi} = \rho_{\Phi^{m-q}\circ\Psi_0}$ , implying that  $\Psi = \Phi^{m-q} \circ \Psi_0$ . Also it is clear  $\Psi_0^2 = \Phi^{2q}$  because they have the same contraction factor. We have proved (iii).

Finally we prove (i). We have already seen that  $\rho_{\Psi} = \pm |\rho_{\Phi}|^{\frac{m}{k}}$  for some  $m \in \mathbb{N}$  for any  $\Psi \in \mathcal{I}_F$ . Set  $\rho = |\rho_{\Phi}|^{\frac{1}{k}}$ . Then  $\rho_{\Psi} = \pm \rho^m$ .

Define  $\mathcal{P}^+ = \{m : \rho^m = \rho_{\Psi} \text{ for some } \Psi \in \mathcal{I}_F\}$  and  $\mathcal{P}^- = \{m : \rho^m = -\rho_{\Psi} \text{ for some } \Psi \in \mathcal{I}_F\}$ . We will show that  $\mathcal{I}_F^+$  is finitely generated. Set  $a = \gcd(\mathcal{P}^+)$ . Let  $\Psi_1, \ldots, \Psi_n \in \mathcal{I}_F$  with  $\rho_{\Psi_j} = \rho^{m_j}$  such that  $\gcd(m_1, m_2, \ldots, m_n) = a$ . By a standard result in elementary number theory every sufficiently large integer  $ma \ge N_0$  can be expressed as  $ma = \sum_{j=1}^n p_j m_j$  with  $p_j \ge 0$ . Thus every  $\Psi \in \mathcal{I}_F^+$  with  $\rho_{\Psi} = \rho^{ma}$ ,  $ma \ge N_0$ , can be expressed as  $\Psi = \prod_{j=1}^n \Psi_j^{p_j}$  since the two IFSs have the same contraction factor. Let  $\{\Psi_{n+1}, \ldots, \Psi_K\} \subseteq \mathcal{I}_F^+$  consist of all elements  $\Psi \in \mathcal{I}_F^+$  with  $\rho_{\Psi} \ge \rho^{N_0}$  that are not already in  $\{\Psi_1, \ldots, \Psi_n\}$ . Then  $\mathcal{I}_F^+ = \langle \Psi_1, \Psi_2, \ldots, \Psi_K \rangle$ , and it is finitely generated.

The proof that  $\mathcal{I}_F$  is finitely generated is virtually identical, and we omit it.

# 3. Non-homogeneous IFS

When we do not require that the contraction factors be homogeneous, the main result in §2 no longer holds. In §5 we give a counterexample showing that the COSC no longer guarantees the existence of a minimal element. For the existence to hold we need stronger assumptions. The following is our main result:

**Theorem 3.1.** Let  $F \subset \mathbb{R}$  be a compact set such that F is the attractor of an IFS  $\Phi = \{\rho_i x + c_i\}_{i=1}^N$ . Assume  $\mathcal{H}^s(F) = (\operatorname{diam} F)^s$ , where s satisfies  $\sum_{i=1}^N |\rho_i|^s = 1$ .

- (i) Let  $\mathcal{G}_F^+$  denote the set of all positive generating IFSs of F with the OSC. Then  $\mathcal{G}_F^+$  contains a minimal element.
- (ii) Let  $\mathcal{G}_F$  denote the set of all generating IFSs of F with the OSC. Suppose that F is not symmetric. Then  $\mathcal{G}_F$  contains a minimal element.
- (iii) Suppose that F is symmetric. Then there exists a  $\Phi_0 = \{\phi_i\}_{i=1}^M$  in  $\mathcal{G}_F$  such that for any  $\Psi \in \mathcal{G}_F$  and each  $\psi \in \Psi$ , there exist  $i_1, \ldots, i_\ell \in \{1, \ldots, M\}$  so that

$$\psi = \phi_{i_1} \circ \cdots \circ \phi_{i_\ell} \quad or \quad \overline{\psi} = \phi_{i_1} \circ \cdots \circ \phi_{i_\ell}$$

where  $\overline{\psi}$  denote the unique affine map satisfying  $\overline{\psi}(a) = \psi(b)$  and  $\overline{\psi}(b) = \psi(a)$ , with  $a = \min(F)$ ,  $b = \max(F)$ .

