ERDŐS SIMILARITY PROBLEM VIA BI-LIPSCHITZ EMBEDDING

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ABSTRACT. The Erdős similarity conjecture asserted that an infinite set of real numbers cannot be affinely embedded into every measurable set of positive Lebesgue measure. The problem is still open, in particular for all fast decaying sequences. In this paper, we relax the problem to the bi-Lipschitz embedding and obtain some sharp criteria about the bi-Lipschitz Erdős similarity problem for strictly decreasing sequences.

1. INTRODUCTION AND MAIN RESULTS

1.1. **Background.** Searching for "copies" of patterns of certain points inside sets of fairly big size has been a central problem in different branches of mathematics. Such a statement can take many different forms depending on what "copies" people are looking for and the notion of the "size". The most natural notion in this regard would be an affine copy. By an **affine copy** of a set $P \subset \mathbb{R}$, we mean a set of the form $\lambda P + t$, where $\lambda, t \in \mathbb{R}$ and $\lambda \neq 0$.

The notion of "size" of sets takes many different forms as well, but one of the most natural choices is undoubtedly the Lebesgue measure. The earliest result about the abundance of patterns for sets of positive measure was due to Steinhaus [Ste20]. Using the Lebesgue density theorem, Steinhaus proved that if P is a finite set of real numbers, then every measurable set $E \subset \mathbb{R}$ of positive Lebesgue measure contains an affine copy of P. Here and afterwards "measurable" always means "Lebesgue measurable". Erdős however believed that infinite sets cannot be that abundant. Following the notion introduced by Kolountzakis [Kol97], we say that a set P is **measure universal** if every measurable set of positive Lebesgue measure contains an affine copy of P. In early 1970's, Erdős (see example [Erd74] or [Erd15]) proposed a conjecture, which is open until today.

Erdős similarity Conjecture: There is no infinite measure universal set.

To validate that the conjecture is true, it would suffice to show that for any positive strictly decreasing sequence $(a_n)_{n=1}^{\infty}$ with $a_n \to 0$, there exists a Lebesgue measurable set E of positive measure such that E contains no affine copy of this sequence. The first progress was made by Eigen [Eig85] and Falconer [Fal84] independently who

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showed that if $(a_n)_{n=1}^\infty$ is a decreasing sequence converging to 0 slowly in the sense that

$$(1.1) \qquad\qquad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1,$$

then $(a_n)_{n=1}^\infty$ is not measure universal. Humke and Laczkovich [HL98] further showed that the condition (1.1) can be weaken to $\limsup_{n\to\infty} a_{n+1}/a_n = 1$ if in addition $(a_n - a_{n+1})$ is monotone decreasing. Kolountzakis [Kol97] showed that an infinite set $A \subset \mathbb{R}$ is not measure universal if for all $n \in \mathbb{N}$ sufficiently large, A contains a subset $\{a_1 > a_2 > \cdots > a_n > 0\}$ such that

$$-\log\left(\min_{1\le i\le n-1} \frac{a_i - a_{i+1}}{a_1}\right) = o(n).$$

His result can be used to recover that of Eigen and Falconer. Using a probabilistic argument, Kolountzakis further showed that the conjecture is almost surely true: for every sequence $A = (a_n)_{n=1}^\infty$, there exists a measurable set E of positive Lebesgue measure such that the set

$$\{(\lambda, t) \in \mathbb{R}^2 : \lambda A + t \subset E\}$$

has two dimensional Lebesgue measure zero. It is still an open question to determine whether any given sequence $(a_n)_{n=1}^\infty$, whose ratio limit is strictly less than 1 (e.g. $a_n = 2^{-n}$), is measure universal. The reader is referred to [CLP22] for some recent progresses and to [Sve00] for other progresses of the problem before year 2000.

There are some other types of infinite sets determined to be measure non-universal via different methods. Bourgain [Bou87] converted the conjecture into a problem about the boundedness of certain operators (see also [Tao21]). He showed that $S_1 + S_2 + S_3$ is not measure universal if all S_i are infinite sets. It is not known if this result is also true about the sum of two infinite sets. Along this direction, Kolountzakis [Kol97] showed a special case that $\{2^{-n^\alpha}\}_{n=1}^\infty + \{2^{-n^\alpha}\}_{n=1}^\infty$ with $\alpha \in (0, 2)$ is not measure universal. More recently, Gallagher, the second named author and Weber [GLW23] showed that Cantor sets of positive Newhouse thickness are not measure universal. This was followed up immediately by Kolountzakis [Kol23] who showed that some Cantor sets of Newhouse thickness zero was also measure non-universal. Nonetheless, despite being uncountable, it is still unknown if all Cantor sets are not measure universal.

Another interesting variant, known as the Erdős similarity problem "in the large", was first initiated by Bradford, Kohut and Mooroogen [BKM22] and later improved in [KP23]. In these work, they showed that for any unbounded sequence of certain restricted increasing rate and $p \in (0, 1)$, there always exists a measurable set E of positive Lebesgue measure such that $\mathcal{L}(E \cap [x, x + 1]) \ge p$ for all $x \in \mathbb{R}$ and E does not contain an affine copy of this sequence. Here $\mathcal{L}$ stands for Lebesgue measure. In a further improvement, Burgin-Goldberg-Keleti-MacMahon-Wang [BGKMW22] turned

a set "in the large" back to a compact set $E \subset \mathbb{R}$ such that 0 is a Lebesgue density point of E but E does not contain any (non-constant) infinite geometric progression.

It is worth to mention that the Erdős similarity problem has arose many different interests and variations of the study under other notions of size, such as Hausdorff dimension. For any countable collection of sets of three points, Keleti [Ke08] constructed a compact subset of the real line with Hausdorff dimension 1 that contains no affine copy of any of the given triplets. On the other hand, there also exists a closed set of real numbers with Hausdorff dimension zero which contains affine copies of all finite sets [DMT60] (such a set must have packing dimension 1, see [SS17, Lemma 5.3]). By assuming certain Fourier decay conditions, Łaba and Pramanik [LP09] showed that a large class of fractal sets contain non-trivial 3-term arithmetic progressions. Meanwhile Shmerkin constructed some Salem sets which contain no 3-term arithmetic progression [Sh17]. A recent in-depth study of the detection of patterns in relation to the Fourier dimension can be found in [LP22].

