WEIGHTED THERMODYNAMIC FORMALISM AND APPLICATIONS

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Abstract. Let (X, T) and (Y, S) be two subshifts so that Y is a factor of X. For any asymptotically sub-additive potential Φ on X and $\mathbf{a} = (a, b) \in \mathbb{R}^2$ with $a > 0$, $b \geq 0$, we introduce the notions of $\mathbf{a}$-weighted topological pressure and $\mathbf{a}$-weighted equilibrium state of Φ. We setup the weighted variational principle. In the case that X, Y are full shifts with one-block factor map, we prove the uniqueness and Gibbs property of $\mathbf{a}$-weighted equilibrium states for almost additive potentials having the bounded distortion properties. Extensions are given to the higher dimensional weighted thermodynamic formalism. As an application, we conduct the multifractal analysis for a new type of level sets associated with Birkhoff averages, as well as for weak Gibbs measures associated with asymptotically additive potentials on self-affine symbolic spaces.

1. Introduction

The classical thermodynamic formalism developed by Sinai, Ruelle, Bowen and Walters plays a fundamental role in statistical mechanics and dynamical systems (see, e.g. [43, 46]). It adapts to describe geometric properties of invariant sets and measures for situations in which the statistics (box counting) carry all the useful geometric information (e.g. the Hausdorff dimension of conformal sets and measures [10, 44], the topological entropy of level sets of Birkhoff averages [8, 37, 38]). However it seems not so efficient when statistical and geometrical point of views reveal different behaviors (e.g. the Hausdorff dimension of non-conformal sets and measures). In this paper we develop the so-called weighted thermodynamic formalism, which may provide a frame for which non-conformal geometry can be understood through natural thermodynamical quantities. This is indeed the case for the dynamics of expanding diagonal endomorphisms of tori. For instance, let $m_1 > m_2 \geq 2$ be two integers, let $K \subset \mathbb{T}^2$ be a self-affine Sierpinski carpet invariant by $T = \mathrm{diag}(m_1, m_2)$, and S denotes the map $y \mapsto m_2 y (\mathrm{mod}\ 1)$; let π be the restriction to K of the second coordinate projection. Let $\mathbf{a} = (a, b) := (1/\log m_1, 1/\log m_2 - 1/\log m_1)$. Our starting point is to substitute the $\mathbf{a}$-weighted entropy $h_\mu^{\mathbf{a}}(T) = a h_\mu(T) + b h_{\mu\circ\pi^{-1}}(S)$ to the classical one in the variational definition of the topological pressure of any continuous potential ϕ; this yields the "$\mathbf{a}$-weighted pressure" $P^{\mathbf{a}}(T, \phi)$ (in this setting, the Hausdorff dimension of K obtained in [34, 6, 28] is $P^{\mathbf{a}}(T, 0)$). Then, we derive the uniqueness and the new Gibbs property for $\mathbf{a}$-weighted equilibrium states associated with any continuous potential

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ϕ satisfying the bounded distortion property, and prove for this case the differentiability of the $\mathbf{a}$-weighted pressure function of ϕ, namely $P^{\mathbf{a}}(T, q\phi)$. This is used to establish a bridge between this weighted thermodynamic formalism and the Hausdorff dimension of invariant subsets of K, thanks to a fundamental result claiming that any invariant measure μ is the limit, in the weak-star topology, of a sequence of $\mathbf{a}$-weighted equilibrium states whose $\mathbf{a}$-weighted entropies converge to that of μ. This property, as well as the Ledrappier-Young type formula $\dim_H \nu = h^{\mathbf{a}}_\nu(T)$ for any ergodic measure ν (see [28]), are exploited to find a sharp lower bound for the Hausdorff dimension of the set of generic points of any invariant measure μ, which turns out to be equal to $h^{\mathbf{a}}_\mu(T)$. It is also exploited to conduct the multifractal analysis of a new family of level sets associated with the Birkhoff averages of ϕ. There, the Hausdorff dimensions of level sets are expressed via the Legendre transform of $P^{\mathbf{a}}(T, q\phi)$.

In fact, our results hold in the more general framework for "self-affine symbolic spaces" and almost additive potentials. Before formulating them, we first give some definitions.

We say that (X, T) is a *topological dynamical system* (TDS) if X is a compact metric space and T is a continuous map from X to X. Assume that (X, T) and (Y, S) are two TDSs such that there is a continuous surjective map $\pi : X \to Y$ with $\pi T = S\pi$, that is, Y is a *factor of* X *with factor map* π. Let $\Phi = (\log \phi_n)_{n=1}^\infty$ be a sequence of functions on X. We say that Φ is a *sub-additive potential* and write $\Phi \in \mathcal{C}_s(X, T)$ if ϕ_n is non-negative continuous for each n and there exists a constant $c > 0$ such that

$$\phi_{n+m}(x) \le c\phi_n(x)\phi_m(T^n x), \quad \forall\, x \in X,\ n, m \in \mathbb{N}.$$

(we admit that ϕ_n takes the value zero). More generally, $\Phi = (\log \phi_n)_{n=1}^\infty$ is said to be an *asymptotically sub-additive potential* and write $\Phi \in \mathcal{C}_{ass}(X, T)$ if for any $\varepsilon > 0$, there exists a sub-additive potential $\Psi = (\log \psi_n)_{n=1}^\infty$ on X such that

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} |\log \phi_n(x) - \log \psi_n(x)| \le \varepsilon,$$

where we take the convention $\log 0 - \log 0 = 0$. Furthermore Φ is called an *asymptotically additive potential* and write $\Phi \in \mathcal{C}_{asa}(X, T)$ if both Φ and $-\Phi$ are asymptotically sub-additive, where $-\Phi$ denotes $(\log(1/\phi_n))_{n=1}^\infty$. In particular, Φ is called *additive* if each ϕ_n is a continuous positive-valued function so that $\phi_{n+m}(x) = \phi_n(x)\phi_m(T^n x)$ for all $x \in X$ and $m, n \in \mathbb{N}$; in this case, there is a continuous real function g on X such that $\phi_n(x) = \exp(\sum_{i=0}^{n-1} g(T^i x))$ for each n.

Let $\Phi = (\log \phi_n)_{n=1}^\infty$ be an asymptotically sub-additive potential on X. Let $\mathbf{a} = (a, b) \in \mathbb{R}^2$ so that $a > 0$ and $b \ge 0$. We introduce

$$P^{\mathbf{a}}(T, \Phi) = \sup\{\Phi_*(\eta) + a h_\eta(T) + b h_{\eta\circ\pi^{-1}}(S) :\ \eta \in \mathcal{M}(X, T)\}, \tag{1.1}$$

where $\mathcal{M}(X, T)$ denotes the collection of T-invariant probability measures on X endowed with the weak-star topology, $h_\eta(T)$ and $h_{\eta\circ\pi^{-1}}(S)$ denote the measure theoretic entropies of η and $\eta \circ \pi^{-1}$ (cf. [46]), and $\Phi_*(\eta)$ is given by

$$\Phi_*(\eta) = \lim_{n\to\infty} \frac{1}{n} \int \log \phi_n(x)\, d\eta(x). \tag{1.2}$$

By subadditivity, the limit in (1.2) always exists (but may take the value $-\infty$). We call $P^{\mathbf{a}}(T, \Phi)$ the $\mathbf{a}$-*weighted topological pressure of* Φ. A measure $\eta \in \mathcal{M}(X, T)$ is called an $\mathbf{a}$-*weighted equilibrium state of* Φ if the supremum in (1.1) is attained at η.

When $\mathbf{a} = (1, 0)$, we write $P^{\mathbf{a}}(T, \Phi)$ simply as $P(T, \Phi)$ and call it the *topological pressure of* Φ. We remark that $P(T, \Phi)$ is a natural generalization of the classical topological pressure of additive functions, and it has been defined in an alternative way via separated sets or open covers in [13].

Let $\nu \in \mathcal{M}(Y, S)$. We say that $\mu \in \mathcal{M}(X, T)$ is a *conditional equilibrium state of* Φ *with respect to* ν if $\mu \circ \pi^{-1} = \nu$ and

$$\Phi_*(\mu) + h_\mu(T) - h_\nu(S) = \sup\{\Phi_*(\eta) + h_\eta(T) - h_\nu(S) :\ \eta \in \mathcal{M}(X, T),\ \eta \circ \pi^{-1} = \nu\}. \tag{1.3}$$

In the remainder part of this section, we assume that X is a subshift over a finite alphabet $\mathcal{A}$, and Y a subshift over a finite alphabet $\mathcal{D}$ together with a one-block factor map $\pi :\ X \to Y$ (see §2.1 for the definitions). Under this setting, the entropy function is upper semi-continuous and hence the supremums in (1.1) and (1.3) are attainable. For $I \in \mathcal{A}^n$, the n-th cylinder set $[I]$ in $\mathcal{A}^{\mathbb{N}}$ is defined as

$$[I] = \{(x_i)_{i=1}^\infty \in \mathcal{A}^{\mathbb{N}} :\ x_1 \dots x_n = I\}.$$

Similarly for $J \in \mathcal{D}^n$, let $[J]$ denote the n-th cylinder set in $\mathcal{D}^{\mathbb{N}}$. Our first result is the following.

Theorem 1.1. *Let* $\mathbf{a} = (a, b) \in \mathbb{R}^2$ *so that* $a > 0$ *and* $b \geq 0$. *Let* $\Phi = (\log \phi_n)_{n=1}^\infty$ *be an asymptotically sub-additive potential on* X, *i.e.* $\Phi \in \mathcal{C}_{ass}(X, T)$. *Define a sequence* $\Psi = (\log \psi_n)_{n=1}^\infty$ *of functions on* Y *by*

$$\psi_n(y) = \sum_{I \in \mathcal{A}^n :\ [I] \cap \pi^{-1}(y) \neq \emptyset} \ \sup_{x \in [I] \cap \pi^{-1}(y)} \phi_n(x)^{\frac{1}{a}}, \quad y \in Y.$$

Set $\frac{a}{a+b}\Psi = \left(\log\left(\psi_n^{\frac{a}{a+b}}\right)\right)_{n=1}^\infty$. *Then* Ψ *and* $\frac{a}{a+b}\Psi$ *are in* $\mathcal{C}_{ass}(Y, S)$, *moreover*

$$\begin{aligned} P^{\mathbf{a}}(T, \Phi) &= (a+b) P\left(S, \frac{a}{a+b}\Psi\right) \\ &= \lim_{n\to\infty} \frac{a+b}{n} \log \sum_{J \in \mathcal{D}^n} \sup_{y \in [J] \cap Y} \psi_n(y)^{\frac{a}{a+b}}. \end{aligned} \tag{1.4}$$

Furthermore, $\mu \in \mathcal{M}(X, T)$ *is an* $\mathbf{a}$-*weighted equilibrium state of* Φ *if and only if* $\nu = \mu \circ \pi^{-1}$ *is an equilibrium state of* $\frac{a}{a+b}\Psi$ *and,* μ *is a conditional equilibrium state of* $\frac{1}{a}\Phi$ *with respect to* ν, *where* $\frac{1}{a}\Phi$ *denotes the potential* $(\log(\phi_n^{1/a}))_{n=1}^\infty$.

Formula (1.4) can be viewed as a kind of weighted variational principle. To further study weighted equilibrium states, we shall put more assumptions on Φ. We say that $\Phi = (\log \phi_n)_{n=1}^{\infty}$ is *almost additive* if ϕ_n is positive and continuous on X for each n and there is a constant $c > 0$ such that

$$\frac{1}{c}\phi_n(x)\phi_m(T^n x) \le \phi_{n+m}(x) \le c\phi_n(x)\phi_m(T^n x), \quad \forall\, x \in X,\ n, m \in \mathbb{N}.$$

For convenience, we denote by $\mathcal{C}_{aa}(X, T)$ the collection of almost additive potentials on X. Furthermore we say that Φ has the *bounded distortion property* if there exists a constant $c > 0$ such that

$$\frac{1}{c}\phi_n(y) \le \phi_n(x) \le c\phi_n(y) \quad \text{whenever } x, y \in X \text{ are in the same } n\text{-th cylinder.} \tag{1.5}$$

Following [11], a full supported Borel probability measure μ on $\mathcal{A}^{\mathbb{N}}$ is called to be *quasi-Bernoulli* if there exists a constant $c > 0$ such that

$$c^{-1} \le \frac{\mu(IJ)}{\mu(I)\mu(J)} \le c, \qquad \forall\, I, J \in \mathcal{A}^* := \bigcup_{i=1}^{\infty} \mathcal{A}^n, \tag{1.6}$$

here and afterwards, we use $\mu(I)$ to denote $\mu([I])$ for $I \in \mathcal{A}^*$, if there is no confusion. For two families $\{a_i\}_{i\in\mathcal{I}}$, $\{b_i\}_{i\in\mathcal{I}}$ of non-negative numbers, we write $a_i \approx b_i$ if there exists $c > 0$ such that $(1/c)b_i \le a_i \le cb_i$ for all $i \in \mathcal{I}$. Our next result is the following.