**Remark.** The condition  $\mathcal{H}^{s}(F) = (b-a)^{s}$  is very strong. In general we always have  $\mathcal{H}^{s}(F) \leq (b-a)^{s}$ . In [15] Marion studied the computation of Hausdorff measure of general self-similar sets. He gave the exact condition for the equality  $\mathcal{H}^{s}(F) = (b-a)^{s}$  to hold. This problem

was also studied by Ayer and Strichartz [1] independently. Actually  $\mathcal{H}^{s}(F) = (b-a)^{s}$  implies that  $\Phi$  satisfies the COSC. Moreover, denote by  $\mu$  the self-similar measure generated by  $\Phi$ with weights  $\{|\rho_{1}|^{s}, \ldots, |\rho_{N}|^{s}\}$ . Note that  $\mu$  is simply the normalized Hausdorff measure  $\mathcal{H}^{s}$ restricted to F. Then the condition  $\mathcal{H}^{s}(F) = (b-a)^{s}$  is equivalent to

(3.1) 
$$\mu([u,v]) \le \left(\frac{u-v}{b-a}\right)^s \quad \text{for all interval } [u,v].$$

Furthermore if  $\Phi$  is a positive IFS satisfying  $\phi_1(a) < \phi_2(a) < \ldots < \phi_N(a)$  then  $\mathcal{H}^s(F) = (b-a)^s$  if and only if

(3.2) 
$$\frac{\sum_{i=m}^{n} |\rho_i|^s}{|\phi_n(b) - \phi_m(a)|^s} \le (b-a)^{-s}, \qquad \forall 1 \le m < n \le N.$$

Equation (3.2) is an easily checkable condition. If we drop the condition  $\mathcal{H}^{s}(F) = (b-a)^{s}$ , then Theorem 3.1 is no longer true. We present a counterexample in §5.

**Proof of Theorem 3.1.** We first prove the following claim:

**Claim.** Let  $\psi_1, \psi_2$  be any two contractive affine map with  $\psi_1(F) \subset F$  and  $\psi_2(F) \subset F$ . Then one of the following cases must happen:

- (A)  $\psi_1(F) \cap \psi_2(F) = \emptyset;$
- (B)  $\psi_1(F) \supseteq \psi_2(F);$

(C) 
$$\psi_2(F) \supseteq \psi_1(F)$$
.

Proof of Claim: Let  $\nu_F$  denote the s-dimensional Hausdorff measure restricted to F. It follows from (4.1) that for all intervals [u, v],

(3.3) 
$$\nu_F([u,v]) \le (u-v)^s.$$

Denote  $[a_1, b_1] := \psi_1([a, b])$  and  $[a_2, b_2] := \psi_2([a, b])$ . There are at most 5 different possible scenarios for these two intervals:

(1)  $[a_1, b_1] \cap [a_2, b_2] = \emptyset;$ (2)  $[a_1, b_1] \supseteq [a_2, b_2];$ (3)  $[a_2, b_2] \supseteq [a_1, b_1];$ (4)  $a_1 < a_2 \le b_1 < b_2;$ (5)  $a_2 < a_1 \le b_2 < b_1.$ 

We prove the claim by examining  $\psi_1(F)$  and  $\psi_2(F)$  in each of the above scenarios.

It is clear that with scenario (1) we have  $\psi_1(F) \cap \psi_2(F) = \emptyset$ . We show that  $\psi_1(F) \supseteq \psi_2(F)$ with scenario (2) by contradiction. Assume it is not true. Then there exists an  $x_0 \in F$  such that dist $(\psi_2(x_0), \psi_1(F)) > 0$ . This means there exists a small cylinder  $E = \phi_{i_1} \circ \cdots \circ \phi_{i_n}(F)$ of the IFS  $\Phi$  containing  $x_0$  such that  $\phi_2(E) \cap \phi_1(F) = \emptyset$ . Note that by the scaling property of the measure  $\nu_F$  we have  $\nu_F(\psi_2(E)) > 0$ . Hence

$$\nu_F([a_1, b_1]) \ge \nu_F(\psi_1(F) \cup \psi_2(F)) > \nu_F(\psi_1(F)).$$

But because  $\nu_F(F) = (b-a)^s$  we also have  $\nu_F(\psi_1(F)) = (b_1 - a_1)^s$  by the scaling property of  $\nu_F$  and the fact that  $\psi_1(F) \subseteq F$ . Therefore  $\nu_F([a_1, b_1]) > (b_1 - a_1)^s$ , a contradiction. Similarly  $\psi_2(F) \supseteq \psi_1(F)$  with scenario (3).