1.2. **Main Results.** In a private communication [Jin23], Xiong Jin proposed another variant of the Erdős similarity problem by considering the bi-Lipschitz copies, instead of the affine copies. Indeed, bi-Lipschitz or C^1 embedding problems have been studied intensively between self-similar sets (see e.g. [DWXX11, FHR15, Alg20]). In this paper we will prove a sharp characterization to decreasing sequences being universal in the sense of having bi-Lipschitz copies.

Recall that a map $f : \mathbb{R} \to \mathbb{R}$ is said to be **bi-Lipschitz** if there exists a constant $L > 1$ such that

$$L^{-1}|x - y| \leq |f(x) - f(y)| \leq L|x - y| \quad \text{for all } x, y \in \mathbb{R}.$$

A **bi-Lipschitz copy** of a set A is the image $f(A)$ where f is a bi-Lipschitz map. Clearly, an affine copy must be a bi-Lipschitz copy. If a bi-Lipschitz copy of A is contained in a set E, we will say that A can be **bi-Lipschitz embedded** into E. We will say that A is **bi-Lipschitz measure universal** if A can be bi-Lipschitz embedded into every measurable set of positive Lebesgue measures. Our first result is the following.

Theorem 1.1. *Let $(a_n)_{n=1}^{\infty}$ be a strictly decreasing sequence of positive numbers with $a_n \to 0$ as $n \to \infty$. If there exists an integer $N \geq 1$ such that*

$$\limsup_{n \to \infty} \frac{a_{n+N}}{a_n} < 1,$$

then for any measurable set $E \subset \mathbb{R}$ with positive Lebesgue measure, there exists a bi-Lipschitz map $f : \mathbb{R} \to \mathbb{R}$ such that $f(a_n) \in E$ for all $n \geq 1$ and $f'(0) = 1$.

This result shows that fast decaying sequences like $(2^{-n})_{n=1}^{\infty}$ is bi-Lipschitz measure universal. Moreover, the derivative condition implies that the map becomes very close to an affine map at the limit point. This suggested some negative evidence of the Erdős similarity conjecture, indicating that the set avoiding affine copies of fast

decaying sequences will not be so easily constructed. Furthermore, as our proof will be based on the Lebesgue density theorem, in comparison to the aforementioned result of [BGKMW22], we can still deduce the existence of bi-Lipschitz copies of geometric sequences around every density point.

In contrast to the above result for fast decaying sequences, our second result states that a slow decaying sequence can not be bi-Lipschitz embedded into all measurable sets of positive Lebesgue measure, which generalizes the aforementioned affine embedding result by Eigen [Eig85] and Falconer [Fal84]. This also provides a new proof of their result.

Theorem 1.2. *Let $(a_n)_{n=1}^\infty$ be a strictly decreasing sequence of positive numbers with $a_n \to 0$ as $n \to \infty$. If*
$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1,$$
then there exists a compact set $E \subset \mathbb{R}$ with positive Lebesgue measure such that $(a_n)_{n=1}^\infty$ can not be bi-Lipschitz embedded into E.

More generally, we provide a classification theorem for a type of decreasing sequences. The condition includes the commonly-known convex deceasing sequences i.e. $a_{n+1} \le \frac{a_n + a_{n+2}}{2}$ for all $n \ge 2$.

Theorem 1.3. *Let $(a_n)_{n=1}^\infty$ be a strictly decreasing sequence of positive numbers with $a_n \to 0$ as $n \to \infty$. Suppose in addition*
$$(1.2) \qquad \sup_{m>n>1} \frac{a_{m-1} - a_m}{a_{n-1} - a_n} < \infty.$$
Then $(a_n)_{n=1}^\infty$ is bi-Lipschitz measure universal if and only if $\limsup_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$.

The 'only if' part of the above theorem generalizes the aforementioned affine embedding result of Humke and Laczkovich [HL98]. It is worth pointing out that the condition (1.2) in Theorem 1.3 can not be dropped. Indeed, there are many examples of decreasing sequences $(a_n)_{n=1}^\infty$ with $\lim_{n\to\infty} a_n = 0$ such that
$$\limsup_{n\to\infty} \frac{a_{n+2}}{a_n} < 1 \quad \text{but} \quad \limsup_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$
For instance, this is the case if
$$a_n = \begin{cases} (2^k - 1)^{-1}, & \text{if } n = 2k-1, \\ 2^{-k}, & \text{if } n = 2k. \end{cases}$$
By Theorem 1.1, the above sequence $(a_n)_{n=1}^\infty$ can be bi-Lipscitz embedded into every measurable set of positive Lebesgue measure, although $\limsup_{n\to\infty} a_{n+1}/a_n = 1$.

Apart from fast decaying decreasing sequence being bi-Lipschitz embedded into every measurable set of Lebesgue measures as shown in Theorem 1.1, we finally demonstrate a type of countable sets with infinitely many limit points are also bi-Lipschitz measure universal.

Theorem 1.4. *Let $A = (a_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that*

$$a_1 + \sum_{n=1}^{\infty} \frac{a_{n+1}}{a_n} < \frac{1}{8}.$$

Then the set

$$F = \bigcup_{n=1}^{\infty} 3^{-n}(1 + A)$$

is bi-Lipschitz measure universal.

Theorem 1.4 has demonstrated that a set with infinitely many limit points could be universally bi-Lipschitz embedded. It will be an interesting question to determine whether or not sets that were previously studied, such as a Minkowski sum of three infinite sets or even a Cantor set, can be bi-Lipschitz measure universal. For Cantor sets, we actually know that if a bi-Lipschitz measure universal Cantor set exists, it must have Newhouse thickness zero. It is because in [GLW23, Theorem 1.5], it has been shown that there exists a set G of full Lebesgue measure not containing any Cantor sets of positive Newhouse thickness. As a bi-Lipschitz image of a Cantor set with positive Newhouse thickness still has positive Newhouse thickness, the existence of the set G immediately implies that Cantor sets with positive Newhouse thickness can not be bi-Lipschitz measure universal.

Moreover, it is worth noting that Theorem 1.4 also implies that the condition in Theorem 1.1 is not necessary to ensure the bi-Lipschitz embedding. To see this, let $A = (4^{-4^n})_{n=1}^{\infty}$. One can check that A satisfies the assumption of Theorem 1.4. Now let A_n be a finite subset of A with cardinality $\geq n$ for each $n \geq 1$. Set $E = \bigcup_{n=1}^{\infty} 3^{-n}(1 + A_n)$. Then E is a decreasing sequence such that $\limsup_{n\to\infty} a_{n+N}/a_n = 1$ for all $N \in \mathbb{N}$ [1], yet E is still bi-Lipschitz measure universal. Hence, the converse of Theorem 1.1 is also not true.