Theorem 1.2. *Assume that $X = \mathcal{A}^{\mathbb{N}}$ and $Y = \mathcal{D}^{\mathbb{N}}$ are two full shifts and $\pi : X \to Y$ is a one-block factor map. Let $\mathbf{a} = (a, b) \in \mathbb{R}^2$ so that $a > 0$ and $b \ge 0$. Let $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathcal{C}_{aa}(X, T)$. Assume that Φ satisfies the bounded distortion property. Then Φ has a unique $\mathbf{a}$-weighted equilibrium state, denoted as μ. The measure μ is quasi-Bernoulli and has the following Gibbs property:*

$$\mu(I) \approx \exp\left(\frac{-nP^{\mathbf{a}}(T, \Phi)}{a+b}\right) \frac{\phi(I)^{1/a}}{\psi(\pi I)^{b/(a+b)}}, \quad I \in \mathcal{A}^n,\ n \in \mathbb{N}, \tag{1.7}$$

where

$$\phi(I) := \sup_{x\in[I]} \phi_n(x) \text{ for } I \in \mathcal{A}^n \quad \text{and} \quad \psi(J) := \sum_{I\in\mathcal{A}^n:\, \pi I = J} \phi(I)^{1/a} \text{ for } J \in \mathcal{D}^n.$$

Furthermore for $\nu := \mu \circ \pi^{-1}$, we have

$$\nu(J) \approx \exp\left(\frac{-nP^{\mathbf{a}}(T, \Phi)}{a+b}\right) \psi(J)^{\frac{a}{a+b}}, \quad J \in \mathcal{D}^n,\ n \in \mathbb{N} \tag{1.8}$$

and

$$\mu(I)^a \nu(\pi I)^b \approx \phi(I) \exp(-nP^{\mathbf{a}}(T, \Phi)), \quad I \in \mathcal{A}^n,\ n \in \mathbb{N}. \tag{1.9}$$

A probability measure μ (not necessarily to be T-invariant) on X is called an $\mathbf{a}$-*weighted Gibbs measure*, if there exists $\Phi \in \mathcal{C}_{aa}(X, T)$ satisfying the bounded distortion property so that (1.7) holds for μ. Clearly, any $\mathbf{a}$-weighted Gibbs measure is quasi-Bernoulli. As an application of Theorem 1.2, we have the following result regarding the regularity property of $P^{\mathbf{a}}(T, \cdot)$.

Theorem 1.3. *Under the assumptions of Theorem 1.2, let* $\Phi_1, \ldots, \Phi_d \in \mathcal{C}_{aa}(X,T)$ *satisfy the bounded distortion property. Then the map* $Q : \mathbb{R}^d \to \mathbb{R}$ *defined as*

$$\mathbf{q} = (q_1, \ldots, q_d) \mapsto P^{\mathbf{a}}\left(T, \sum_{i=1}^{d} q_i \Phi_i\right),$$

is C^1 *over* $\mathbb{R}^d$ *with*

$$\nabla Q(q_1, \ldots, q_d) = ((\Phi_1)_*(\mu_{\mathbf{q}}), \ldots, (\Phi_d)_*(\mu_{\mathbf{q}})),$$

where ∇ *denotes the gradient and* $\mu_{\mathbf{q}}$ *is the unique* $\mathbf{a}$*-weighted equilibrium state of* $\sum_{i=1}^{d} q_i \Phi_i$.

Using Theorems 1.2 and 1.3, we derive the following two results, which play key roles in the multifractal analysis on self-affine symbolic spaces, and are of independent interest.

Theorem 1.4. *Assume that* $X = \mathcal{A}^{\mathbb{N}}$ *and* $Y = \mathcal{D}^{\mathbb{N}}$ *are two full shifts and* $\pi : X \to Y$ *is a one-block factor map. Let* $\mathbf{a} = (a, b) \in \mathbb{R}^2$ *so that* $a > 0$ *and* $b \geq 0$. *Then for each fully supported measure* $\eta \in \mathcal{M}(X,T)$ *and each* $n \in \mathbb{N}$, *there is a unique measure* $\mu = \mu(\mathbf{a}, \eta, n)$ *in* $\mathcal{M}(X,T)$ *attaining the following supremum*

$$\sup\{a h_\mu(T) + b h_{\mu\circ\pi^{-1}}(S) : \ \mu(I) = \eta(I) \text{ for all } \omega \in \mathcal{A}^n\}.$$

Furthermore $\mu(\mathbf{a}, \eta, n)$ *is the* $\mathbf{a}$*-weighted equilibrium state of certain* $\Phi \in \mathcal{C}_{aa}(X,T)$ *with the bounded distortion property, and hence* $\mu(\mathbf{a}, \eta, n)$ *is a fully supported quasi-Bernoulli measure.*

Theorem 1.5. *Under the condition of Theorem 1.4, for any* $\eta \in \mathcal{M}(X,T)$, *there exists a sequence of* $\mathbf{a}$*-weighted Gibbs measures* $(\mu_n)_{n=1}^{\infty} \subset \mathcal{M}(X,T)$ *converging to* η *in the weak-star topology such that*

$$a h_{\mu_n}(T) + b h_{\mu_n\circ\pi^{-1}}(S) \geq a h_\eta(T) + b h_{\eta\circ\pi^{-1}}(S).$$

Furthermore,

$$\lim_{n\to\infty} a h_{\mu_n}(T) + b h_{\mu_n\circ\pi^{-1}}(S) = a h_\eta(T) + b h_{\eta\circ\pi^{-1}}(S).$$

Remark 1.6. If we take $\mathbf{a} = (1,1)$, due to the upper semi-continuity of the entropy, for any $\mu \in \mathcal{M}(X,T)$, Theorem 1.5 yields a sequence of quasi-Bernoulli measures $(\mu_n)_{n=1}^{\infty}$ which converges to μ in the weak-star topology, such that we have both $\lim_{n\to\infty} h_{\mu_n}(T) = h_\mu(T)$ and $\lim_{n\to\infty} h_{\mu_n\circ\pi^{-1}}(S) = h_{\mu\circ\pi^{-1}}(S)$. Moreover, one can deduce from Theorem 1.2 that for any $\mathbf{a} = (a,b)$ with $a > 0$ and $b \geq 0$, each invariant quasi-Bernoulli measure is the $\mathbf{a}$-weighted equilibrium state of some almost additive potential satisfying the bounded distortion property.

Now we present our results about the multifractal analysis on self-affine symbolic spaces. In the remainder part of the section, we always assume that $X = \mathcal{A}^{\mathbb{N}}$ and $Y = \mathcal{D}^{\mathbb{N}}$ are

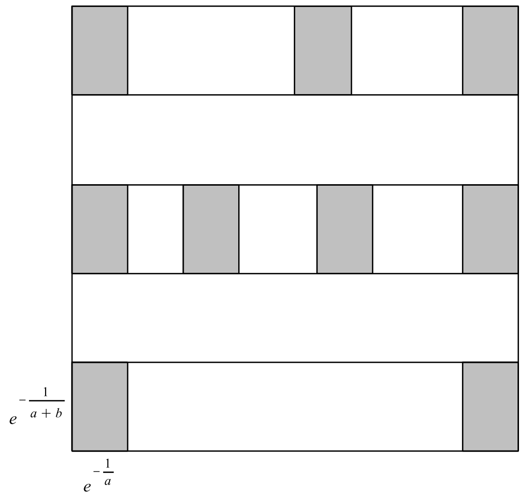

FIGURE 1

two full shifts and $\pi:\ X\to Y$ is a one-block factor map. Endow X with a metric $d_{\mathbf{a}}$ as follows:

$$d_{\mathbf{a}}(x,y)=\max\left(e^{-|x\wedge y|/a},\ e^{-|\pi x\wedge \pi y|/(a+b)}\right),$$

where $|x\wedge y|=\inf\{k\geq 1:\ x_k\neq y_k\}-1$ and $|\pi x\wedge \pi y|=\inf\{k\geq 1:\ \pi x_k\neq \pi y_k\}-1$. The space X, endowed with the metric $d_{\mathbf{a}}$, is called a *self-affine full shift*. Indeed if

$$e^{-\frac{1}{a}}\cdot\sup_{j\in\mathcal{D}}\#\pi^{-1}\{j\}<1 \quad\text{and}\quad e^{-\frac{1}{a+b}}\cdot\#\mathcal{D}<1, \tag{1.10}$$

the space $(X,d_{\mathbf{a}})$ is Lipschitz equivalent to a planar self-affine set generated by a linear iterated function system $\{S_i\}_{i\in\mathcal{A}}$ with

$$S_i(x,y)=\left(e^{-\frac{1}{a}}x+c_i,\ e^{-\frac{1}{a+b}}y+d_{\pi(i)}\right),\quad i\in\mathcal{A},$$

where $(c_i)_{i\in\mathcal{A}}$ and $(d_j)_{j\in\mathcal{D}}$ are chosen so that $S_i([0,1]^2)$'s are rectangles inside $[0,1]^2$ distributed as in Figure 1. Such sets belong to a broader class of self-affine sets studied by Lalley and Gatzouras in [31].

For $\mu\in\mathcal{M}(X,T)$, define the set of generic points of μ as

$$\mathcal{G}(\mu)=\left\{x\in X:\lim_{n\to\infty}\frac{S_ng(x)}{n}=\int g d\mu,\ \forall\, g\in C(X)\right\}, \tag{1.11}$$

where $C(X)$ denotes the collection of real continuous functions on X, and $S_ng(x)=\sum_{i=0}^{n-1}g(T^ix)$.

At first, we deal with the Hausdorff dimension of the sets of generic points of invariant measures. When $\mathbf{a}=(1,0)$, this result is well known (cf. [8, 12, 39, 18]).

Theorem 1.7. *Let $\mu\in\mathcal{M}(X,T)$. We have $\mathcal{G}(\mu)\neq\emptyset$ and $\dim_H\mathcal{G}(\mu)=ah_\mu(T)+bh_{\mu\circ\pi^{-1}}(S)$.*

Next we consider the level sets for Birkhoff averages of asymptotically additive potentials on X. Let $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_d) \in \mathcal{C}_{asa}(X,T)^d$, where $\Phi_i = (\log \phi_{n,i})_{n=1}^{\infty}$. For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, define

$$E_{\mathbf{\Phi}}(\alpha) = \big\{x \in X : \lim_{n\to\infty} \frac{1}{n} \log \phi_{n,i}(x) = \alpha_i \text{ for } 1 \le i \le d\big\}.$$

For $\mu \in \mathcal{M}(X,T)$, write

$$\mathbf{\Phi}_*(\mu) = ((\Phi_1)_*(\mu), \ldots, (\Phi_d)_*(\mu)).$$

Define

$$L_{\mathbf{\Phi}} = \{\mathbf{\Phi}_*(\mu) : \mu \in \mathcal{M}(X,T)\}.$$

Theorem 1.8. *For $\alpha \in \mathbb{R}^d$, $E_{\mathbf{\Phi}}(\alpha) \neq \emptyset$ if and only if $\alpha \in L_{\mathbf{\Phi}}$. Furthermore for $\alpha \in L_{\mathbf{\Phi}}$, we have*

$$\begin{aligned}\dim_H E_{\mathbf{\Phi}}(\alpha) &= \max\{a h_\mu(T) + b h_{\mu\circ\pi^{-1}}(S) : \ \mu \in \mathcal{M}(X,T),\ \mathbf{\Phi}_*(\mu) = \alpha\} \\ &= \inf\{P^{\mathbf{a}}(T, \mathbf{q}\cdot\mathbf{\Phi}) - \alpha\cdot\mathbf{q} : \ \mathbf{q} \in \mathbb{R}^d\},\end{aligned}$$

where $\mathbf{q}\cdot\mathbf{\Phi}$ denotes the potential $\sum_{i=1}^d q_i\Phi_i$ for $\mathbf{q} = (q_1, \ldots, q_d)$, and $\alpha\cdot\mathbf{q}$ denotes the standard inner product of α and $\mathbf{q}$. Moreover, if L_Φ is not reduced to a singleton, then $\{x \in X : \lim_{n\to\infty} \mathbf{\Phi}_n(x)/n$ does not exists$\}$ is of full Hausdorff dimension.