Now we prove that scenarios (4) and (5) never occur. Assume this is false. Without loss of generality we assume that scenario (4) has occurred. Then

$$\nu_F([a_1, b_2]) = \nu_F([a_1, b_1]) + \nu_F([a_2, b_2]) - \nu_F([a_2, b_1])$$
  
=  $(b_1 - a_1)^{\alpha} + (b_2 - a_2)^{\alpha} - \nu_F([a_2, b_1])$   
 $\geq (b_1 - a_1)^{\alpha} + (b_2 - a_2)^{\alpha} - (b_1 - a_2)^{\alpha}$   
 $> (b_2 - a_1)^{\alpha},$ 

a contradiction. Note that in the last inequality we have employed the fact that  $(x + y)^{\alpha} + (y + z)^{\alpha} - y^{\alpha} > (x + y + z)^{\alpha}$  with  $x = a_2 - a_1 > 0$ ,  $y = b_1 - a_2 \ge 0$  and  $z = b_2 - b_1 > 0$ . So have completed the proof of the claim.

Going back to the proof, suppose that  $\Phi_0 = \{\phi_i\}_{i=1}^M$  is an element in  $\mathcal{G}_F$  (resp.  $\mathcal{G}_F^+$ ) with the smallest integer M. By the claim  $\Phi_0$  satisfies the SC. To prove the theorem, it suffices to prove that if  $\psi(x) = \rho x + b$  is an affine map (reps.  $\rho > 0$ ) satisfying  $\psi(F) \subset F$ , then

$$\psi(F) = \phi_{i_1} \circ \cdots \circ \phi_{i_\ell}(F)$$

for some indexes  $i_1, \ldots, i_\ell \in \{1, \ldots, M\}$ .

First we assert that  $\psi(F) \subseteq \phi_i(F)$  for some index *i*. To see this, denote by  $\Lambda$  the set of all indexes *j* so that  $\psi(F) \cap \phi_j(F) \neq \emptyset$ . We only need to show that  $\Lambda$  is a singleton. Assume it is not true. Then by the claim we have  $\psi(F) \supseteq \bigcup_{j \in \Lambda} \phi_j(F)$ , and thus  $\psi(F) = \bigcup_{j \in \Lambda} \phi_j(F)$ . It follows that  $\{\psi, \phi_{j'}\}$  with  $j' \in \{1, \ldots, M\} \setminus \Lambda$  constitutes an IFS for *F*, which contradicts the minimality of *M*. Now let  $\ell$  be the largest integer such that

$$\psi(F) \subseteq \phi_{i_1} \circ \cdots \circ \phi_{i_\ell}(F)$$

for some indexes  $i_1, \ldots, i_{\ell}$ . We show that  $\psi(F) = \phi_{i_1} \circ \cdots \circ \phi_{i_{\ell}}(F)$  as required. Denote  $\hat{\psi} := \phi_{i_{\ell}}^{-1} \circ \cdots \circ \phi_{i_1}^{-1} \circ \psi$ . Then  $\hat{\psi}(F) \subseteq F$ . Assume that  $\psi(F) \neq \phi_{i_1} \circ \cdots \circ \phi_{i_{\ell}}(F)$ , that is,  $\hat{\psi}(F) \neq F$ . Then again  $\hat{\psi}(F) \subseteq \phi_{i_{\ell+1}}(F)$  for some index  $i_{\ell+1}$ . Therefore  $\psi(F) \subseteq \phi_{i_1} \circ \cdots \circ \phi_{i_{\ell+1}}(F)$ , contradicting the maximality of  $\ell$ .

Observe that by the scaling property of  $\nu_F$  again,  $\psi(F) = \phi_{i_1} \circ \cdots \circ \phi_{i_\ell}(F)$  implies that the two maps on both side of the equality must have the same contraction factor in absolute values. Therefore  $\hat{\psi} = x + c$  or -x + c for some c. If  $\hat{\psi} = x + c$  then  $\hat{\psi}(F) = F$  yields c = 0, so  $\psi = \phi_{i_1} \circ \cdots \circ \phi_{i_\ell}$ . In the case of  $\mathcal{G}_F^+$  this is the only possibility. If  $\hat{\psi} = -x + c$  then  $\hat{\psi}(F) = F$  implies F is symmetric and  $\hat{\psi} = \overline{\psi}$ . The proof of the theorem is now complete.

# 4. LOGARITHMIC COMMENSURABILITY OF CONTRACTION FACTORS

In this section we prove Theorem 1.1 and Corollary 1.2. The most difficult part of the proof by far is for part (i) of Theorem 1.1, which is rather tedious and technical, requiring delicate estimates and analysis. We first prove a stronger form of part (ii) of Theorem 1.1. A compact set F is said to satisfy the *no interval condition* if  $F \not\supseteq [\min(F), \min(F) + \varepsilon]$  for any  $\varepsilon > 0$ .