The proof of Theorem 1.4 involves a study about universally bi-Lipschitz embedding with uniform bi-Lipschitz bound. A quantitative result will be given in Theorem 4.1.

The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3, we prove Theorems 1.2 and 1.3. We will prove Theorem 1.4 with a study of uniform universal bi-Lipschitz embedding in Section 4.

2. THE PROOF OF THEOREM 1.1

Let $\mathcal{L}$ denote the Lebesgue measure on $\mathbb{R}$. The proof of Theorem 1.1 is based on the following.

Lemma 2.1. *Let $N \in \mathbb{N}$ and $\delta \in (0, 1)$. Let E be a Lebesgue measurable subset of $\mathbb{R}$ and $I = [\delta u, v]$, where*

$$(2.1) \qquad\qquad 0 < \delta^{N-1} v \leq u \leq v.$$

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[1] $\liminf_{n\to\infty} a_{n+1}/a_n < 1$ still holds, so it does not contradict Theorem 1.2.

Suppose that

$$(2.2) \qquad \rho := \frac{\mathcal{L}(I \setminus E)}{\mathcal{L}(I)} < N^{-2}\delta^N.$$

Then for any $x_1, \ldots, x_N \in [u, v]$, there exists $0 \le t \le (1-\delta)N^2\delta^{-N}\rho u$ such that $x_1 - t, \ldots, x_N - t \in E \cap I$.

Proof. Let $x_1, \ldots, x_N \in [u, v]$. By (2.2), $N^2\delta^{-N}\rho < 1$. Hence

$$x_1 - t, \ldots, x_N - t \in [\delta u, v] = I$$

for every $t \in \left[0, (1-\delta)N^2\delta^{-N}\rho u\right]$. Below we show by contradiction that there always exists $t \in \left[0, (1-\delta)N^2\delta^{-N}\rho u\right]$ such that $x_i - t \in E$ for all $1 \le i \le N$.

Suppose on the contrary that the above conclusion is false. Then for each $t \in \left[0, (1-\delta)N^2\delta^{-N}\rho u\right]$, there exists i such that $x_i - t \in I \setminus E$, or equivalently, $t \in x_i - (I \setminus E)$. Hence

$$\left[0, (1-\delta)N^2\delta^{-N}\rho u\right] \subset \bigcup_{i=1}^{N} \left(x_i - (I \setminus E)\right).$$

It follows that

$$(2.3) \qquad (1-\delta)N^2\delta^{-N}\rho u \le N\mathcal{L}(I \setminus E).$$

However,

$$
\begin{aligned}
\mathcal{L}(I \setminus E) &= \mathcal{L}(I)\rho && \text{(by (2.2))} \\
&= (v - \delta u)\rho \\
&\le \left(\delta^{-(N-1)} - \delta\right)\rho u && \text{(by (2.1))} \\
&= (1-\delta)\left(\delta + \cdots + \delta^N\right)\delta^{-N}\rho u \\
&< (1-\delta)N\delta^{-N}\rho u,
\end{aligned}
$$

leading to a contradiction with (2.3). $\qquad\qquad\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Since $(a_n)_{n=1}^{\infty}$ is strictly decreasing and $\limsup_{n\to\infty} \frac{a_{n+N}}{a_n} < 1$, there exists $\delta \in (0, 1)$ such that

$$(2.4) \qquad \frac{a_{n+N}}{a_n} < \delta^N \quad \text{for all } n \ge 1.$$

Let $(n_k)_{k=0}^{\infty}$ be the increasing sequence of positive integers given by $n_0 = 1$ and

$$(2.5) \qquad \{n_k \colon k \ge 1\} = \{n \ge 1 \colon a_{n+1}/a_n < \delta\}.$$

We claim that

$$(2.6) \qquad n_{k+1} - n_k \le N \quad \text{for all } k \ge 1.$$

Otherwise if $n_{k+1} - n_k > N$ for some $k \ge 1$, then by (2.5),

$$a_{j+1}/a_j \ge \delta$$

for $j = n_k + 1, \ldots, n_k + N$, implying that

$$\frac{a_{n_k+1+N}}{a_{n_k+1}} = \prod_{j=n_k+1}^{n_k+N} \frac{a_{j+1}}{a_j} \geq \delta^N,$$

which contradicts (2.4). This proves (2.6).

To simplify the notation, for $k \geq 1$ we write

(2.7) $$u_k = a_{n_k}, \quad v_k = a_{n_{k-1}+1}, \quad \text{and} \quad I_k = [\delta u_k, v_k].$$

By (2.5) and (2.6),

(2.8) $$\frac{v_{k+1}}{u_k} = \frac{a_{n_k+1}}{a_{n_k}} < \delta,$$

and

(2.9) $$1 \geq \frac{u_k}{v_k} = \frac{a_{n_k}}{a_{n_{k-1}+1}} = \prod_{j=n_{k-1}+1}^{n_k-1} \frac{a_{j+1}}{a_j} \geq \delta^{n_k - n_{k-1} - 1} \geq \delta^{N-1}.$$

It follows from (2.8) that $I_k \cap I_{k+1} = \varnothing$. As $(v_k)_{k=1}^{\infty}$ is monotone decreasing, the intervals I_k are disjoint.

Let $E \subset \mathbb{R}$ be measurable with positive Lebesgue measure. Replacing E by its suitable translation if necessary, we may assume that 0 is a Lebesgue density point of E, that is,

$$\lim_{r \to 0} \frac{\mathcal{L}([0, r] \cap E)}{r} = 1.$$

Then $\lim_{r \to 0} \mathcal{L}([0, r] \setminus E)/r = 0$. Since $\lim_{k \to \infty} v_k = 0$, it follows that

$$\frac{\mathcal{L}(I_k \setminus E)}{\mathcal{L}(I_k)} \leq \frac{\mathcal{L}([0, v_k] \setminus E)}{\mathcal{L}(I_k)} = \frac{\mathcal{L}([0, v_k] \setminus E)}{v_k} \cdot \frac{v_k}{v_k - \delta u_k} \leq \frac{\mathcal{L}([0, v_k] \setminus E)}{v_k} \cdot \frac{1}{1 - \delta} \to 0$$

as $k \to \infty$. Write

(2.10) $$\rho_k := \frac{\mathcal{L}(I_k \setminus E)}{\mathcal{L}(I_k)}, \quad k \geq 1.$$

Choose a large integer p such that

(2.11) $$\rho_k < N^{-2} \delta^N \quad \text{for all } k \geq p.$$

Since 0 is a density point of E we may also assume that $\mathcal{L}(E \cap (v_p, \infty)) > 0$.