The above theorem can be extended in an elaborated way. Let $\mathbf{\Phi}^{(1)}, \mathbf{\Phi}^{(2)} \in \mathcal{C}_{asa}(X,T)^d$, where $\mathbf{\Phi}^{(j)} = \big(\Phi_1^{(j)}, \ldots, \Phi_d^{(j)}\big)$ with $\Phi_i^{(j)} = \big(\log \phi_{n,i}^{(j)}\big)_{n=1}^{\infty} \in \mathcal{C}_{asa}(X,T)$. Let $\mathbf{c} = (c_1, c_2) \in \mathbb{R}^2$, where $c_1, c_2 > 0$. Denote for $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$,

$$E_{\mathbf{\Phi}^{(1)},\mathbf{\Phi}^{(2)},\mathbf{c}}(\alpha) = \Big\{x \in X : \lim_{n\to\infty} \sum_{j=1}^{2} \frac{1}{\lfloor c_j n\rfloor} \log \phi^{(j)}_{\lfloor c_j n\rfloor, i}(x) = \alpha_i \text{ for } 1 \le i \le d\Big\},$$

where $\lfloor c_j n\rfloor$ denotes the integral part of $c_j n$.

Theorem 1.9. *Under the above setting, set $\mathbf{\Phi} = \sum_{j=1}^2 \mathbf{\Phi}^{(j)}$. Then for $\alpha \in \mathbb{R}^d$,*

$$E_{\mathbf{\Phi}^{(1)},\mathbf{\Phi}^{(2)},\mathbf{c}}(\alpha) \neq \emptyset \iff E_{\mathbf{\Phi}}(\alpha) \neq \emptyset \iff \alpha \in L_{\mathbf{\Phi}}.$$

Furthermore for $\alpha \in L_{\mathbf{\Phi}}$, we have

$$\begin{aligned}\dim_H E_{\mathbf{\Phi}^{(1)},\mathbf{\Phi}^{(2)},\mathbf{c}}(\alpha) &= \dim_H E_{\mathbf{\Phi}}(\alpha) \\ &= \max\{a h_\mu(T) + b h_{\mu\circ\pi^{-1}}(S) : \ \mu \in \mathcal{M}(X,T),\ \mathbf{\Phi}_*(\mu) = \alpha\} \\ &= \inf\{P^{\mathbf{a}}(T, \mathbf{q}\cdot\mathbf{\Phi}) - \alpha\cdot\mathbf{q} : \ \mathbf{q} \in \mathbb{R}^d\}.\end{aligned}$$

Moreover, if L_Φ is not reduced to a singleton, then $X \setminus \bigcup_{\alpha\in L_{\mathbf{\Phi}}} E_{\mathbf{\Phi}^{(1)},\mathbf{\Phi}^{(2)},\mathbf{c}}(\alpha)$ is of full Hausdorff dimension.

The level sets $E_{\mathbf{\Phi}^{(1)},\mathbf{\Phi}^{(2)},\mathbf{c}}(\alpha)$ do depend on $\mathbf{c}$ (see Example 5.7). However, by Theorem 1.9, $\dim_H E_{\mathbf{\Phi}^{(1)},\mathbf{\Phi}^{(2)},\mathbf{c}}(\alpha)$ does not depends on $\mathbf{c}$. It is quite interesting. As a natural application, we shall use Theorem 1.9 to study the multifractal analysis of certain measures on X. Let $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathcal{C}_{asa}(X,T)$. A probability measure μ is called an $\mathbf{a}$-*weighted* *weak Gibbs measure of* Φ if there exists a sequence $(\kappa_n)_{n=1}^\infty$ of positive numbers with $\lim_{n\to\infty}(1/n)\log\kappa_n = 0$, such that

$$A(I)/\kappa_n \le \mu(I) \le \kappa_n A(I), \quad I \in \mathcal{A}^n,$$

where $A(I) := \exp\left(\frac{-nP^{\mathbf{a}}(T,\Phi)}{a+b}\right)\frac{\phi(I)^{1/a}}{\psi(\pi I)^{b/(a+b)}}$ is the term in the right hand side of (1.7). We recover the usual weak Gibbs measures when $\mathbf{a} = (1, 0)$ and Φ is the sequence of Birkhoff sums associated with a continuous potential over X (cf. [49, 29]). Our last theorem is the following.

Theorem 1.10. *Let* $\Phi = (\log\phi_n)_{n=1}^\infty \in \mathcal{C}_{asa}(X,T)$. *Then there exists at least one* **a***-weighted weak Gibbs measure of* Φ. *Let* μ *be such a measure. For* $\alpha \ge 0$ *we define*

$$E_\mu(\alpha) = \Big\{x \in X : \lim_{r\to 0^+}\frac{\log\mu(B(x,r))}{\log r} = \alpha\Big\}.$$

Let $\Psi_1 = (\log\mu(x_{|n}))_{n=1}^\infty$, $\Psi_2 = (\log\mu\circ\pi^{-1}(\pi x_{|n}))_{n=1}^\infty$, *and* $\Psi = a\Psi_1 + b\Psi_2$, *where* $x_{|n} := x_1\dots x_n$ *for* $x = (x_i)_{i=1}^\infty \in X$. *Then* Ψ_1, Ψ_2 *and* Ψ *belong to* $\mathcal{C}_{asa}(X,T)$. *Furthermore, let* $L_\mu = L_{-\Psi} = \{-\Psi_*(\lambda) : \lambda \in \mathcal{M}(X,T)\}$. *Then, for all* $\alpha \ge 0$, $E_\mu(\alpha) \ne \emptyset$ *if and only if* $\alpha \in L_\mu$. *For* $\alpha \in L_\mu$, *we have*

$$\begin{aligned}\dim_H E_\mu(\alpha) &= \sup\{ah_\lambda(T) + bh_{\lambda\circ\pi^{-1}}(S) :\ \lambda \in \mathcal{M}(X,T),\ \Psi_*(\lambda) = -\alpha\}\\ &= \inf\{P^{\mathbf{a}}(T, q\Psi) + \alpha q :\ q \in \mathbb{R}\}.\end{aligned}$$

Remark 1.11. It is worth mentioning that the *concatenation of measures* play a crucial role in our geometric results. At first, the computations of Hausdorff dimensions are based on a kind of constructions of Moran measures obtained by the concatenation of quasi-Bernoulli measures. This method strongly depends on Theorem 1.5. In the classical case for which $b = 0$, one can construct either Moran measures by concatenating Markov measures (see e.g. [12]), or Moran sets directly (see for instance [17, 19]). This second approach seems not efficient when $b \ne 0$.

Also, the existence of (weighted) weak Gibbs measures for a given asymptotically additive potential Φ is obtained by concatenating (weighted) Gibbs measures associated with Hölder potentials converging to Φ.

Remark 1.12. (1) We mention that (1.4) is obtained independently in [48] for $\Phi = 0$.

(2) Theorem 1.2 has been partially extended in [21] to the case that X is a subshift satisfying specification. For example, the uniqueness of weighted equilibrium states is proved for almost additive potentials with the bounded distortion condition. This solves a question of Gaztouras and Peres about the uniqueness of invariant measures of maximizing weighted entropy (cf. [24, Problem 3]).

(3) Special cases of Theorems 1.8 and 1.10 have been obtained in [2] and [30, 2] respectively when $d = 1$ and under the bounded distortion assumption, except for the endpoints of the spectra which are not captured by the methods developed in these papers. Moreover, those methods cannot be extended to the case of general

almost additive potentials. Also, the results on multifractal analysis of Birkhoff averages and quasi-Bernoulli measures in those papers are not unified, while it is the case in the self-similar case $b = 0$. The weighted thermodynamic formalism introduced in this paper makes it possible to have a simple and unified presentation of the results concerning both questions.

(4) Reduced to the case $b = 0$, Theorems 1.8-1.9 cover the previous works on the multifractal analysis of almost additive potentials and related measures on symbolic spaces with the standard metric (see [40, 37, 5, 17, 19, 4] and references therein).

(5) Following the works achieved in [30, 36, 1] for almost additive potentials satisfying the bounded distortion property, it is possible to conduct the multifractal analysis of the projections of weak Gibbs measures on the planar self-affine sets described above when conditions (1.10) hold. We will not discuss such geometrical realizations in this paper.

(6) It is worth to point out that Falconer gave a variational formula for the Hausdorff dimension for "almost all" self-affine sets under some assumptions [15], and for this case Käenmäki showed the existence of ergodic measures of full Hausdorff dimension on the typical self-affine sets [27]. See [26] for a related result on the multifractal analysis.

The paper is organized as follows. Some definitions and known results on sub-additive thermodynamic formalism on subshifts are given in Section 2. The proofs of Theorems 1.1–1.5 on the weighted thermodynamic formalism are given in Section 3. In Section 4, we present the higher dimensional weighted thermodynamic formalism. Since the proofs of the result are very similar to those used in the 2-dimensional case, we omit them. Then, in Section 5 we present and prove the extensions to the higher dimensional case of Theorems 1.7–1.10. Indeed, for these results, the higher dimensional case is more involved, due to the upper bound estimates for Hausdorff dimensions.

2. Sub-additive thermodynamical formalism on subshifts

In this section, we present some definitions and known results about sub-additive thermodynamical formalism on subshifts.

2.1. **One-sided subshifts over finite alphabets.** Let $p \geq 2$ be an integer and $\mathcal{A} = \{1, \dots, p\}$. Denote

$$\mathcal{A}^{\mathbb{N}} = \{(x_i)_{i=1}^{\infty} : \ x_i \in \mathcal{A} \text{ for } i \geq 1\}.$$

Then $\mathcal{A}^{\mathbb{N}}$ is compact endowed with the product discrete topology ([33]). We say that (X, T) is a *subshift over* $\mathcal{A}$, if X is a compact subset of $\mathcal{A}^{\mathbb{N}}$ and $T(X) \subseteq X$, where T is the left shift map on $\mathcal{A}^{\mathbb{N}}$ defined as

$$T((x_i)_{i=1}^{\infty}) = (x_{i+1})_{i=1}^{\infty}, \quad \forall\ (x_i)_{i=1}^{\infty} \in \mathcal{A}^{\mathbb{N}}.$$

In particular, (X,T) is called the *full shift over* $\mathcal{A}$ if $X=\mathcal{A}^{\mathbb{N}}$. For any $n\in\mathbb{N}$ and $I\in\mathcal{A}^n$, we write

$$[I]=\{(x_i)_{i=1}^{\infty}\in\mathcal{A}^{\mathbb{N}}:\ x_1\dots x_n=I\}$$

and call it an *n-th cylinder* in $\mathcal{A}^{\mathbb{N}}$.

Let (X,T) and (Y,S) be two subshifts over finite alphabets $\mathcal{A}$ and $\mathcal{D}$, respectively. We say that Y is a *factor* of X, if there is a continuous surjective map $\pi:\ X\to Y$ such that $\pi T=S\pi$. Here π is called a *factor map*. Furthermore π is called a *one-block factor map* if there exists a map $\pi:\ \mathcal{A}\to\mathcal{D}$ such that

$$\pi\left((x_i)_{i=1}^{\infty}\right)=(\pi(x_i))_{i=1}^{\infty}, \qquad \forall\ (x_i)_{i=1}^{\infty}\in X.$$

It is well known (see, e.g. [33, Proposition 1.5.12]) that each factor map $\pi:\ X\to Y$ between two subshifts X and Y, will become a one-block factor map if we enlarge the alphabet for X and recode X appropriately.