**Theorem 4.1.** Let  $\Phi = \{\phi_i(x) := \rho_i x + c_i\}_{i=1}^N$  be an IFS in  $\mathbb{R}$  with attractor F satisfying the no interval condition. Assume that  $x_0 := \min(F) \in \phi_1(F)$  but  $x_0 \notin \phi_j(F)$  for all j > 1, and  $\rho_1 > 0$ . Let  $\psi(x) = \lambda x + b$  such that  $x_0 \in \psi(F) \subset F$  and  $\lambda > 0$ . Then  $\log \lambda / \log \rho_1 \in \mathbb{Q}$ .

**Proof.** Since  $\rho_1, \lambda > 0$  it is clear that  $x_0$  is a fixed point of  $\phi_1$  and  $\psi$ , i.e.  $x_0 = \phi_1(x_0) = \psi(x_0)$ . By making a translation  $F' = F - x_0$  it is easy to see that we may without loss of generality assume that  $x_0 = \min(F) = 0$ , which forces  $\phi_1(x) = \rho_1 x$  and  $\psi(x) = \lambda x$ .

Observe that  $0 \in \phi_1(F)$  but dist  $(0, \phi_j(F)) \ge \delta$  for some  $\delta > 0$  for all j > 1. This means  $0 \in \phi_1^m(F)$  but dist  $(0, \phi(F)) \ge \rho_1^m \delta$  for all other  $\phi \in \Phi^m$ . Hence

(4.1) 
$$\rho_1^{-m}F = \rho_1^{-m} \bigcup_{\phi \in \Phi^m} \phi(F) = F \cup \Big(\bigcup_{\phi \in \Phi^m \setminus \{\phi_1^m\}} \rho_1^{-m} \phi(F)\Big),$$

and dist  $(0, \rho_1^{-m} \phi(F)) \ge \delta$  for all  $\phi \in \Phi^m \setminus \{\phi_1^m\}$ . Since  $\psi^n(F) = \lambda^n F \subset F$ , we have

(4.2) 
$$\rho_1^{-m}\lambda^n F \subseteq \rho_1^{-m}F = F \cup \Big(\bigcup_{\phi \in \Phi^m \setminus \{\phi_1^m\}} \rho_1^{-m}\phi(F)\Big),$$

Now,  $[0, \delta] \not\subseteq F$  by the no interval condition. So there exists an interval  $I_0 \subseteq (0, \delta) \setminus F$ . Assume that  $\log \lambda / \log \rho_1 \notin \mathbb{Q}$ . Then  $\{-m \log \rho_1 + n \log \lambda\}$  is dense in  $\mathbb{R}$ , and hence  $\rho_1^{-m} \lambda^n$  is dense in  $\mathbb{R}^+$ . In particular we may choose m, n such that  $\rho_1^{-m} \lambda^n \max(F) \in I_0$ . For such m, n (4.2) is clearly violated, yielding a contradiction.

We remark that Theorem 4.1 does not require the IFS to satisfy the OSC. Clearly the no interval condition is satisfied if  $\dim_H F < 1$ . If a homogeneous IFS  $\Phi$  satisfying the OSC and  $\dim_H F_{\Phi} = 1$ , then the no interval condition is equivalent to  $F_{\Phi}$  is not a finite union of intervals:

**Proposition 4.2.** Let  $\Phi$  be a homogeneous IFS with the OSC. Suppose that  $F_{\Phi}$  does not satisfy the no interval condition. Then  $\rho_{\Phi} = \frac{1}{p}$  for some integer p and  $F_{\Phi}$  is a finite union of intervals.

**Proof.** This is proved in Lagarias and Wang [10], using a result of Odlyzko [16]. In fact, the structure of  $\Phi$  is known.

We now prove part (i) of Theorem 1.1. This is done by breaking it down into several lemmas.

**Lemma 4.3.** Under the assumptions of Theorem 1.1, there exists a positive number t (depending on F) such that

(4.3) 
$$\mathcal{H}^{s}(F \cap [a,b]) \leq t \ (b-s)^{s}, \qquad \forall \ [a,b] \subset \mathbb{R}$$

**Proof.** It is implied in the proof of Theorem 8.6 in Falconer [8].

As a result of the above lemma, we introduce

 $d_{\max} = \sup\{\mathcal{H}^s(F \cap [a,b])/(b-a)^s : [a,b] \subset \mathbb{R}\},\$ 

and clearly  $0 < d_{\text{max}} < \infty$ . The following lemma plays a central role in the proof of Theorem 1.1.