Next we construct a strictly decreasing sequence $(b_n)_{n=1}^{\infty}$ of positive numbers such that $b_n \in E$ for $n \geq 1$, $\lim_{n \to \infty} b_n = 0$, and moreover,

(2.12) $$\lim_{n \to \infty} \frac{b_n - b_{n+1}}{a_n - a_{n+1}} = 1.$$

To this end, we first arbitrarily choose a strictly decreasing sequence $(b_j)_{j=1}^{n_p}$ of positive numbers from $E \cap (v_p, \infty)$. Then for each $k \geq p + 1$, by (2.9), (2.11) and Lemma 2.1 (in which we take $I = I_k$), we can find $0 \leq t_k \leq (1 - \delta) N^2 \delta^{-N} \rho_k u_k$ such that

$$a_j - t_k \in E \cap I_k \quad \text{for all } n_{k-1} + 1 \leq j \leq n_k.$$

Define $b_j = a_j - t_k$ for $n_{k-1}+1 \leq j \leq n_k$. In this way, we obtain the sequence $(b_n)_{n=1}^\infty$. See Figure 1 for an illustration of our construction.
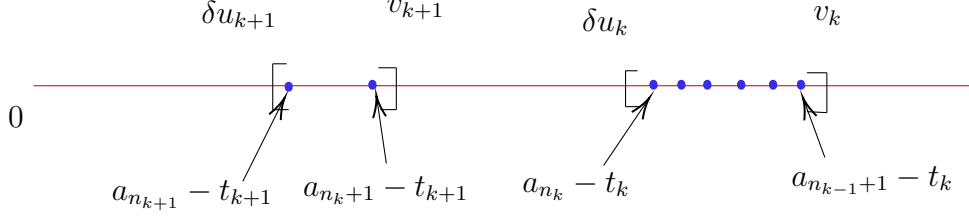

FIGURE 1. An illustration of the intervals I_k and I_{k+1} and the points b_n that we are going to choose. Inside each interval, there are at most N points.

Clearly, $b_n \in E$ for $n \geq 1$ and $\lim_{n\to\infty} b_n = 0$. To see (2.12), by the definition of $(b_n)_{n=1}^\infty$, we see that for each $k \geq p+1$,

$$(2.13) \qquad \frac{b_j - b_{j+1}}{a_j - a_{j+1}} = \begin{cases} 1, & \text{if } n_{k-1}+1 \leq j \leq n_k - 1, \\ \dfrac{a_{n_k} - a_{n_k+1} + t_{k+1} - t_k}{a_{n_k} - a_{n_k+1}}, & \text{if } j = n_k. \end{cases}$$

Recall that $a_{n_k+1} < \delta a_{n_k}$ (see (2.5)), and $0 \leq t_k \leq (1-\delta)N^2\delta^{-N}\rho_k a_{n_k}$, so

$$\frac{|t_{k+1} - t_k|}{a_{n_k} - a_{n_k+1}} \leq \frac{t_k + t_{k+1}}{a_{n_k} - a_{n_k+1}} \leq (1-\delta)^{-1}\frac{t_k + t_{k+1}}{a_{n_k}} \leq N^2\delta^{-N}(\rho_k + \rho_{k+1}) \to 0$$

as $k \to \infty$. Combining this with (2.13) yields (2.12).

Finally we show that the sequence $(b_n)_{n=1}^\infty$ is the image of $(a_n)_{n=1}^\infty$ under a bi-Lipschtiz map. To see it, define a mapping $f : \mathbb{R} \to \mathbb{R}$ by

$$f(x) = \begin{cases} x, & \text{if } x \leq 0, \\ \dfrac{a_n - x}{a_n - a_{n+1}} \cdot b_{n+1} + \dfrac{x - a_{n+1}}{a_n - a_{n+1}} \cdot b_n, & \text{if } a_{n+1} \leq x \leq a_n, \\ x - a_1 + b_1, & \text{if } x > a_1. \end{cases}$$

Clearly $f(a_n) = b_n$ for $n \geq 1$, and f is a continuous piecewise linear map, with slope 1 on $(-\infty, 0]\bigcup(a_1, \infty)$, and $(b_n - b_{n+1})/(a_n - a_{n+1})$ on $[a_{n+1}, a_n]$ for $n \geq 1$. By (2.12), $f'(0) = 1$, and all these slopes are positive, uniformly bounded away from zero, and from above by a constant, thus f is bi-Lipschitz on $\mathbb{R}$. $\qquad\square$

3. The proofs of Theorems 1.2 and 1.3

In this section we prove Theorems 1.2 and 1.3. We begin with the following.

Lemma 3.1. *Let $(a_n)_{n=1}^\infty$ be a strictly decreasing sequence of positive numbers such that*

$$\lim_{k\to\infty} a_n = 0 \quad and \quad \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1.$$

Then there exists a subsequence $(a_{n_k})_{k=1}^{\infty}$ such that $\lim_{k \to \infty} a_{n_{k+1}}/a_{n_k} = 1$ and

$$(3.1) \qquad a_{n_k} - a_{n_{k+1}} \leq 2(a_{n_m} - a_{n_{m+1}}) \quad \text{for all} \ \ k, m \in \mathbb{N} \ \text{with} \ \ k > m.$$

Proof. Write for $n \in \mathbb{N}$,

$$t_n = \sup\{a_p - a_{p+1} : \ p \geq n\}.$$

Clearly, (t_n) is monotone decreasing. Since the sequence (a_n) is strictly decreasing with limit 0, the supremum in the above equality is attainable for each n; that is, for each n there exists $p(n) \in \mathbb{N}$ such that

$$p(n) \geq n \quad \text{and} \quad t_n = a_{p(n)} - a_{p(n)+1}.$$

Next we inductively define a subsequence $(n_k)_{k=1}^{\infty}$ of natural numbers. Set $n_1 = 1$. Suppose $n_1, \ldots, n_k$ have been defined. Then we define

$$(3.2) \qquad n_{k+1} = \inf\{p \in \mathbb{N} : p > n_k, \ a_{n_k} - a_p \geq t_{n_k}\}.$$

Since $a_{n_k} - a_{p(n_k)+1} \geq a_{p(n_k)} - a_{p(n_k)+1} = t_{n_k}$, it follows from (3.2) that

$$n_k < n_{k+1} \leq p(n_k) + 1 < \infty.$$

Continuing this process, we obtain the sequence $(n_k)_{k=1}^{\infty}$.