2.2. **Sub-additive thermodynamical formalism.** For the reader's convenience we recall some definitions. Let (X,T) be a subshift over a finite alphabet $\mathcal{A}$. A sequence $\Phi=(\log\phi_n)_{n=1}^{\infty}$ is called a *sub-additive potential* on X and write $\Phi\in\mathcal{C}_s(X,T)$, if each ϕ_n is a non-negative continuous function on X and there exists $c>0$ such that

$$\phi_{n+m}(x)\le c\phi_n(x)\phi_m(T^nx), \quad \forall\ x\in X,\ n,m\in\mathbb{N}.$$

More generally, $\Phi=(\log\phi_n)_{n=1}^{\infty}$ is said to be an *asymptotically sub-additive potential* and write $\Phi\in\mathcal{C}_{ass}(X,T)$ if for any $\varepsilon>0$, there exists a sub-additive potential $\Psi=(\log\psi_n)_{n=1}^{\infty}$ on X such that

$$\limsup_{n\to\infty}\frac{1}{n}\sup_{x\in X}|\log\phi_n(x)-\log\psi_n(x)|\le\varepsilon,$$

where we take the convention $\log 0-\log 0=0$. Furthermore Φ is called an *asymptotically additive potential* and write $\Phi\in\mathcal{C}_{asa}(X,T)$ if both Φ and $-\Phi$ are asymptotically sub-additive, where $-\Phi$ denotes $(\log(1/\phi_n))_{n=1}^{\infty}$.

Let $\mathcal{M}(X,T)$ denote the set of T-invariant Borel probability measures on X endowed with the weak-star topology. For $\mu\in\mathcal{M}(X,T)$, let $h_\mu(T)$ denote the measure-theoretic entropy of μ with respect to T, and write

$$\Phi_*(\mu)=\lim_{n\to\infty}\frac{1}{n}\int_X\log\phi_n(x)\,d\mu(x). \tag{2.1}$$

The existence of the limit (which may take value $-\infty$) in (2.1) follows from the sub-additivity of Φ. The following lemma will be useful.

Lemma 2.1 ([22]). *Let $\Phi=(\log\phi_n)_{n=1}^{\infty}\in\mathcal{C}_{ass}(X,T)$. Then we have the following properties.*

(i) *Let* $\mu \in \mathcal{M}(X,T)$. *The limit* $\lambda_\Phi(x) := \lim_{n\to\infty} \frac{1}{n}\log\phi_n(x)$ *exists (which may take value* $-\infty$*) for* μ*-a.e.* $x \in X$, *and* $\int \lambda_\Phi(x)\, d\mu(x) = \Phi_*(\mu)$. *When* μ *is ergodic,* $\lambda_\Phi(x) = \Phi_*(\mu)$ *for* μ*-a.e.* $x \in X$.

(ii) *The map* $\Phi_* : \mathcal{M}(X,T) \to \mathbb{R}\cup\{-\infty\}$ *is upper semi-continuous, and there is* $C \in \mathbb{R}$ *such that for all* $\mu \in \mathcal{M}(X,T)$, $\lambda_\Phi(x) \le C$ μ*-a.e and* $\Phi_*(\mu) \le C$. *If* $\Phi \in \mathcal{C}_{asa}(X,T)$, Φ_* *is continuous on* $\mathcal{M}(X,T)$.

(iii) $\Phi \in \mathcal{C}_{asa}(X,T)$ *if and only if for any* $\varepsilon > 0$, *there exists a continuous function* g *on* X *such that*

$$\limsup_{n\to\infty} \frac{1}{n} \sup_{x\in X} |\log\phi_n(x) - S_n g(x)| \le \varepsilon,$$

where $S_n g(x) := \sum_{j=0}^{n-1} g(T^j x)$.

Remark 2.2. According to Lemma 2.1(iii), for $\mu \in \mathcal{M}(X,T)$, the set $\mathcal{G}(\mu)$ of generic points of μ defined as in (1.11) is just equal to

$$\left\{x \in X : \lim_{n\to\infty} \frac{\log\phi_n(x)}{n} = \Phi_*(\mu), \quad \forall\, \Phi = (\log\phi_n)_{n=1}^\infty \in \mathcal{C}_{asa}(X,T)\right\}.$$

For $\Phi = (\log\phi_n)_{n=1}^\infty \in \mathcal{C}_{ass}(X,T)$, and a compact set $K \subseteq X$, define

$$P_n(T,\Phi,K) = \sum_{I\in\mathcal{A}^n,\ [I]\cap K\neq\emptyset}\ \sup_{x\in[I]\cap K} \phi_n(x). \tag{2.2}$$

and

$$P(T,\Phi,K) = \limsup_{n\to\infty} \frac{1}{n}\log P_n(T,\Phi,K). \tag{2.3}$$

The following variational principle was proved in [13] when $\Phi \in \mathcal{C}_s(X,T)$. As pointed in [22], it holds also for $\Phi \in \mathcal{C}_{ass}(X,T)$.

Proposition 2.3. *Let* $P(T,\Phi,X)$ *be defined as above. Then for any* $\Phi \in \mathcal{C}_{ass}(X,T)$, *we have the following variational principle:*

$$P(T,\Phi,X) = \sup\{\Phi_*(\mu) + h_\mu(T) :\ \mu \in \mathcal{M}(X,T)\}. \tag{2.4}$$

We call $P(T,\Phi) := P(T,\Phi,X)$ the *topological pressure of* Φ.

Remark 2.4. When $\Phi = (\log\phi_n)_{n=1}^\infty$ is an additive potential, i.e.,

$$\phi_n(x) = \exp\left(\sum_{i=0}^{n-1} \phi(T^i x)\right)$$

for a continuous function ϕ on X, the above proposition comes to the Ruelle-Walters variational principle for additive topological pressures (see e.g. [42, 43, 45]).

We say that $\mu \in \mathcal{M}(X,T)$ is an *equilibrium state* of Φ if the supremum in (2.4) is attained at μ. Note that $\Phi_*(\cdot)$ is upper semi-continuous on $\mathcal{M}(X,T)$ (cf. Lemma 2.1(ii)), and so is $h_{(\cdot)}(T)$ for subshifts. Hence Φ has at least one equilibrium state. In the following, we consider the case when Φ has a unique equilibrium state.

We say that $\Phi = (\log \phi_n)_{n=1}^{\infty}$ is *almost additive* if ϕ_n is positive and continuous on X for each n and there is a constant $c > 0$ such that

$$\frac{1}{c}\phi_n(x)\phi_m(T^n x) \le \phi_{n+m}(x) \le c\phi_n(x)\phi_m(T^n x), \quad \forall\, x \in X,\ n, m \in \mathbb{N}.$$

For convenience, we denote by $\mathcal{C}_{aa}(X,T)$ the collection of almost-additive potentials on X. Clearly $\mathcal{C}_{aa}(X,T) \subset \mathcal{C}_{asa}(X,T)$.

For $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathcal{C}_{ass}(X,T)$, we say that Φ has the *bounded distortion property* if there exists a constant $c > 0$ such that

$$\frac{1}{c}\phi_n(y) \le \phi_n(x) \le c\phi_n(y) \quad \text{whenever } x, y \in X \text{ are in the same } n\text{-th cylinder.}$$

Proposition 2.5. *Let (X,T) be a full shift or mixing subshift of finite type. Let $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathcal{C}_{aa}(X,T)$. Assume that Φ has the bounded distortion property. Then Φ has a unique equilibrium state μ. Furthermore, there exists a constant $c > 0$ such that for any $n \in \mathbb{N}$ and $x = (x_i)_{i=1}^{\infty} \in X$,*

$$c^{-1} \le \frac{\mu([x_1 \dots x_n])}{\exp(-nP(T,\Phi))\ \phi_n(x)} \le c.$$

Proposition 2.5 was first proved in [23, 20] for special almost additive potentials given by

$$\phi_n(x) = \|M(x)M(Tx)\dots M(T^{n-1}x)\|, \quad n \in \mathbb{N},$$

where M is a Hölder continuous function taking values in the set of $d \times d$ positive matrices. It was completed into the present form by Barreira [3] and Mummert [35] independently. We remark that Proposition 2.5 extends the classical theory about equilibrium states for additive continuous potentials with the bounded distortion property (cf. Bowen [9]).

2.3. **Relativized sub-additive thermodynamic formalism.** Let $\pi : X \to Y$ be a one-block factor map between two subshifts (X,T) and (Y,S). The following relativized variational principle was proved in [47] for sub-additive potentials under a general random setting by using an idea in [13]. It does hold for $\Phi \in \mathcal{C}_{ass}(X,T)$ by modifying the proof in [47] slightly. This extends the relativized variational principle of Ledrappier and Walters [32] for additive potentials.

Proposition 2.6. *Let $\Phi \in \mathcal{C}_{ass}(X,T)$ and $\nu \in \mathcal{M}(Y,S)$. Then*

$$\sup\{\Phi_*(\mu) + h_\mu(T) - h_\nu(S)\} = \int_Y P(T,\Phi,\pi^{-1}(y))\, d\nu(y), \tag{2.5}$$

where the supremum is taken over the set of $\mu \in \mathcal{M}(X,T)$ such that $\mu \circ \pi^{-1} = \nu$, $P(T,\Phi,\pi^{-1}(y))$ is defined as in (2.3).

By the upper semi-continuity of $\Phi_*(\cdot)$ and $h_{(\cdot)}(T)$ on $\mathcal{M}(X,T)$, the supremum in (2.5) is attainable. Any measure $\mu \in \mathcal{M}(X,T)$ for which the supremum in (2.5) is attained at μ is called a *conditional equilibrium state of* Φ *with respect to* ν.

3. Weighted thermodynamic formalism

3.1. **The proof of Theorem 1.1.** Throughout this section, we assume that X is a subshift over $\mathcal{A}$, Y a subshift over $\mathcal{D}$ and $\pi: X \to Y$ a one-block factor map. The following lemma plays a key role in the proof of Theorem 1.1.

Lemma 3.1. *Let* $\Phi = (\log \phi_n(x))_{n=1}^{\infty} \in \mathcal{C}_{ass}(X,T)$ *and* $\nu \in \mathcal{M}(Y,S)$. *Then we have*

$$\sup\{\Phi_*(\mu) + h_\mu(T) - h_\nu(S) :\ \mu \in \mathcal{M}(X,T),\ \mu\circ\pi^{-1} = \nu\} = \Psi_*(\nu), \tag{3.1}$$

where $\Psi = (\log \psi_n)_{n=1}^{\infty} \in \mathcal{C}_{ass}(Y,S)$ *is defined by*

$$\psi_n(y) = \sum_{I\in\mathcal{A}^n:\ [I]\cap\pi^{-1}(y)\neq\emptyset} \ \sup_{x\in[I]\cap\pi^{-1}(y)} \phi_n(x).$$

Proof. By Proposition 2.6, the left-hand side of (3.1) equals $\int P(T,\Phi,\pi^{-1}(y))\, d\nu(y)$. However by (2.3)-(2.2),

$$P(T,\Phi,\pi^{-1}(y)) = \limsup_{n\to\infty} \frac{1}{n} \log P_n(T,\Phi,\pi^{-1}(y))$$

and

$$P_n(T,\Phi,\pi^{-1}(y)) = \sum_{I\in\mathcal{A}^n:\ [I]\cap\pi^{-1}(y)} \ \sup_{x\in[I]\cap\pi^{-1}(y)} \phi_n(x).$$

Clearly $\psi_n(y) = P_n(T,\Phi,\pi^{-1}(y))$. It is direct to check that $\Psi = (\log\psi_n)_{n=1}^{\infty} \in \mathcal{C}_{ass}(Y,S)$. Hence by Lemma 2.1,

$$\Psi_*(\nu) = \int \limsup_{n\to\infty} \frac{1}{n}\log\psi_n(y)\, d\nu(y) = \int P(T,\Phi,\pi^{-1}(y))\, d\nu(y).$$

This finishes the proof of the lemma. □

Proof of Theorem 1.1. Clearly we have

$$\begin{aligned}
&\sup\{\Phi_*(\mu) + a h_\mu(T) + b h_{\mu\circ\pi^{-1}}(S) :\ \mu \in \mathcal{M}(X,T)\} \\
&= \sup\{\Phi_*(\mu) + a h_\mu(T) + b h_\nu(S) :\ \nu\in\mathcal{M}(Y,S),\ \mu \in \mathcal{M}(X,T),\ \mu\circ\pi^{-1} = \nu\} \\
&= \sup\{A(\nu) + (a+b) h_\nu(S) :\ \nu \in \mathcal{M}(Y,S)\},
\end{aligned} \tag{3.2}$$

where $A(\nu) := a \sup\{\frac{1}{a}\Phi_*(\mu) + h_\mu(T) - h_\nu(S) :\ \mu\in\mathcal{M}(X,T),\ \mu\circ\pi^{-1} = \nu\}$.