**Lemma 4.4.** There exist an interval [a, b] and an integer k > 0 such that

(i) 
$$[a,b] \cap F \neq \emptyset$$
.  
(ii)  $[x - |\rho_{\Phi}|^{k} \operatorname{diam} F, x + |\rho_{\Phi}|^{k} \operatorname{diam} F] \cap F = \emptyset$  for  $x = a, b$ .  
(iii) Denote  $\mathcal{M} = \{\mathbf{i} \in \{1, \cdots, N\}^{k} : \phi_{\mathbf{i}}(F) \subset [a,b]\}$  and  $M = \#\mathcal{M}$ , then  
(4.4)  $(M + 1/2) |\rho_{\Phi}|^{ks} \mathcal{H}^{s}(F) > d_{\max} (b-a)^{s}$ .

**Proof.** Denote  $\rho = |\rho_{\Phi}|$ . Since 0 < s < 1, using L'Hospital's rule we have

$$\lim_{x \to 0} \frac{(1+ux)^s - 1}{x^s} = 0, \qquad \forall \ u > 0$$

Therefore there exist  $\ell \in \mathbb{N}$  and  $\varepsilon > 0$  such that

(4.5) 
$$\frac{1}{2}\rho^{\ell s}\mathcal{H}^{s}(F) - \varepsilon > d_{\max}\left((1+8\rho^{\ell-1}\mathrm{diam}F)^{s} - 1\right).$$

By the definition of  $d_{\max}$  there exists an interval [c, d] such that  $[c, d] \cap F \neq \emptyset$  and

$$\mathcal{H}^{s}(F \cap [c,d]) \ge (d_{\max} - \varepsilon) (d-c)^{s}.$$

Let r be the integer so that  $\rho^{r+1} < d - c \leq \rho^r$ . Then we have

$$\mathcal{H}^{s}(F \cap [c,d]) + \frac{1}{2}\rho^{(r+\ell)s}\mathcal{H}^{s}(F) > (d_{\max} - \varepsilon) (d-c)^{s} + \frac{1}{2}\rho^{\ell s}\mathcal{H}^{s}(F) (d-c)^{s}$$

$$\geq \left(d_{\max} - \varepsilon + \frac{1}{2}\rho^{\ell s}\mathcal{H}^{s}(F)\right) (d-c)^{s}$$

$$\geq d_{\max} \left(1 + 8\rho^{\ell-1}\mathrm{diam}F\right)^{s} (d-c)^{s}$$

$$\geq d_{\max} \left(d-c+8\rho^{\ell-1}(d-c)\mathrm{diam}F\right)^{s}$$

$$\geq d_{\max} \left(d-c+8\rho^{\ell+r}\mathrm{diam}F\right)^{s}.$$

That is

(4.6) 
$$\mathcal{H}^{s}(F \cap [c,d]) + \frac{1}{2}\rho^{(r+\ell)s}\mathcal{H}^{s}(F) > d_{\max} \left(d-c+8\rho^{\ell+r}\mathrm{diam}F\right)^{s}.$$

Define  $k = \ell + r$  and  $[a, b] = [c - 2\rho^k \operatorname{diam} F, d + 2\rho^k \operatorname{diam} F]$ . We show that [a, b] and k satisfy (i), (ii) and (iii). Part (i) is obvious since  $[a, b] \supset [c, d]$ . Assume that (ii) is not true. Then

$$F \cap \left( [c - 3\rho^k \operatorname{diam} F, \ c - \rho^k \operatorname{diam} F] \cup [d + \rho^k \operatorname{diam} F, \ d + 3\rho^k \operatorname{diam} F] \right) \neq \emptyset.$$

Therefore there exists at least one  $\mathbf{i} \in \{1, \dots, N\}^k$  such that

$$\Phi_I(F) \subset [c - 4\rho^k \operatorname{diam} F, c] \cup [d, d + 4\rho^k \operatorname{diam} F]$$

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It follows from (4.6) that

$$\mathcal{H}^{s}(F \cap [c - 4\rho^{k} \operatorname{diam} F, \ d + 4\rho^{k} \operatorname{diam} F]) \geq \mathcal{H}^{s}(F \cap [c, d]) + \rho^{ks} \mathcal{H}^{s}(F)$$
  
 
$$> d_{\max}(d - c + 8\rho^{k} \operatorname{diam} F)^{s},$$

which leads to a contradiction. This finishes the proof of part (ii). Observe that  $\bigcup_{i \in \mathcal{M}} \phi_i(F) \supseteq F \cap [c, d]$ . Thus

$$M \ \rho^{ks} \ \mathcal{H}^s(F) \ge \mathcal{H}^s(F \cap [c,d]).$$

Hence by (4.6),

$$(M+1/2) \rho^{ks} \mathcal{H}^{s}(F) \geq \mathcal{H}^{s}(F \cap [c,d]) + 1/2 \rho^{ks} \mathcal{H}^{s}(F)$$
$$> d_{\max}(d-c+8 \rho^{\ell+r} \operatorname{diam} F)^{s}$$
$$> d_{\max}(b-a)^{s},$$

proving part (iii).