By (3.2), $a_{n_k} - a_{n_{k+1}} \geq t_{n_k}$. Moreover, by (3.2) and the definition of t_{n_k},

$$a_{n_k} - a_{n_{k+1}} = (a_{n_k} - a_{n_{k+1}-1}) + (a_{n_{k+1}-1} - a_{n_{k+1}}) \leq t_{n_k} + t_{n_k} = 2t_{n_k}.$$

That is,

$$(3.3) \qquad t_{n_k} \leq a_{n_k} - a_{n_{k+1}} \leq 2t_{n_k} \quad \text{for all } k \geq 1.$$

It follows that for any $k, m \in \mathbb{N}$ with $k > m$,

$$a_{n_k} - a_{n_{k+1}} \leq 2t_{n_k} \leq 2t_{n_m} \leq 2(a_{n_m} - a_{n_{m+1}}).$$

Finally we show that $\lim_{k \to \infty} a_{n_{k+1}}/a_{n_k} = 1$, which is equivalent to

$$\lim_{k \to \infty} (a_{n_k} - a_{n_{k+1}})/a_{n_k} = 0.$$

Notice that by (3.3), $a_{n_k} - a_{n_{k+1}} \leq 2t_{n_k} = 2(a_{p(n_k)} - a_{p(n_k)+1})$, and $a_{n_k} \geq a_{p(n_k)}$. Hence

$$0 \leq \frac{a_{n_k} - a_{n_{k+1}}}{a_{n_k}} \leq \frac{2(a_{p(n_k)} - a_{p(n_k)+1})}{a_{p(n_k)}} = 2\left(1 - \frac{a_{p(n_k)+1}}{a_{p(n_k)}}\right) \to 0,$$

as desired. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \square$

Proposition 3.2. *Let $(a_n)_{n=1}^{\infty}$ be a strictly decreasing sequence of positive numbers with $a_n \to 0$ as $n \to \infty$. Suppose that*

$$\sup_{m>n>1} \frac{a_{m-1} - a_m}{a_{n-1} - a_n} < \infty, \quad \text{and} \quad \limsup_{n \to \infty} \frac{a_{n+1}}{a_n} = 1.$$

Then there exists a compact set $E \subset \mathbb{R}$ with positive Lebesgue measure such that $(a_n)_{n=1}^{\infty}$ can not be bi-Lipschitz embedded into E.

Proof. By our assumption, there exists $C > 1$ such that

(3.4) $$a_n - a_{n+1} \leq C(a_m - a_{m+1}) \quad \text{for all } n, m \text{ with } n > m.$$

Since

$$\liminf_{n \to \infty} \frac{a_n - a_{n+1}}{a_n} = 1 - \limsup_{n \to \infty} \frac{a_{n+1}}{a_n} = 0,$$

we can choose a strictly increasing sequence $(n_k)_{k=1}^{\infty}$ of natural numbers such that

$$\frac{a_{n_k} - a_{n_k+1}}{a_{n_k}} \leq k^{-2} 4^{-k} \quad \text{for } k \geq 1.$$

For each $k \geq 1$, let ℓ_k be the smallest integer $\geq k/a_{n_k}$, and let $\delta_k = k(a_{n_k} - a_{n_k+1})$, Clearly, it holds that

$$\frac{1}{\ell_k} \leq \frac{a_{n_k}}{k} < \frac{2}{\ell_k}$$

and

(3.5) $$\ell_k \delta_k \leq \frac{2k}{a_{n_k}} \cdot k(a_{n_k} - a_{n_k+1}) \leq 2 \cdot 4^{-k} \quad \text{for } k \geq 1.$$

Define a sequence $(E_k)_{k=1}^{\infty}$ of compact subsets of $[0,1]$ by

$$E_k = [0,1] \setminus \bigcup_{j=0}^{\ell_k} \left(\frac{j}{\ell_k} - \frac{\delta_k}{2}, \frac{j}{\ell_k} + \frac{\delta_k}{2} \right).$$

It is easy to see that for each k, E_k is the union of ℓ_k disjoint intervals of length $(1 - \delta_k \ell_k)/\ell_k$, with a gap of length δ_k between any two adjacent intervals.

Set $E = \bigcap_{k=1}^{\infty} E_k$. Then E is a compact set with Lebesgue measure

$$\mathcal{L}(E) \geq 1 - \sum_{k=1}^{\infty} \mathcal{L}([0,1] \setminus E_k) \geq 1 - \sum_{k=1}^{\infty} \ell_k \delta_k \geq 1 - \sum_{k=1}^{\infty} 2 \cdot 4^{-k} = \frac{1}{3} > 0,$$

where we have used (3.5) in the third inequality. Below we show by contradiction that $(a_n)_{n=1}^{\infty}$ can not be embedded into E by a bi-Lipschitz map.

Suppose on the contrary that $(a_n)_{n=1}^{\infty}$ can be embedded into E by a bi-Lipschitz map $f : \mathbb{R} \to \mathbb{R}$. Let $b_n = f(a_n)$ for $n \geq 1$ and $b_{\infty} = \lim_{n \to \infty} b_n$. Then $b_n, b_{\infty} \in E$. Clearly $b_{\infty} = f(0)$, and $(b_n)_{n=1}^{\infty}$ is strictly monotone increasing or monotone decreasing. Since f is bi-Lipschitz, there exists a constant $L > 1$ such that

$$L^{-1} \leq \frac{|b_n - b_m|}{a_n - a_m} \leq L \quad \text{for all } n, m \in \mathbb{N}, n \neq m.$$

In particular, this implies that

(3.6) $$L^{-1} \leq \frac{|b_n - b_{n+1}|}{a_n - a_{n+1}} \leq L \quad \text{and} \quad L^{-1} \leq \frac{|b_n - b_{\infty}|}{a_n} \leq L.$$

Now fix an integer $k > CL$. Then by (3.6) and (3.4), for all $m \geq n_k$,

(3.7) $$|b_m - b_{m+1}| \leq L(a_m - a_{m+1}) \leq CL(a_{n_k} - a_{n_k+1}) < k(a_{n_k} - a_{n_k+1}) = \delta_k.$$

Meanwhile by (3.6),

$$(3.8) \qquad |b_{n_k} - b_\infty| \geq \frac{a_{n_k}}{L} > \frac{a_{n_k}}{k} \geq \frac{1}{\ell_k}.$$

Notice that $(b_m)_{m=n_k}^\infty \subset E \subset E_k$. Recall that E_k is the union of ℓ_k disjoint intervals of length $(1 - \delta_k \ell_k)/\ell_k$, with a gap of length δ_k between any two adjacent intervals. By (3.7), the sequence $(b_m)_{m=n_k}^\infty$ must be entirely contained in a component interval of E_k. This forces that $|b_m - b_\infty| \leq (1 - \delta_k \ell_k)/\ell_k$, which clearly contradicts (3.8). $\qquad\square$

Now we ready to prove Theorems 1.2 and 1.3.