By Lemma 3.1, we have $A(\nu) = a\Psi_*(\nu)$, where $\Psi = (\log\psi_n)_{n=1}^{\infty} \in \mathcal{C}_{ass}(Y,S)$ is defined as

$$\psi_n(y) = \sum_{I\in\mathcal{A}^n:\ [I]\cap\pi^{-1}(y)\neq\emptyset} \ \sup_{x\in[I]\cap\pi^{-1}(y)} \phi_n(x)^{1/a}.$$

Hence by (3.2) and Proposition 2.3, we have

$$
\begin{aligned}
&\sup\{\Phi_*(\mu) + ah_\mu(T) + bh_{\mu\circ\pi^{-1}}(S) :\ \mu \in \mathcal{M}(X,T)\} \\
&\quad = \sup\{a\Psi_*(\nu) + (a+b)h_\nu(S) :\ \nu \in \mathcal{M}(Y,S)\} \\
&\quad = (a+b)\sup\left\{\frac{a}{a+b}\Psi_*(\nu) + h_\nu(S) :\ \nu \in \mathcal{M}(Y,S)\right\} \qquad (3.3)\\
&\quad = (a+b)P\left(S, \frac{a}{a+b}\Psi\right) = \lim_{n\to\infty} \frac{a+b}{n} \log \sum_{J\in\mathcal{D}^n} \sup_{y\in[J]\cap Y} \psi_n(y)^{\frac{a}{a+b}}.
\end{aligned}
$$

This proves the first part of Theorem 1.1. The second part follows directly from (3.3) and (3.2). □

3.2. **The proof of Theorem 1.2.** Throughout this section, we assume that $X = \mathcal{A}^{\mathbb{N}}$ and $Y = \mathcal{D}^{\mathbb{N}}$ are two full shifts over finite alphabets, and $\pi :\ X \to Y$ is a one-block factor map. To prove Theorem 1.2, we need some auxiliary results.

Lemma 3.2. *Assume that $\Phi \in \mathcal{C}_{aa}(X,T)$ and that Φ satisfies the bounded distortion property. Let $\nu \in \mathcal{M}(Y,S)$. Then $\int_Y P(T,\Phi,\pi^{-1}(y))\, d\nu(y) = \Psi_*(\nu)$, where $\Psi = (\log\psi_n)_{n=1}^\infty \in \mathcal{C}_{aa}(Y,S)$ is given by*

$$
\psi_n(y) = \sum_{I\in\mathcal{A}^n:\ \pi I = y_1\dots y_n} \sup_{x\in[I]} \phi_n(x), \quad \forall\, y = (y_i)_{i=1}^\infty \in Y.
$$

Furthermore

$$
\sup\{\Phi_*(\mu) + h_\mu(T) - h_\nu(S)\} = \Psi_*(\nu),
$$

where the supremum is taken over the set of $\mu \in \mathcal{M}(X,T)$ such that $\mu\circ\pi^{-1} = \nu$.

Proof. It follows directly from Lemma 3.1 and the bounded distortion property of Φ. □

Proposition 3.3. *Assume that $\Phi \in \mathcal{C}_{aa}(X,T)$ and Φ satisfies the bounded distortion property. Let $\nu \in \mathcal{M}(Y,S)$ so that ν has the quasi-Bernoulli property. Then Φ has a unique conditional equilibrium state μ with respect to ν. Furthermore there is a constant $c > 0$ such that*

$$
c^{-1} \le \frac{\mu(I)}{\nu(\pi I)\phi(I)/\psi(\pi I)} \le c, \quad \forall\, n\in\mathbb{N},\ I\in\mathcal{A}^n,\ J\in\mathcal{D}^n, \qquad (3.4)
$$

where $\phi(I) := \sup_{x\in[I]}\phi_n(x)$ for $I \in \mathcal{A}^n$ and $\psi(J) := \sum_{I\in\mathcal{A}^n:\ \pi I = J}\phi(I)$ for $J \in \mathcal{D}^n$.

Proof. We first construct $\mu \in \mathcal{M}(X,T)$ such that $\mu\circ\pi^{-1} = \nu$ and μ satisfies (3.4). Here we adopt an idea from [23]. Since $\Phi \in \mathcal{C}_{aa}(X,T)$ and Φ satisfies the bounded distortion property, it is direct to check that ϕ and ψ are quasi-Bernoulli in the sense that

$$
\phi(I_1I_2) \approx \phi(I_1)\phi(I_2), \quad I_1, I_2 \in \mathcal{A}^* = \bigcup_{n\ge1}\mathcal{A}^n,
$$

and

$$
\psi(J_1J_2) \approx \psi(J_1)\psi(J_2), \quad J_1, J_2 \in \mathcal{D}^* = \bigcup_{n\ge1}\mathcal{D}^n,
$$

where for two families of positive numbers (a_n) and (b_n), we write $(a_n) \approx (b_n)$ if a_n/b_n is bounded from below and above by some positive constants.

For each integer $n > 0$, let $\mathcal{B}_n$ be the σ-algebra generated by the cylinders $[I]$ in X, $I \in \mathcal{A}^n$. We define a sequence of probability measures $(\mu_n)_{n=1}^{\infty}$ on $\mathcal{B}_n$ by

$$\mu_n(I) = \nu(\pi I)\phi(I)/\psi(\pi I), \qquad \forall\ I \in \mathcal{A}^n.$$

Then there is a subsequence $(\mu_{n_k})_{k\geq 1}$ converging in the weak-star topology to a probability measure $\widetilde{\mu}$. We claim that $\widetilde{\mu}$ satisfies (3.4). To see this, for any $I \in \mathcal{A}^n$ and $p > n$, we have

$$\begin{aligned} \mu_p(I) &= \sum_{I_1 \in \mathcal{A}^{p-n}} \mu_p(II_1) = \sum_{I_1 \in \mathcal{A}^{p-n}} \nu(\pi(II_1))\phi(II_1)/\psi(\pi(II_1)) \\ &\approx \frac{\nu([\pi I])\phi(I)}{\psi(\pi I)} \sum_{I_1 \in \mathcal{A}^{p-n}} \frac{\nu(\pi I_1)\phi(I_1)}{\psi(\pi I_1)} = \nu(\pi I)\phi(I)/\psi(\pi I). \end{aligned}$$

Letting $p = n_k \uparrow \infty$, we obtain $\widetilde{\mu}(I) \approx \nu(\pi I)\phi(I)/\psi(\pi I)$, as desired.

Let μ be a limit point of the sequence $\frac{1}{n}\left(\widetilde{\mu} + \widetilde{\mu}\circ T^{-1} + \ldots + \widetilde{\mu}\circ T^{-(n-1)}\right)$ in the weak-star topology. Then $\mu \in \mathcal{M}(X,T)$ (cf. [46, Theorem 6.9]). Note that for any $I \in \mathcal{A}^n$ and $p \geq 0$,

$$\begin{aligned} \widetilde{\mu}\circ T^{-p}(I) &= \sum_{I_1 \in \mathcal{A}^{p}} \widetilde{\mu}(I_1 I) \approx \sum_{I_1 \in \mathcal{A}^{p}} \nu(\pi(I_1 I))\phi(I_1 I)/\psi(\pi(I_1 I)) \\ &\approx \frac{\nu(\pi I)\phi(I)}{\psi(\pi I)} \sum_{I_1 \in \mathcal{A}^{p}} \frac{\nu(\pi I_1)\phi(I_1)}{\psi(\pi I_1)} = \nu(\pi I)\phi(I)/\psi(\pi I). \end{aligned}$$

Hence we have $\mu(I) \approx \nu(\pi I)\phi(I)/\psi(\pi I)$. It is clear that μ is quasi-Bernoulli. Hence μ is ergodic (cf. [46, Theorem 1.5(iv)]). Also, by construction, we have $\mu\circ\pi^{-1}(\pi I) \approx \nu(\pi I)$ ($I \in \bigcup_{n\geq 1} \mathcal{A}^n$). Since both $\mu\circ\pi^{-1}$ and ν are ergodic, we have $\mu\circ\pi^{-1} = \nu$.

Next we show that μ is a conditional equilibrium state of Φ with respect to ν. Write for $n \in \mathbb{N}$,

$$t_n = -\left(\sum_{I\in\mathcal{A}^n} \mu(I)\log\mu(I)\right) + \left(\sum_{J\in\mathcal{D}^n} \nu(J)\log\nu(J)\right).$$

Then $(t_n)_{n\geq 1}$ is sub-additive in the sense that $t_{n+m} \leq t_n + t_m$ for any $n, m \in \mathbb{N}$ (cf. [14, Lemma 1]), and hence

$$h_\mu(T) - h_\nu(S) = \lim_{n\to\infty} t_n/n = \inf_{n\in\mathbb{N}} t_n/n. \tag{3.5}$$

For two families of real numbers $\{a_i\}_{i\in\mathcal{I}}$ and $\{b_i\}_{i\in\mathcal{I}}$, we write $a_i = b_i + O(1)$ if there is a constant $c > 0$ such that $|a_i - b_i| \leq c$ for each $i \in \mathcal{I}$. By the quasi-Bernoulli property of

μ and ν, we have

$$\begin{aligned}
&\int \log \phi_n(x)\, d\mu(x) + t_n \\
&= O(1) + \left(\sum_{I\in\mathcal{A}^n} \mu(I)\log\phi(I) - \mu(I)\log\mu(I)\right) \\
&\qquad\qquad\qquad + \sum_{J\in\mathcal{D}^n} \nu(J)\log\nu(J) \\
&= O(1) + \sum_{J\in\mathcal{D}^n} \nu(J) \sum_{I\in\mathcal{A}^n\colon \pi I = J} \frac{\mu(I)}{\nu(J)} \log \frac{\phi(I)\nu(J)}{\mu(I)} \\
&= O(1) + \sum_{J\in\mathcal{D}^n} \nu(J) \sum_{I\in\mathcal{A}^n\colon \pi I = J} \frac{\mu(I)}{\nu(J)} \log \psi(J) \quad \text{(by (3.4))} \\
&= O(1) + \sum_{J\in\mathcal{D}^n} \nu(J)\log\psi(J),
\end{aligned}$$

Dividing both sides by n and letting $n \to \infty$, we obtain

$$\Phi_*(\mu) + h_\mu(T) - h_\nu(S) = \Psi_*(\nu).$$

Hence by Lemma 3.2, μ is a conditional equilibrium state of Φ with respect to ν.