**Proof of Theorem 1.1.** Note that  $\mathcal{I}_F$  and  $\mathcal{I}_F^+$  are abelian as a result of Proposition 2.4. Now we only need to prove part (i) of the theorem, as part (ii) is clearly a corollary of Lemma 4.1 and Proposition 4.2.

Let [a, b], k,  $\mathcal{M}$  and  $\mathcal{M}$  be given as in Lemma 4.4. Assume that Theorem 1.1 is false, that is,  $\log |\lambda| / \log |\rho_{\Phi}| \notin \mathbb{Q}$ . We derive a contradiction.

Let  $\varepsilon > 0$  be a small number such that  $(1-\varepsilon)^s (M+1) \ge (M+1/2)$ . Since  $\log |\lambda| / \log |\rho_{\Phi}| \notin \mathbb{Q}$ , there exist  $m, n \in \mathbb{N}$  such that

$$1 - \varepsilon < |\rho_{\Phi}|^m / |\lambda|^n < 1.$$

Define  $J = \psi^n([a, b])$ . We show that

(4.7) 
$$\mathcal{H}^s(J \cap F) > d_{\max} |\mathrm{diam} J|^s,$$

which contradicts the maximality of  $d_{\text{max}}$ .

To show (4.7), let

$$\tilde{J} := \psi^n \left[ a + |\rho_{\Phi}|^k \operatorname{diam} F, \ b - |\rho_{\Phi}|^k \operatorname{diam} F \right].$$

By Lemma 4.4,

$$\begin{split} \tilde{J} \cap F &\supseteq \quad \tilde{J} \cap \psi^n(F) \\ &= \quad \psi^n \left( \left[ a + |\rho_{\Phi}|^k \operatorname{diam} F, \ b - |\rho_{\Phi}|^k \operatorname{diam} F \right] \cap F \right) \\ &= \quad \psi^n \left( \bigcup_{\mathbf{i} \in \mathcal{M}} \phi_{\mathbf{i}}(F) \right). \end{split}$$

Hence

(4.8) 
$$\mathcal{H}^{s}(\tilde{J} \cap F) \geq \mathcal{H}^{s}\left(\psi^{n}\left(\bigcup_{\mathbf{i} \in \mathcal{M}} \phi_{\mathbf{i}}(F)\right)\right) = M |\lambda|^{ns} |\rho_{\Phi}|^{ks} \mathcal{H}^{s}(F).$$

Define

$$\mathcal{R} := \left\{ \mathbf{i} \in \{1, \cdots, N\}^{m+k} : \phi_{\mathbf{i}}(F) \cap \tilde{J} \neq \emptyset \right\}$$

and  $R = \#\mathcal{R}$ . Then  $\phi_{\mathbf{i}}(F) \subset J$  for any  $\mathbf{i} \in \mathcal{R}$ , and  $\bigcup_{\mathbf{i} \in \mathcal{R}} \phi_{\mathbf{i}}(F) \supset \tilde{J} \cap F$ . Thus

$$\mathcal{H}^{s}(J \cap F) \geq \mathcal{H}^{s}(\bigcup_{\mathbf{i} \in \mathcal{R}} \phi_{\mathbf{i}}(F)) = R |\rho_{\Phi}|^{(m+k)s} \mathcal{H}^{s}(F) \geq \mathcal{H}^{s}(\tilde{J} \cap F)$$

Combining the second inequality with (4.8) we obtain R > M and thus  $R \ge M + 1$ . Hence we have

$$\begin{aligned} \mathcal{H}^{s}(J \cap F) &\geq (M+1) |\rho_{\Phi}|^{(m+k)s} \mathcal{H}^{s}(F) \\ &> (M+1) (1-\varepsilon)^{s} |\lambda|^{ns} |\rho_{\Phi}|^{ks} \mathcal{H}^{s}(F) \\ &> (M+1/2) |\lambda|^{ns} |\rho_{\Phi}|^{ks} \mathcal{H}^{s}(F) \\ &> d_{\max} |\lambda|^{ns} (b-a)^{s} \quad (\text{ by } (4.4)) \\ &= d_{\max} |J|^{s}. \end{aligned}$$

This is a contradiction, proving part (i) of the theorem.