Proof of Theorem 1.2. It follows directly from Lemma 3.1 and Proposition 3.2. $\qquad\square$

Proof of Theorem 1.3. The sufficiency part of the theorem follows from Theorem 1.1 and the necessity part follows from Proposition 3.2. $\qquad\square$

4. THE PROOF OF THEOREM 1.4

We will prove Theorem 1.4 in this section. To start, we will need to study whether we can bi-Lipschitz embed certain sequences with bi-Lipschitz bounds independent of the measurable sets. The following theorem provides a quantitative result in this direction, which may be of independent interest for other future study.

Theorem 4.1. *Let $A = (a_n)_{n=1}^\infty$ be a sequence of positive numbers such that*

$$(4.1) \qquad \delta := a_1 + \sum_{n=1}^\infty \frac{a_{n+1}}{a_n} < 1/8.$$

Then for any measurable set $E \subset [0,1]$ with $\mathcal{L}(E) > \frac{1}{2} + 4\delta$, there exists a bi-Lipschitz map $f : \mathbb{R} \to \mathbb{R}$ such that $f(A) \subset E$ and

$$\frac{1}{2}|x - y| \leq |f(x) - f(y)| \leq \frac{3}{1 - \delta}|x - y|.$$

We remark that the requirement that the Lebesgue measure is uniformly bounded away from zero is necessary. Indeed, let $E = [0, \varepsilon_0]$. Clearly, any sequences $A = (a_n)_{n=1}^\infty$ converging to zero can be bi-Lipschitz embedded into E via a map f. But then $|f(a_1) - f(0)| \leq \varepsilon_0$ which means that the bi-Lipschitz lower bound must be less than ε_0/a_1. So there cannot be any uniform lower bound over all measurable sets of positive measures.

The key to the proof of Theorem 4.1 is the following lemma, which asserts that, for any measurable set $E \subset [a, b]$ with sufficiently large density, there are two small subintervals of $[a, b]$ with pre-specified length and distance, such that the restrictions of E on these two intervals satisfy certain density conditions.

Lemma 4.2. *Let $t \in (1/2, 1)$ and let $0 < \varepsilon < t - 1/2$. Suppose that M is an even integer such that $M > 2/\varepsilon$. Then for every interval $I = [a, b]$ and every Lebesgue measurable set E with $\mathcal{L}(E \cap I) \geq t \cdot \mathcal{L}(I)$, there exists $1 \leq j \leq M - 2$ such that*

$$\frac{\mathcal{L}(E \cap I_j)}{\mathcal{L}(I_j)} \geq t - \varepsilon \quad and \quad \mathcal{L}(E \cap I_{j+2}) > 0,$$

where $I_j = \left[a + \frac{j-1}{M}(b-a), a + \frac{j}{M}(b-a)\right]$.

Proof. Let $p_j = \mathcal{L}(E \cap I_j)/\mathcal{L}(I_j)$, then $p_j \in [0, 1]$. Moreover,

$$\sum_{j=1}^{M} \mathcal{L}(E \cap I_j) = \mathcal{L}(E \cap I) \geq t \cdot \mathcal{L}(I).$$

Splitting the sum into j being odd or even, we must have

$$\sum_{j=1}^{M/2} \mathcal{L}(E \cap I_{2j}) \geq \frac{t}{2} \cdot \mathcal{L}(I) \qquad \text{or} \qquad \sum_{j=1}^{M/2} \mathcal{L}(E \cap I_{2j-1}) \geq \frac{t}{2} \cdot \mathcal{L}(I).$$

We may assume without loss of generality that the first case holds. Since $\mathcal{L}(I_j) = \frac{1}{M}\mathcal{L}(I)$, it follows that

$$(4.2) \qquad\qquad\qquad \sum_{j=1}^{M/2} p_{2j} \geq \frac{M}{2}t.$$

Let $N = M/2$. We now claim that there exists $k \in \{1, 2, ..., N - 1\}$ such that

$$p_{2k} \geq t - \varepsilon \quad \text{and} \quad p_{2k+2} > 0.$$

This will complete the proof. To justify the claim, we suppose on the contrary that the claim is false. Then for all $k \in \{1, ..., N - 1\}$, we have

$$\text{either } p_{2k} < t - \varepsilon, \quad \text{or } p_{2k} \geq t - \varepsilon \text{ and } p_{2k+2} = 0.$$

We now define

$$\Lambda_1 = \{k : p_{2k} \geq t - \varepsilon\}, \quad \Lambda_2 = \{k + 1 : k \in \Lambda_1\}.$$

Then Λ_1 and Λ_2 are disjoint subset of $\{1, \ldots, N\}$. Write also $\Lambda = \Lambda_1 \cup \Lambda_2$. Then $\#\Lambda_1 = \#\Lambda_2$ and hence $\#\Lambda = 2 \cdot \#\Lambda_1$. Here $\#$ denotes the cardinality of a set. Note that

$$\sum_{k=1}^{N} p_{2k} \leq \left(\sum_{k \in \Lambda} p_{2k}\right) + \left(\sum_{k \in \{1,...,N-1\}\backslash\Lambda} p_{2k}\right) + p_{2N}$$

$$\leq \frac{1}{2} \cdot \#\Lambda + (t - \varepsilon) \cdot (N - \#\Lambda) + 1$$

$$\leq (t - \varepsilon)N + 1$$

$$< tN,$$

where in the last two inequalities, we have used the assumptions $t - 1/2 > \varepsilon$ and $N = M/2 > 1/\varepsilon$. But this contradicts (4.2). $\qquad\square$

We also need a preliminary lemma.