In the end, we prove that μ is the unique conditional equilibrium state of Φ with respect to ν. Here we adopt an idea due to Bowen (cf. [9, p. 34–36]). Assume that $\mu' \neq \mu$ is another conditional equilibrium state of Φ with respect to ν. That is, $\mu' \circ \pi^{-1} = \nu$ and

$$\Phi_*(\mu') + h_{\mu'}(T) - h_\nu(S) = \Psi_*(\nu). \tag{3.6}$$

Without loss of generality we may assume that μ' is ergodic (otherwise, we may consider the ergodic decomposition of μ'). Then μ' and μ are totally singular to each other. Hence for each $\varepsilon > 0$ and sufficiently large n, there exists a set F_n which is the union of some n-th cylinders in X, such that

$$\mu(F_n) < \varepsilon \quad \text{and} \quad \mu'(F_n) > 1 - \varepsilon. \tag{3.7}$$

It is direct to check that

$$\left| n\Psi_*(\nu) - \sum_{J\in\mathcal{D}^n} \nu(J)\log\psi(J) \right| = O(1) \quad \text{and}$$

$$\text{for } \lambda \in \{\mu, \mu'\},\ \left| n\Phi_*(\lambda) - \sum_{I\in\mathcal{A}^n} \lambda(I)\log\phi(I) \right| = O(1).$$

Hence for $\lambda \in \{\mu, \mu'\}$ we have

$$\begin{aligned}
n\Phi_*(\lambda) &= \sum_{I\in\mathcal{A}^n} \lambda(I)\log\phi(I) + O(1)\\
&= \sum_{I\in\mathcal{A}^n} \lambda(I)\log\frac{\mu(I)\psi(\pi I)}{\nu(\pi I)} + O(1)\\
&= \left(\sum_{I\in\mathcal{A}^n} \lambda(I)\log\mu(I)\right) + \left(\sum_{J\in\mathcal{D}^n} \nu(J)\log\frac{\psi(J)}{\nu(J)}\right) + O(1)\\
&= \left(\sum_{I\in\mathcal{A}^n} \lambda(I)\log\mu(I)\right) - \left(\sum_{J\in\mathcal{D}^n} \nu(J)\log\nu(J)\right) + n\Psi_*(\nu) + O(1).
\end{aligned}$$

Hence, by (3.6) and applying (3.5) to μ' we have

$$\begin{aligned}
0 &\le n\Phi_*(\mu') - \left(\sum_{I\in\mathcal{A}^n} \mu'(I)\log\mu'(I)\right) + \left(\sum_{J\in\mathcal{D}^n} \nu(J)\log\nu(J)\right) - n\Psi_*(\nu)\\
&= \sum_{I\in\mathcal{A}^n} \Big[-\mu'(I)\log\mu'(I) + \mu'(I)\log\mu(I)\Big] + O(1)\\
&= \sum_{[I]\subset F_n} \Big[-\mu'(I)\log\mu'(I) + \mu'(I)\log\mu(I)\Big]\\
&\quad + \sum_{[I]\subset X\setminus F_n} \Big[-\mu'(I)\log\mu'(I) + \mu'(I)\log\mu(I)\Big] + O(1)\\
&\le \mu'(F_n)\log\mu(F_n) + \mu'(X\setminus F_n)\log\mu(X\setminus F_n) + 2\sup_{0\le s\le 1}(-s\log s) + O(1),
\end{aligned}$$

where for the last inequality, we use the elementary inequality (cf. [9, Lemma 1.24])

$$\sum_{i=1}^k(-p_i\log p_i + p_i\log a_i) \le s\log\sum_{i=1}^k a_i - s\log s, \quad s := \sum_{i=1}^k p_i,\ p_i\ge 0.$$

It leads to a contradiction since by (3.7), $\mu'(F_n)\log\mu(F_n)\to-\infty$ as $\varepsilon\to 0$. This finishes the proof of Proposition 3.3. □

Proof of Theorem 1.2. Assume that $\Phi = (\log\phi_n)_{n=1}^\infty \in \mathcal{C}_{aa}(X,T)$ satisfies the bounded distortion property. Let $\mathbf{a} = (a,b)\in\mathbb{R}^2$ so that $a>0$ and $b\ge 0$.

Write $\phi(I) = \sup_{x\in[I]}\phi_n(x)$ for $I\in\mathcal{A}^n$ and $\psi(J) = \sup_{I\in\mathcal{A}^n:\,\pi I=J}\phi(I)^{1/a}$ for $J\in\mathcal{D}^n$. Define $\Psi = (\log\psi_n)_{n=1}^\infty$ by $\psi_n(y) = \psi(y_1\dots y_n)$. By the assumption on Φ, we have $\Psi\in\mathcal{C}_{aa}(Y,S)$. By Theorem 1.1 we have

$$P^{\mathbf{a}}(T,\Phi) = (a+b)P\Big(S, \frac{a}{a+b}\Psi\Big) = \lim_{n\to\infty}\frac{a+b}{n}\log\sum_{J\in\mathcal{D}^n}\Big(\sum_{I\in\mathcal{A}^n:\,\pi I=J}\phi(I)^{\frac{1}{a}}\Big)^{\frac{a}{a+b}}.$$

Let μ be an $\mathbf{a}$-weighted equilibrium state of Φ and $\nu = \mu\circ\pi^{-1}$. By Theorem 1.1, ν is an equilibrium state of $\frac{a}{a+b}\Psi$ and μ is a conditional equilibrium state of $\frac{1}{a}\Phi$ with respect to

ν. Since $\frac{a}{a+b}\Psi \in \mathcal{C}_{aa}(Y,S)$ and satisfies the bounded distortion property, by Proposition 2.5, ν is unique and it satisfies the Gibbs property:

$$\nu(J) \approx \exp\left(-nP\left(S, \frac{a}{a+b}\Psi\right)\right)\psi(J)^{\frac{a}{a+b}} = \exp\left(\frac{-nP^{\mathbf{a}}(T,\Phi)}{a+b}\right)\psi(J)^{\frac{a}{a+b}}$$

for $n \in \mathbb{N}$ and $J \in \mathcal{D}^n$. This proves (1.8). Since ν is quasi-Bernoulli, applying Proposition 3.3 to the potential $\frac{1}{a}\Phi$, we see that μ is unique and satisfies the Gibbs property:

$$\mu(I) \approx \phi(I)^{\frac{1}{a}}\nu(\pi I)/\psi(\pi I) \approx \exp\left(\frac{-nP^{\mathbf{a}}(T,\Phi)}{a+b}\right)\frac{\phi(I)^{1/a}}{\psi(\pi I)^{b/(a+b)}}$$

for $n \in \mathbb{N}$ and $I \in \mathcal{A}^n$. This proves (1.7). Note that (1.9) follows directly from (1.7) and (1.8). This finishes the proof of Theorem 1.2. □

3.3. **The proof of Theorem 1.3.** To prove Theorem 1.3, we need the following result which is just based on classical convex analysis.

Proposition 3.4 ([22], Proposition 2.3)**.** *Let Z be a compact convex subset of a topological vector space which satisfies the first axiom of countability (i.e., there is a countable base at each point) and $U \subseteq \mathbb{R}^d$ a non-empty open set. Suppose $f : U \times Z \to \mathbb{R} \cup \{-\infty\}$ is a map satisfying the following conditions:*

(i) *$f(\mathbf{q}, z)$ is convex in $\mathbf{q}$;*
(ii) *$f(\mathbf{q}, z)$ is affine in z;*
(iii) *f is upper semi-continuous over $U \times Z$;*
(iv) *$g(\mathbf{q}) := \sup_{z\in Z} f(\mathbf{q}, z) > -\infty$ for any $\mathbf{q} \in U$.*

For each $\mathbf{q} \in U$, denote $\mathcal{I}(\mathbf{q}) := \{z \in Z : f(\mathbf{q}, z) = g(\mathbf{q})\}$. Then

$$\partial g(\mathbf{q}) = \bigcup_{z\in\mathcal{I}(\mathbf{q})} \partial f(\mathbf{q}, z),$$

where $\partial f(\mathbf{q}, z)$ denotes the subdifferential of $f(\cdot, z)$ at $\mathbf{q}$.

Proof of Theorem 1.3. In Proposition 3.4, we let $U = \mathbb{R}^d$, $Z = \mathcal{M}(X,T)$, and define $f : U \times Z \to \mathbb{R}$ by

$$f(\mathbf{q}, \mu) = \sum_{i=1}^{d} q_i(\Phi_i)_*(\mu) + ah_\mu(T) + bh_{\mu\circ\pi^{-1}}(S), \qquad \mathbf{q} = (q_1, \ldots, q_d).$$

Set $g(\mathbf{q}) = \sup_{z\in Z} f(\mathbf{q}, z) = P^{\mathbf{a}}(T, \sum_{i=1}^d q_i\Phi_i)$. Since $\Phi_i \in \mathcal{C}_{aa}(X,T)$, $\mu \mapsto (\Phi_i)_*(\mu)$ is continuous on $\mathcal{M}(X,T)$ (see Lemma 2.1(ii)). Thus, f and g satisfy the assumptions (i)-(iv) in Proposition 3.4. However by Theorem 1.2, $\mathcal{I}(\mathbf{q}) = \{\mu_{\mathbf{q}}\}$ is a singleton for each $\mathbf{q} \in \mathbb{R}^d$. By Proposition 3.4, $\nabla g(\mathbf{q}) = ((\Phi_1)_*(\mu_{\mathbf{q}}), \ldots, (\Phi_d)_*(\mu_{\mathbf{q}}))$. Since g is convex and differentiable on $\mathbb{R}^d$, it is C^1 on $\mathbb{R}^d$ (see, e.g. [41, Corollary 25.5.1]). This finishes the proof of Theorem 1.3.

□

3.4. **The proofs of Theorems 1.4 and 1.5.**

Proof of Theorem 1.4. Fix $n \in \mathbb{N}$. Denote by Ω_n the collection of probability vectors $\mathbf{p} = (p(\omega))_{\omega \in \mathcal{A}^n}$ in $\mathbb{R}^{\mathcal{A}^n}$ satisfying

$$\sum_{\varepsilon \in \mathcal{A}} p(\varepsilon x_2 \dots x_n) = \sum_{\varepsilon \in \mathcal{A}} p(x_2 \dots x_n \varepsilon) \quad \text{for any word } x_2 \dots x_n \in \mathcal{A}^{n-1}.$$

It is clear that Ω_n is a convex compact subset of $\mathbb{R}^{\mathcal{A}^n}$. In fact, Ω_n is the image of the following map

$$\eta \in \mathcal{M}(X,T) \mapsto (\eta(I))_{I \in \mathcal{A}^n}.$$

(cf. [17, p. 232]). Define a function $f : \Omega_n \to \mathbb{R}$ by

$$f(\mathbf{p}) = \sup\{a h_\eta(T) + b h_{\eta\circ\pi^{-1}}(S) : \ \eta \in \mathcal{M}(X,T) : \ (\eta(I))_{I\in\mathcal{A}^n} = \mathbf{p}\}. \tag{3.8}$$

The following properties of f can be checked directly.

Lemma 3.5. *The map $f : \Omega_n \to \mathbb{R}$ is concave, bounded and upper semi-continuous.*

Extend f to a function on $\mathbb{R}^{\mathcal{A}^n}$ by

$$f(\mathbf{p}) = -\infty \quad \text{ for } \mathbf{p} \in \mathbb{R}^{\mathcal{A}^n} \backslash \Omega_n$$

and define $f^* : \mathbb{R}^{\mathcal{A}^n} \to \mathbb{R}$ by

$$f^*(\mathbf{q}) = \sup\left\{f(\mathbf{p}) + \mathbf{p}\cdot\mathbf{q} : \ \mathbf{p} \in \mathbb{R}^{\mathcal{A}^n}\right\} = \sup\left\{f(\mathbf{p}) + \mathbf{p}\cdot\mathbf{q} : \ \mathbf{p} \in \Omega_n\right\}, \tag{3.9}$$

where $\mathbf{p}\cdot\mathbf{q}$ denotes the standard inner product of $\mathbf{p}$ and $\mathbf{q}$ in $\mathbb{R}^{\mathcal{A}^n}$. Since f is a bounded upper semi-continuous concave function on Ω_n, we obtain

$$f(\mathbf{p}) = \inf\left\{f^*(\mathbf{q}) - \mathbf{p}\cdot\mathbf{q} : \ \mathbf{q} \in \mathbb{R}^{\mathcal{A}^n}\right\}, \qquad \mathbf{p} \in \Omega_n \tag{3.10}$$

by using the duality principle in convex analysis (cf. [41, Theorem 12.2]). By (3.8) and (3.9), we have

Lemma 3.6. *For $\mathbf{q} = (q(I))_{I\in\mathcal{A}^n} \in \mathbb{R}^{\mathcal{A}^n}$,*

$$\begin{aligned} f^*(\mathbf{q}) &= \sup\left\{\left(\sum_{I\in\mathcal{A}^n} q(I) \int \chi_{[I]}\, d\eta\right) + a h_\eta(T) + b h_{\eta\circ\pi^{-1}}(S) : \ \eta \in \mathcal{M}(X,T)\right\} \\ &= P^{\mathbf{a}}\left(T, \sum_{\omega\in\mathcal{A}^n} q(I)\Phi_I\right), \end{aligned}$$

where $\chi_{[I]}$ denotes the indicator function of $[I]$, and Φ_I denotes the additive potential $\left(\sum_{i=0}^{m-1} \chi_{[I]}(T^i x)\right)_{m=1}^{\infty}$. Furthermore denote by μ the $\mathbf{a}$-weighted equilibrium state of $\sum_{I\in\mathcal{A}^n} q(I)\Phi_I$ and let $\mathbf{p} = (\mu(I))_{I\in\mathcal{A}^n}$. Then $\mathbf{p} \in ri(\Omega_n)$ and $f(\mathbf{p}) = a h_\mu(T) + b h_{\mu\circ\pi^{-1}}(S)$, where $ri(A)$ denotes the relative interior of a convex set A.