# 5. Counterexamples and Open Questions

In this section we present various counterexamples, including a counterexample to Mattila's question. We also propose some open questions.

Let us first give an example to show that the condition COSC in Theorem 2.1 cannot be replaced with the SC.

**Example 5.1.** Let *F* be the attractor of the IFS  $\Phi = \{\frac{1}{16}(x+a): a \in A\}$  where  $A = \{0, 1, 64, 65\}$ . It is not difficult to check that  $\Phi$  satisfies the SC but does not satisfy the

COSC. We prove that  $\mathcal{I}_F^+$  does not contain a minimal element by contradiction. Assume this is not true. Let  $\Phi_0 = \{\rho x + c_i\}_{i=1}^N$  be the minimal element of  $\mathcal{I}_F^+$ . By the dimension formula and Corollary 1.2,  $\log \rho / \log 16^{-1} = \log N / \log 4 \in \mathbb{Q}$ . Therefore N = 2 and  $\rho = \frac{1}{4}$ or N = 4 and  $\rho = \frac{1}{16}$ . But it is easy to check that if N = 2 then the IFS  $\Phi_0$  must satisfy the COSC, but  $\Phi$  does not, a contradiction to Theorem 2.1. Hence we must have N = 4and hence  $\Phi_0 = \Phi$  by Lemma 2.3. Now let  $\Psi = \frac{1}{64}(x + b)$ :  $a \in \mathcal{B}$  where q = 64 and  $\mathcal{B} = \{0, 1, 16, 17, 256, 257, 272, 273\}$ . One can check directly  $\mathcal{B} + q\mathcal{B} = \mathcal{A} + p\mathcal{A} + p^2\mathcal{A}$ . Thus  $\Psi^2 = \Phi^3$ , which implies  $\Psi \in \mathcal{I}_F$ . However  $\Psi$  is not derived from  $\Phi$ , which leads to a contradiction. Hence  $\mathcal{I}_F^+$  does not contain a minimal element.

Now we give a counterexample showing that the COSC no longer guarantees the existence of a minimal element if we don't require that the contraction factors be homogeneous.

**Example 5.2.** Let F be the attractor of the IFS  $\Phi = \{\frac{1}{10}(x+a) : a \in \mathcal{A}\}$  where  $\mathcal{A} = \{0, 1, 5, 6\}$ . As before let  $\mathcal{G}_F^+$  denotes the set of all positive generating IFSs of F satisfying the OSC. We claim that  $\Phi$  satisfies the COSC and  $\mathcal{G}_F^+$  does not contain a minimal element.

To prove this fact we need the following lemma:

**Lemma 5.1.** Let  $\Phi$  be a positive IFS satisfying the COSC with attractor  $F = F_{\Phi}$ . If  $\phi$  is an affine map such that  $\phi(F) \subseteq F$ , then there exists an interval [a,b] such that  $a,b \in F$ ,  $(a,b) \cap F \neq \emptyset$  and

$$\phi(F) \cap [a,b] = F \cap [a,b].$$

**Proof.** Denote  $s = \dim_H F$ . Following Ayer and Strichartz [1], for all interval I we call  $d(I) = \mathcal{H}^s(I \cap F)/|I|^s$  the *density* of I. It is known (see e.g. [1, Theorem 2.4]) that under the condition of the lemma, the maximum density is attained for an interval J. Clearly the endpoints of J and  $\phi(J)$  belong to F. Note that

$$d(\phi(J)) = \frac{\mathcal{H}^s(\phi(J) \cap F)}{|\phi(J)|^s} \ge \frac{\mathcal{H}^s(\phi(J) \cap \phi(F))}{|\phi(J)|^s} = d(J).$$

By the maximality of d(J), we have  $d(\phi(J)) = d(J)$  and thus  $\mathcal{H}^s(\phi(J) \cap F) = \mathcal{H}^s(\phi(J) \cap \phi(F))$ .  $\phi(F)$ ). We conclude  $\phi(J) \cap F = \phi(J) \cap \phi(F)$ . Assume it is not true. Then there exists an  $x_0 \in \operatorname{int}(\phi(J)) \cap F$  such that  $x_0 \notin \phi(F)$ . Therefore there exists a small cylinder  $E = \phi_{i_1} \circ \cdots \circ \phi_{i_n}(F)$  of the IFS  $\Phi$  containing  $x_0$  such that  $E \subset \phi(J)$  and  $E \cap \phi(F) = \emptyset$ . Thus  $\mathcal{H}^s(\phi(J) \cap F) - \mathcal{H}^s(\phi(J) \cap \phi(F)) \geq \mathcal{H}^s(E) > 0$ , a contradiction. Denote  $[a, b] = \phi(J)$ . Then [a, b] satisfies the desired properties. This completes the proof of the lemma.