Lemma 4.3. *Let $(a_n)_{n=1}^\infty$ be a sequence of positive numbers such that*

$$\delta := a_1 + \sum_{n=1}^\infty \frac{a_{n+1}}{a_n} \leq \frac{1}{4}.$$

Let

$$M_1 = 2\left\lceil \frac{1}{2a_1} \right\rceil \quad and \quad M_{n+1} = 2\left\lceil \frac{1}{2M_1 \cdots M_n a_{n+1}} \right\rceil$$

for all $n \geq 1$, where $\lceil x \rceil$ denotes the smallest integer larger than x. Then

(4.3) $$\frac{a_n}{2} \leq \frac{1}{M_1 \cdots M_n} \leq a_n \quad for\ all\ n \geq 1,$$

and

(4.4) $$\sum_{n=1}^\infty \frac{1}{M_n} < 2\delta.$$

Proof. We first prove (4.3) by induction on n. By the definition of M_1,

$$M_1 \geq \frac{2}{2a_1} = \frac{1}{a_1}.$$

On the other hand,

$$M_1 \leq 2 \cdot \left(\frac{1}{2a_1} + 1 \right) = \frac{1}{a_1} + 2 \leq \frac{2}{a_1},$$

since $1/a_1 \geq 4$. Thus (4.3) holds for $n = 1$.

Now suppose this is true for $n = k$. By the definition of M_{k+1},

$$M_1 \cdots M_k \cdot M_{k+1} \geq M_1 \cdots M_k \cdot \frac{2}{2M_1 \cdots M_k a_{k+1}} = \frac{1}{a_{k+1}}.$$

On the other hand,

$$\begin{aligned}
M_1 \cdots M_k \cdot M_{k+1} &\leq M_1 \cdots M_k \cdot 2 \cdot \left((2M_1 \cdots M_k a_{k+1})^{-1} + 1 \right) \\
&= \frac{1}{a_{k+1}} + 2M_1 \cdots M_k \\
&\leq \frac{1}{a_{k+1}} + \frac{4}{a_k} \qquad \text{(by induction hypothesis)} \\
&\leq \frac{1}{a_{k+1}} + \frac{a_k}{a_{k+1}} \cdot \frac{1}{a_k} \qquad \text{(since } a_k/a_{k+1} \geq 4\text{)} \\
&= \frac{2}{a_{k+1}}.
\end{aligned}$$

Hence, (4.3) holds for $n = k + 1$. This completes the proof of (4.3).

Finally, we deduce (4.4) from (4.3). Indeed, by (4.3),

$$\sum_{n=1}^{\infty} \frac{1}{M_n} = \frac{1}{M_1} + \frac{M_1}{M_1 M_2} + \cdots + \frac{M_1 \cdots M_{n-1}}{M_1 \cdots M_n} + \cdots$$

$$\leq a_1 + \frac{2a_2}{a_1} + \cdots + \frac{2a_n}{a_{n-1}} + \cdots$$

$$< 2\delta. \qquad \qquad \square$$

Proof of Theorem 4.1. Let $(a_n)_{n=1}^{\infty}$ be a sequence of positive numbers satisfying (4.1). Let

$$M_1 = 2 \left\lceil \frac{1}{2a_1} \right\rceil \quad \text{and} \quad M_{n+1} = 2 \left\lceil \frac{1}{2M_1 \cdots M_n a_{n+1}} \right\rceil$$

for all $n \geq 1$. By Lemma 4.3, the sequence $(M_n)_{n=1}^{\infty}$ satisfies (4.3) and (4.4).

For a given measurable set $E \subset [0, 1]$ with $\mathcal{L}(E) > \frac{1}{2} + 4\delta$, pick $\eta > 0$ such that

$$\mathcal{L}(E) > \frac{1}{2} + 2(2 + \eta)\delta.$$

Define $\varepsilon_n = (2 + \eta)/M_n$. Then $M_n > 2/\varepsilon_n$. Let $t = 1/2 + (2 + \eta) \cdot 2\delta$. By (4.4), we have

(4.5) $$\sum_{i=1}^{\infty} \varepsilon_i < t - 1/2.$$

Let $\Delta_0 = [0, 1]$. We now construct inductively two sequence $\{\Delta_k\}_{k=1}^{\infty}$ and $\{\Delta_k'\}_{k=1}^{\infty}$ of intervals such that the following properties hold for all $k \geq 1$:

(i) $\Delta_{k-1} \supset \Delta_k \cup \Delta_k'$;
(ii) for some integer $1 \leq j_k \leq M_1 \cdots M_k - 2$,

$$\Delta_k = \frac{1}{M_1 \cdots M_k} \cdot [j_k - 1, j_k] \quad \text{and} \quad \Delta_k' = \frac{1}{M_1 \cdots M_k} \cdot [j_k + 1, j_k + 2];$$

(iii) $\dfrac{\mathcal{L}(E \cap \Delta_k)}{\mathcal{L}(\Delta_k)} \geq t - \sum_{i=1}^{k} \varepsilon_i$ and $\mathcal{L}(E \cap \Delta_k') > 0$.

To see this, we first apply Lemma 4.2 on Δ_0 to obtain $1 \leq j_1 \leq M_1 - 2$ such that

$$\frac{\mathcal{L}(E \cap \Delta_1)}{\mathcal{L}(\Delta_1)} \geq t - \varepsilon_1 \quad \text{and} \quad \mathcal{L}(E \cap \Delta_1') > 0,$$

where $\Delta_1 = \left[\frac{j_1 - 1}{M_1}, \frac{j_1}{M_1} \right]$ and $\Delta_1' = \left[\frac{j_1 + 1}{M_1}, \frac{j_1 + 2}{M_1} \right]$. Clearly, all properties hold for $k = 1$ with Δ_1 and Δ_1'.

Suppose that for $k \geq 1$, we have chosen Δ_k and Δ_k' with all the properties hold. We subdivide Δ_k into M_{k+1} intervals. By the fact $M_{k+1} \geq 2/\varepsilon_{k+1}$ and (4.5), we can apply Lemma 4.2 to obtain Δ_{k+1} and Δ_{k+1}' inside Δ_k such that Property (iii) holds. Property (i) holds by our construction. Furthermore, Property (ii) also holds by our choice of Δ_{k+1} and Δ_{k+1}' in Lemma 4.2.

By the second part of Property (iii), we can pick $b_k \in E \cap \Delta'_k$ for each $k \geq 1$. Then $b_{k+1} \in \Delta'_{k+1} \subset \Delta_k$ by Property (i). By Property (ii), we have

$$\frac{1}{M_1 \cdots M_k} \leq b_k - b_{k+1} \leq \frac{3}{M_1 \cdots M_k}.$$

Combining with (4.3), we have

$$\frac{1}{2} \leq \frac{b_k - b_{k+1}}{a_k - a_{k+1}} \leq \frac{3a_k}{a_k - \delta a_k} = \frac{3}{1-\delta}.$$

By linearly interpolating all points between (a_k, b_k), we obtain a bi-Lipschitz map f with bi-Lipschitz constants $1/2$ and $3/(1-\delta)$ and $f(A) \subset E$. This completes the proof. $\qquad\square$

We are now ready to prove Theorem 1.4 in the introduction.