By Lemma 3.6 and Theorem 1.3, f^* is differentiable on $\mathbb{R}^{\mathcal{A}^n}$. Hence by Corollary 26.4.1 in [41] and (3.10), for any $\mathbf{p} \in \mathrm{ri}(\Omega_n)$, there exists $\mathbf{q} \in \mathbb{R}^{\mathcal{A}^n}$ such that

$$\nabla f^*(\mathbf{q}) = \mathbf{p}.$$

It is easy to check that $\mathrm{ri}(\Omega_n)$ consists of the strictly positive vectors in Ω_n. However, by Lemma 3.6 and Theorem 1.3,

$$\nabla f^*(\mathbf{q}) = (\mu(I))_{I\in\mathcal{A}^n},$$

where $\mu = \mu_{\mathbf{q}}$ is the $\mathbf{a}$-weighted equilibrium state of $\sum_{I\in\mathcal{A}^n} q(I)\Phi_I$. By Theorem 1.2, $\mu_{\mathbf{q}}$ is quasi-Bernoulli. Thus for each positive vector $\mathbf{p}$ in Ω_n, there exists a quasi-Bernoulli measure $\mu_{\mathbf{q}}$ such that $(\mu_{\mathbf{q}}(I))_{I\in\mathcal{A}^n} = \mathbf{p}$. By Lemma 3.6, we do have

$$\begin{aligned} &ah_{\mu_{\mathbf{q}}}(T) + bh_{\mu_{\mathbf{q}}\circ\pi^{-1}}(S) = f(\mathbf{p}) \\ &\qquad = \sup\{ah_\eta(T) + bh_{\eta\circ\pi^{-1}}(S) : \ \eta \in \mathcal{M}(X,T) : \ (\eta(I))_{I\in\mathcal{A}^n} = \mathbf{p}\}. \end{aligned}$$

Furthermore, the measure μ which attains the supremum is unique, because each such a measure is a $\mathbf{a}$-weighted equilibrium state of $\sum_{I\in\mathcal{A}^n} q(I)\Phi_I$. This finishes the proof of Theorem 1.4. □

Proof of Theorem 1.5. First assume that η is fully supported. Let $\mu_n = \mu(\mathbf{a}, \eta, n)$ as in Theorem 1.4. Then the sequence $(\mu_n)_{n=1}^\infty$ is desired in Theorem 1.5.

Now consider the general case. Let $\eta_n = (1 - 1/n)\eta + (1/n)\eta_0$, where η_0 denotes the Parry measure on X. Clearly, η_n is fully supported. Denote $\mu_n' = \mu(\mathbf{a}, \eta_n, n)$. Then $(\mu_n')_{n=1}^\infty$ is desired. □

4. Higher dimensional weighted thermodynamic formalism

In this section, we present the higher dimensional versions of our main results. Since the proofs are essentially identical to those in the two dimensional case, we just omit them.

Let $k \geq 2$. Assume that (X_i, T_i) $(i = 1, \ldots, k)$ are subshifts over finite alphabets $\mathcal{A}_i$ such that X_{i+1} is a factor of X_i with a one-block factor map $\pi_i : \ X_i \to X_{i+1}$ for $i = 1, \ldots, k-1$. For convenience, we use π_0 to denote the identity map on X_1. Define $\tau_i : \ X_1 \to X_{i+1}$ by $\tau_i = \pi_i \circ \pi_{i-1} \circ \cdots \circ \pi_0$ for $i = 0, 1, \ldots, k-1$.

Let $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{R}^k$ so that $a_1 > 0$ and $a_i \geq 0$ for $i > 1$. For $\Phi \in \mathcal{C}_{ass}(X_1, T_1)$. We define the $\mathbf{a}$-*weighted topological pressure of* Φ as

$$P^{\mathbf{a}}(T_1, \Phi) = \sup\left\{\Phi_*(\mu) + h_\mu^{\mathbf{a}}(T_1) : \ \mu \in \mathcal{M}(X_1, T_1)\right\},$$

where $h_\mu^{\mathbf{a}}(T_1)$ is the $\mathbf{a}$-weighted topological entropy defined as

$$h_\mu^{\mathbf{a}}(T_1) = \sum_{i=1}^k a_i h_{\mu\circ\tau_{i-1}^{-1}}(T_i).$$

Clearly the supremum is attainable. Each measure μ which attains the supremum is called an $\mathbf{a}$-weighted equilibrium state of Φ.

For $i = 1, \ldots, k-1$, we define $\theta_i : \mathcal{C}_{ass}(X_i, T_i) \to \mathcal{C}_{ass}(X_{i+1}, T_{i+1})$ by $(\log \phi_n)_{n=1}^\infty \mapsto (\log \psi_n)_{n=1}^\infty$, where

$$\psi_n(y) = \left(\sum_{I \in \mathcal{A}_i^n :\ [I] \cap \pi_i^{-1}(y) \neq \emptyset} \ \sup_{x \in [I] \cap \pi_i^{-1}(y)} \phi_n(x)^{1/A_i} \right)^{A_i}$$

for $y \in X_{i+1}$, with $A_i = a_1 + \cdots + a_i$. In particular, let $\mathcal{S}_{ass}$ denote the collection of asymptotically sub-additive additive (scalar) sequences $(\log c_n)_{n=1}^\infty$ (a sequence $(\log c_n)_{n=1}^\infty$, where $c_n \geq 0$, is called *asymptotically sub-additive* if, for any $\varepsilon > 0$, there exists a sequence $(d_n)_{n=1}^\infty$, so that $0 \leq d_{n+m} \leq d_n d_m$ and $\limsup_{n\to\infty} \frac{1}{n} |\log c_n - \log d_n| < \varepsilon$). Let $\theta_k : \mathcal{C}_{ass}(X_k, T_k) \to \mathcal{S}_{ass}$ be defined as $(\log \phi_n)_{n=1}^\infty \mapsto (\log c_n)_{n=1}^\infty$, where

$$c_n = \left(\sum_{I \in \mathcal{A}_k^n} \sup_{x \in [I]} \phi_n(x)^{1/A_k} \right)^{A_k}.$$

As an extension of Theorem 1.1, we have

Theorem 4.1. (i) $P^{\mathbf{a}}(T_1, \Phi) = \lim_{n\to\infty} (1/n) \log c_n$, *where* $(c_n)_{n=1}^\infty = \theta_k \circ \cdots \circ \theta_1(\Phi)$.

(ii) *For any* $1 \leq i \leq k-1$, $P^{\mathbf{a}}(T_1, \Phi) = P^{(\sum_{j=1}^i a_j,\, a_{i+1}, \ldots, a_k)}(T_{i+1},\ \theta_i \circ \cdots \circ \theta_1(\Phi))$.

(iii) $\mu \in \mathcal{M}(X_1, T_1)$ *is an* $\mathbf{a}$*-weighted equilibrium state of* Φ *if and only if* $\mu \circ \tau_{k-1}^{-1}$ *is an equilibrium state of* $\frac{\theta_{k-1} \circ \cdots \circ \theta_1(\Phi)}{a_1 + \cdots + a_k}$ *and, for* $i = k-2, k-3, \ldots, 0$, $\mu \circ \tau_i^{-1}$ *is a conditional equilibrium state of* $\frac{\theta_i \circ \cdots \circ \theta_1(\Phi)}{a_1 + \cdots + a_{i+1}}$ *with respective to* $\mu \circ \tau_{i+1}^{-1}$.

In the remaining part of this section, we assume that X_i is the full shift over $\mathcal{A}_i$ for each $i \in \{1, \ldots, k\}$. For $i = 1, \ldots, k-1$, we redefine $\theta_i : \mathcal{C}_{asa}(X_i, T_i) \to \mathcal{C}_{asa}(X_{i+1}, T_{i+1})$ by $(\log \phi_n)_{n=1}^\infty \mapsto (\log \psi_n)_{n=1}^\infty$, where

$$\psi_n(y) = \left(\sum_{I \in \mathcal{A}_i^n :\ [I] \cap \pi_i^{-1}(y) \neq \emptyset} \ \sup_{x \in [I]} \phi_n(x)^{1/A_i} \right)^{A_i}$$

for $y \in X_{i+1}$. In particular, let $\mathcal{S}_{asa}$ denote the collection of asymptotically additive (scalar) sequences $(\log c_n)_{n=1}^\infty$. Let $\theta_k : \mathcal{C}_{asa}(X_k, T_k) \to \mathcal{S}_{asa}$ be defined as $(\log \phi_n)_{n=1}^\infty \mapsto (\log c_n)_{n=1}^\infty$, where

$$c_n = \left(\sum_{I \in \mathcal{A}_k^n} \sup_{x \in [I]} \phi_n(x)^{1/A_k} \right)^{A_k}.$$

For $\Phi = (\log \phi_n)_{n=1}^\infty \in \mathcal{C}_{asa}(X_1, T_1)$, write

$$\begin{aligned} &(\log \phi_n^{(i)})_{n=1}^\infty := \theta_i \circ \cdots \circ \theta_1(\Phi), \quad i = 1, \ldots, k, \\ &(\log \phi_n^{(0)})_{n=1}^\infty := (\log \phi_n)_{n=1}^\infty \text{ and} \\ &\phi^{(i)}(J) := \sup\{\phi_n^{(i)}(y) :\ y \in [J]\} \end{aligned} \tag{4.1}$$

for any n-th cylinder $[J] \subset X_{i+1}$, $i = 0, \dots, k-1$. Then, we define the $\mathbf{a}$*-weighted potential associated with* Φ by

$$\Phi^{\mathbf{a}} = (\log \phi_n^{\mathbf{a}})_{n=1}^{\infty}, \text{ where } \phi_n^{\mathbf{a}}(x) = \phi^{(0)}(x_{|n})^{1/A_1} \prod_{i=1}^{k-1} \phi^{(i)}(\tau_i(x_{|n}))^{1/A_{i+1}-1/A_i}, \tag{4.2}$$

where $A_i = a_1 + \cdots + a_i$. Since there exists a sequence $(g^{(p)})_{p\geq 1}$ of Hölder potentials such that $\lim_{p\to 0} \limsup_{n\to\infty} \|\Phi_n - S_n g^{(p)}\|_\infty / n = 0$ (see Lemma 2.1(iii)), it is easily seen that all the potentials $(\log \phi^{(i)}(\tau_{i-1}(\cdot_{|n})))_{n=1}^{\infty}$ and $(\log \phi_n^{\mathbf{a}})_{n=1}^{\infty}$ belong to $\mathcal{C}_{asa}(X, T)$.

As an analogue of Theorems 1.2-1.5, we have

Theorem 4.2. (i) *Let* $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathcal{C}_{asa}(X_1, T_1)$. *Then*

$$P^{\mathbf{a}}(T_1, \Phi) = \lim_{n\to\infty} (1/n) \log c_n,$$

where $(\log c_n)_{n=1}^{\infty} = \theta_k \circ \cdots \circ \theta_1(\Phi)$.

(ii) *Assume* $\Phi \in \mathcal{C}_{aa}(X_1, T_1)$ *and* Φ *has the bounded distortion property. Then there is a unique* $\mathbf{a}$*-weighted equilibrium state* μ *of* Φ. *The measure* μ *is fully supported and quasi-Bernoulli, and it satisfies the following Gibbs property*

$$\mu(I) \approx \exp\Big(\frac{-nP}{A_k}\Big) \phi_n^{\mathbf{a}}(I), \quad I \in \mathcal{A}_1^n, \tag{4.3}$$

where $P = P^{\mathbf{a}}(T_1, \Phi)$. *Consequently, for* $i = 2, \dots, k$,

$$\mu_i(\tau_{i-1} I) \approx \exp\Big(\frac{-nP}{A_k}\Big) \phi^{(i-1)}(\tau_{i-1} I)^{1/A_i} \prod_{j=i}^{k-1} \phi^{(j)}(\tau_j I)^{1/A_{j+1}-1/A_j}, \quad I \in \mathcal{A}_1^n, \tag{4.4}$$

where $\mu_i := \mu \circ \tau_{i-1}^{-1}$. *Furthermore,*

$$\phi_n(x) \exp(-nP) \approx \prod_{i=1}^{k} \mu_i(\tau_{i-1} x_{|n})^{a_i} \quad \text{for } x \in X_1,\ n \geq 1,$$

A Borel probability measure μ (not necessarily invariant) on X satisfying (4.3) is called *an* $\mathbf{a}$*-weighted Gibbs measure for* Φ.