Now we return to the proof of our claim in Example 5.2. One can check directly that F is symmetric with F = 0 and  $\max F = 2/3$ . Since F < F + 1 < F + 5 < F + 6, F satisfies the COSC. Moreover one can check that each connected component of  $[0, 2/3]\setminus F$  has length  $\frac{1}{3} \cdot 10^{-n}$  for some integer  $n \ge 0$ .

Now assume that  $\phi$  is an arbitrary contractive affine map with  $\phi(F) \subset F$ . We assert that  $|\rho_{\phi}| = 10^{-k}$  for some  $n \in \mathbb{N}$ . To prove this assertion, by Lemma 5.1 we know that there exist  $a, b \in F$  such that  $(a, b) \cap F \neq \emptyset$  and  $[a, b] \cap \phi(F) = [a, b] \cap F$ . By the similarity of  $\phi$ , each connected branch of  $[a, b] \setminus \phi(F)$  has length  $\frac{|\rho_{\phi}|}{3} \cdot 10^{-n}$  for some integer n. However at the same time it must have length  $\frac{1}{3} \cdot 10^{-m}$  for some integer m since  $[a, b] \setminus \phi(F) = [a, b] \setminus F$ . This implies that  $|\rho_{\phi}| = 10^{-k}$  for some  $k \in \mathbb{N}$ . Particularly  $|\rho_{\phi}| \leq 10^{-1}$ .

Now we prove  $\mathcal{G}_F^+$  does not contain a minimal element by contradiction. Assume that  $\mathcal{G}_F^+$  contains a minimal element, say  $\Phi_0$ . Any map in  $\Phi_0$  must have contraction factor no greater than  $\frac{1}{10}$ , hence we must have  $\Phi_0 = \Phi$ , for this is the only way  $\Phi$  can be an iteration of  $\Phi_0$ . just take  $\Phi_0 = \Phi$ . Consider

$$\Psi := \left\{ \frac{x}{100}, \ \frac{x+1}{100}, \ \frac{x+1/2}{10}, \ \frac{x+15}{100}, \ \frac{x+16}{100}, \ \frac{x+5}{10}, \ \frac{x+6}{10} \right\}.$$

Its attractor is also F. The similarity dimension is the Haudorff dimension for  $\Psi$  so  $\Psi$  satisfies the OSC. Thus  $\Psi \in \mathcal{G}_F$ . This is a contradiction because the map  $\psi := \frac{x+1/2}{10}$  or  $\overline{\psi}$  (defined in Theorem 3.1) is not a composition of the elements in  $\Phi$ . This completes the proof.

**Remark**. Example 5.2 also shows that the condition  $\mathcal{H}^{s}(F) = (\operatorname{diam} F)^{s}$  cannot be dropped.

**Example 5.3.** In this example we consider the question raised by Mattila: Is it true that any self-similar subset F of the middle-third Cantor set C is trivial, in the sense that F has a generating IFS that is derived from the generating IFS { $\phi_0, \phi_1$ } of C given in (1.1)?

We give a negative answer here by constructing a counterexample. For now, let  $\Phi = \{\frac{1}{9}x, \frac{1}{9}(x+\frac{2}{27}), \frac{1}{9}(x+\frac{2}{3^6})\}$ . Then by looking at the ternary expansion of the elements in  $F_{\Phi}$  it is easy to see that  $F_{\Phi} \subset C$ . But clearly  $\Phi$  cannot be derived from the original IFS given in (1.1).

**Open Question 1.** We pose the following question concerning the symmetry of a selfsimilar set: Let  $\Phi$  and  $\Psi$  be two homogeneous IFSs satisfying the OSC, with  $\rho_{\Phi} = -\rho_{\Psi}$  and  $F_{\Phi} = F_{\Psi}$ . Does it follow that F is symmetric?

This is answered in affirmative under the strong assumption of COSC. But is it true in general? If so, then the results in part (ii) and (iii) of Theorem 1.3 will be much cleaner.

It should be pointed out that this is *not* true for self-similar measures. We'll leave to the readers to construct a counterexample.

**Open Question 2.** We do not have a good way to generalize our results to higher dimensions.

The challenge here is to generalize the Logarithmic Commensurability Theorem to higher dimensions for affine IFSs. There is a possibility to do it for similation IFSs.

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