Proof of Theorem 1.4. Let $A = (a_n)_{n=1}^{\infty}$ be a sequence of positive numbers such that

$$\delta := a_1 + \sum_{n=1}^{\infty} \frac{a_{n+1}}{a_n} < \frac{1}{8}.$$

Let E be a measurable set of positive measure. Replacing E by a suitable translation of itself, we may assume that 0 is a Lebesgue density point of E. By the Lebesgue density theorem, there exists $N \in \mathbb{N}$ such that for all $n > N$,

$$\frac{\mathcal{L}(E \cap [3^{-n}, 2 \cdot 3^{-n}])}{3^{-n}} > \frac{1}{2} + 4\delta.$$

Let $g_n(x) = 3^n x - 1$ and consider

$$E_n = g_n(E \cap [3^{-n}, 2 \cdot 3^{-n}]).$$

Then $E_n \subset [0,1]$ and $\mathcal{L}(E_n) > \frac{1}{2} + 4\delta$ for all $n > N$. Using Theorem 4.1, for each $n > N$, we can find a bi-Lipschitz map $f_n : \mathbb{R} \to \mathbb{R}$ whose bi-Lipschitz lower and upper bounds are $1/2$ and $3/(1-\delta)$ respectively and $f_n(A) \subset E_n$. Let $h_n = g_n^{-1} \circ f_n \circ g_n$. A direct check shows that h_n is also bi-Lipschitz on $\mathbb{R}$ with bi-Lipschitz lower and upper bounds $1/2$ and $3/(1-\delta)$ respectively. Moreover,

$$h_n \left(3^{-n}(1+A) \right) \subset E \cap [3^{-n}, 2 \cdot 3^{-n}].$$

Define the map $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = h_n(x) \quad \text{if } x \in [3^{-n}, 3^{-n}(1+a_1)] \text{ for some } n > N.$$

Then we extend h continuously by a linear function on each of the intervals in the complement, i.e., h is a straight line on each interval $[3^{-n-1}(1+a_1), 3^{-n}]$ with $n > N$ and on the unbounded intervals, we simply define it with a straight line of slope 1.

We first claim that h is a bi-Lipschitz function. Indeed, h_n are all bi-Lipschitz with uniform bounds $1/2$ and $3/(1-\delta)$. It suffices to show that the slopes of h

on the intervals $[3^{-n-1}(1 + a_1), 3^{-n}]$ are uniformly bounded. To see this, because $h_n(3^{-n}(1 + A)) \subset E \cap [3^{-n}, 2 \cdot 3^{-n}]$,

$$h_n(3^{-n}) \in [3^{-n}, 2 \cdot 3^{-n}], \quad h_n(3^{-n-1}(1 + a_1)) \in [3^{-n-1}, 2 \cdot 3^{-n-1}].$$

the slope of h on $[3^{-n-1}(1 + a_1), 3^{-n}]$ is therefore lying in the interval

$$\left[\frac{3^{-n} - 2 \cdot 3^{-n-1}}{3^{-n-1}(2 - a_1)}, \ \frac{2 \cdot 3^{-n} - 3^{-n-1}}{3^{-n-1}(2 - a_1)} \right] = \left[\frac{1}{2 - a_1}, \ \frac{5}{2 - a_1} \right].$$

Hence, h is a bi-Lipschitz function with bi-Lipschitz lower bound

$$\min\left(\frac{1}{2}, \frac{1}{2 - a_1} \right) = \frac{1}{2}$$

and upper bound

$$\max\left(\frac{3}{1 - \delta}, \frac{5}{2 - a_1} \right) = \frac{3}{1 - \delta},$$
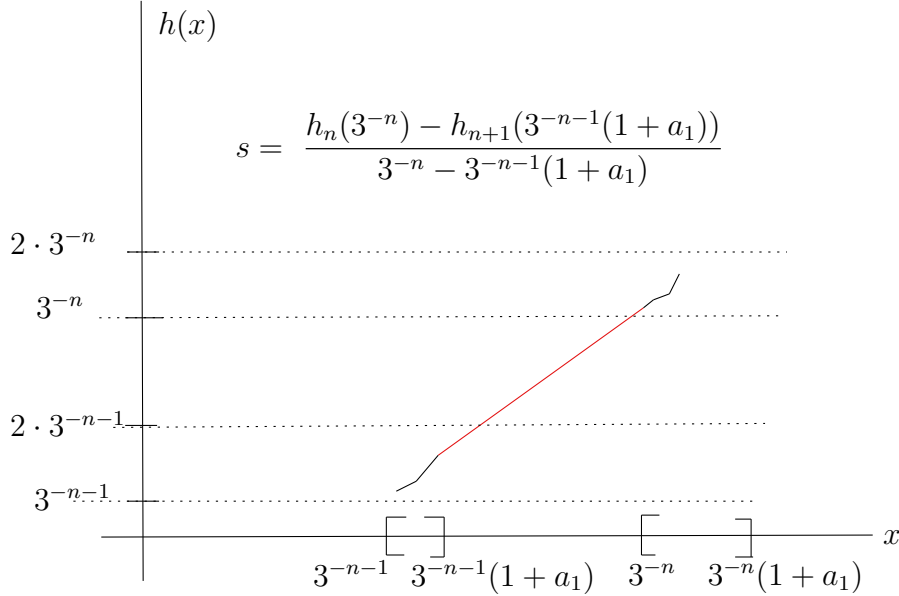
since $0 < a_1 < \delta < 1/8$.

FIGURE 2. An illustration of $h(x)$ for $3^{-n-1} \leq x \leq 3^{-n}(1 + a_1)$ with slope in $[3^{-n-1}(1 + a_1), 3^{-n}]$ equal to s.

By our construction of h, $h\left(\bigcup_{n > N} 3^{-n}(1 + A) \right) \subset E$. We now consider the map $H(x) = h(3^{-N}x)$. Clearly, H is still a bi-Lipschitz map and

$$H\left(\bigcup_{n=1}^{\infty} 3^{-n}(1 + A) \right) = h\left(\bigcup_{n > N} 3^{-n}(1 + A) \right) \subset E.$$

This completes the proof. □

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