Theorem 4.3. *Let* $\Phi_1, \dots, \Phi_d \in \mathcal{C}_{aa}(X, T)$ *satisfy the bounded distortion property. Then the map* $Q : \mathbb{R}^d \to \mathbb{R}$ *defined as*

$$(q_1, \dots, q_d) \mapsto P^{\mathbf{a}}\left(T, \sum_{i=1}^{d} q_i \Phi_i\right),$$

is C^1 *over* $\mathbb{R}^d$ *with*

$$\nabla Q(q_1, \dots, q_d) = ((\Phi_1)_*(\mu_{\mathbf{q}}), \dots, (\Phi_d)_*(\mu_{\mathbf{q}})),$$

where $\mu_{\mathbf{q}}$ *is the unique* $\mathbf{a}$*-weighted equilibrium state of* $\sum_{i=1}^{d} q_i \Phi_i$.

Theorem 4.4. *For each fully supported measure $\eta \in \mathcal{M}(X_1, T_1)$ and each $n \in \mathbb{N}$, there is a unique measure $\mu = \mu(\mathbf{a}, \eta, n)$ in $\mathcal{M}(X_1, T_1)$ attaining the following supremum*

$$\sup\left\{h_\mu^{\mathbf{a}}(T_1) : \ \mu(I) = \eta(I) \text{ for all } n\text{-th cylinder } [I] \in X_1\right\}.$$

Furthermore $\mu(\mathbf{a}, \eta, n)$ is the $\mathbf{a}$-weighted equilibrium state of certain $\Phi \in \mathcal{C}_{aa}(X_1, T_1)$ with the bounded distortion property, and hence $\mu(\mathbf{a}, \eta, n)$ is a fully supported quasi-Bernoulli measure on X_1.

Theorem 4.5. *For any $\eta \in \mathcal{M}(X_1, T_1)$, there exists $(\mu_n)_{n=1}^\infty \subset \mathcal{M}(X_1, T_1)$ converging to η in the weak-star topology such that for each n, μ_n is quasi-Bernoulli and*

$$h_{\mu_n}^{\mathbf{a}}(T_1) \geq h_\mu^{\mathbf{a}}(T_1).$$

Furthermore,

$$\lim_{n\to\infty} h_{\mu_n}^{\mathbf{a}}(T_1) = h_\mu^{\mathbf{a}}(T_1).$$

Remark 4.6. If we take $\mathbf{a} = (1, \ldots, 1)$, due to the upper semi-continuity of the entropy, for any $\mu \in \mathcal{M}(X, T)$, Theorem 4.5 yields a sequence of quasi-Bernoulli measures $(\mu_n)_{n=1}^\infty$ which converges to μ in the weak-star topology, such that we have both $\lim_{n\to\infty} h_{\mu_n}(T) = h_\mu(T)$ and $\lim_{n\to\infty} h_{\mu_n\circ\pi^{-1}}(S) = h_{\mu\circ\pi^{-1}}(S)$. Moreover, one can deduce from Theorem 4.2 that for any $\mathbf{a} = (a_1, \ldots, a_k)$ with $a_1 > 0$ and $a_i \geq 0$ for $i \geq 2$, each invariant quasi-Bernoulli measure is the $\mathbf{a}$-weighted equilibrium state of some almost additive potential satisfying the bounded distortion property.

Definition 4.7. We say that two almost additive potentials $\Phi = (\log\phi_n)_{n=1}^\infty$ and $\Psi = (\log\psi_n)_{n=1}^\infty$ are *cohomologous* if $\sup_n \|\log\phi_n - \log\psi_n\|_\infty < \infty$. If there exists $C \in \mathbb{R}$ such that $\log\psi_n = Cn$, we say that Φ is *cohomologous to a constant.*

The following proposition is a direct consequence of Theorem 4.2.

Proposition 4.8. *Let Φ, $\Psi \in \mathcal{C}_{aa}(X, T)$ satisfy the bounded distortion property. Then, Φ and Ψ share the same $\mathbf{a}$-weighted equilibrium state if and only if $\Phi - \Psi$ is cohomologous to a constant.*

Next theorem is reminiscent from Sections 4.6 and 4.7 of [43].

Theorem 4.9. *Let $\Phi_1, \ldots, \Phi_d \in \mathcal{C}_{aa}(X, T)$ satisfy the bounded distortion property. Let V be the vector subspace of those $\mathbf{q}$ such that $\sum_{i=1}^d q_i\Phi_i$ is cohomolohous to a constant. The map Q defined in Theorem 4.3 is strictly convex if and only if $V = \{\mathbf{0}\}$. Moreover, Q is affine on any affine subspace of $\mathbb{R}^d$ parallel to V. In particular, if $d = 1$ and Q is not strictly convex, it is affine.*

An immediate corollary is

Corollary 4.10. *Let $\Phi_1, \ldots, \Phi_d \in \mathcal{C}_{aa}(X,T)$ satisfy the bounded distortion property. Let $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_d)$. The convex set $\{((\Phi_1)_*(\mu), \ldots, (\Phi_d)_*(\mu)) : \mu \in \mathcal{M}(X,T)\}$ is reduced to a singleton if and only if each Φ_i is cohomologous to a constant.*

Proof of Proposition 4.8. Suppose that Q is affine on a non-trivial segment $[\mathbf{q}, \mathbf{q}']$. For every $t \in [0,1]$ we have

$$\begin{aligned} Q(\mathbf{q} + t(\mathbf{q}' - \mathbf{q})) &= Q(\mathbf{q}) + t\nabla Q(\mathbf{q}) \cdot (\mathbf{q}' - \mathbf{q}) \\ &= Q(\mathbf{q}) + t\sum_{i=1}^{d} (q_i' - q_i)(\Phi_i)_*(\mu_{\mathbf{q}}). \end{aligned}$$

Since $Q(\mathbf{q}) = \sum_{i=1}^d q_i(\Phi_i)_*(\mu_{\mathbf{q}}) + \sum_{i=1}^k a_i h_{\mu\circ\tau_{i-1}^{-1}}(T_i)$, we have

$$Q(\mathbf{q} + t(\mathbf{q}' - \mathbf{q})) = \sum_{i=1}^{d} (q_i + t(q_i' - q_i))(\Phi_i)_*(\mu_{\mathbf{q}}) + \sum_{i=1}^{k} a_i h_{\mu\circ\tau_{i-1}^{-1}}(T_i).$$

Consequently, $\mu_{\mathbf{q}}$ is the unique $\mathbf{a}$-weighted equilibrium state of $\sum_{i=1}^d (q_i + t(q_i' - q_i))\Phi_i$, for each $t \in [0,1]$. Due to Proposition 4.8, this implies that $\sum_{i=1}^d (q_i' - q_i)\Phi_i$ is cohomologous to a constant, hence $V \neq \{\mathbf{0}\}$.

Conversely, assume that $V \neq \{\mathbf{0}\}$. Then, the same argument as above can be used to prove that Q is affine on any affine subspace of $\mathbb{R}^d$ parallel to V. □

5. Multifractal analysis on higher dimensional self-affine symbolic spaces

Let $k \geq 2$. Assume that (X_i, T_i) $(i = 1, \ldots, k)$ are full shifts over $\mathcal{A}_i$ such that X_{i+1} is a factor of X_i with a one-block factor map $\pi_i : X_i \to X_{i+1}$ for $i = 1, \ldots, k-1$. For convenience, we use π_0 to denote the identity map on X_1. Define $\tau_i : X_1 \to X_{i+1}$ by $\tau_i = \pi_i \circ \pi_{i-1} \circ \cdots \circ \pi_0$ for $i = 0, 1, \ldots, k-1$. We simply write (X,T) for (X_1, T_1).

For $x = (x_i)_{i=1}^\infty \in X$ and $n \geq 1$, $x_{|n}$ denotes the word $x_1 \cdots x_n$.

We endow the set X with a "self-affine" metric as follows. We fix $\mathbf{a} = (a_1, \ldots, a_k) \in \mathbb{R}^k$ with $a_1 > 0$ and $a_i \geq 0$ for $i > 1$, and we define the ultrametric distance

$$d_{\mathbf{a}}(x,y) = \max\left(e^{-|\tau_{i-1}(x)\wedge\tau_{i-1}(y)|/(a_1+\cdots+a_i)} : 1 \leq i \leq k\right).$$

For $1 \leq i \leq k$ and $n \in \mathbb{N}$, let

$$\ell_i(n) = \min\{p \in \mathbb{N} : p \geq (a_1 + \cdots + a_i)n/a_1\},$$

and by convention set $\ell_0(n) = 0$. It is easy to check that

Lemma 5.1. *In $(X, d_{\mathbf{a}})$, the closed ball centered at x of radius e^{-n/a_1} is given by*

$$B(x, e^{-n/a_1}) = \left\{y \in X : \tau_{i-1}(y) \in \tau_{i-1}(x_{|\ell_i(n)}) \text{ for all } 1 \leq i \leq k\right\}.$$

The following result estimates the value of an $\mathbf{a}$-weighted Gibbs measure on a ball in $(X, d_{\mathbf{a}})$.

Lemma 5.2. *Let $\Phi = (\log \phi_n)_{n=1}^{\infty} \in \mathcal{C}_{aa}(X,T)$ satisfy the bounded distortion property. Let μ denote the $\mathbf{a}$-weighted Gibbs measure of Φ. Then we have the following estimate:*

$$\mu(B(x, e^{-n/a_1})) \approx \exp\Big(\frac{-nP^{\mathbf{a}}(T,\Phi)}{a_1}\Big)\phi_n(x)^{1/a_1}\prod_{j=1}^{k-1}\frac{\phi^{(j)}(\tau_j(x_{|\ell_{j+1}(n)}))^{1/A_{j+1}}}{\phi^{(j)}(\tau_j(x_{|\ell_j(n)}))^{1/A_j}},$$

where $\phi^{(j)}$, $j = 0,\dots,k-1$, are defined as in (4.1), *and $A_j = a_1 + \cdots + a_j$.*

Proof. Let $x = (x_i)_{i=1}^{\infty} \in X$ and $n \geq 1$. For $i = 1,\dots,k$, write $I_i = x_{\ell_{i-1}(n)+1}\cdots x_{\ell_k(n)}$. Let B denote $B(x, e^{-n/a_1})$. By Lemma 5.1, $B = \{y : \forall\ 1 \leq i \leq k,\ \tau_{i-1}(y) \in \tau_{i-1}([I_1 \dots I_i])\}$. Since μ is quasi-Bernoulli (cf. Theorem 4.2(ii)), we have $\mu(B) \approx \prod_{i=1}^{k}\mu_i(\tau_{i-1}I_i)$, where $\mu_i = \mu \circ \tau_{i-1}^{-1}$. Let us transform this expression by using (4.4). Since each word I_i is of length $\ell_i(n) - \ell_{i-1}(n)$ and by construction $\ell_k(n)/A_k - n/a_1 = O(1/n)$, (4.4) yields

$$\begin{aligned}
\mu(B) &\approx \exp\Big(\frac{-\ell_k(n)P^{\mathbf{a}}(T,\Phi)}{A_k}\Big)\prod_{i=1}^{k}\phi^{(i-1)}(\tau_{i-1}I_i)^{1/A_i}\prod_{j=i}^{k-1}\phi^{(j)}(\tau_j I_i)^{1/A_{j+1}-1/A_j}\\
&\approx \exp\Big(\frac{-nP^{\mathbf{a}}(T,\Phi)}{a_1}\Big)\Big(\prod_{i=0}^{k-1}\phi^{(i-1)}(\tau_{i-1}I_i)^{1/A_i}\Big)\prod_{j=1}^{k-1}\prod_{i=1}^{j}\phi^{(j)}(\tau_j I_i)^{1/A_{j+1}-1/A_j}\\
&\approx \exp\Big(\frac{-nP^{\mathbf{a}}(T,\Phi)}{a_1}\Big)\phi^{(0)}(I_1)^{1/a_1}\prod_{j=1}^{k-1}\frac{\phi^{(j)}(\tau_j(I_1\cdots I_{j+1}))^{1/A_{j+1}}}{\phi^{(j)}(\tau_j(I_1\cdots I_j))^{1/A_j}}\\
&\approx \exp\Big(\frac{-nP^{\mathbf{a}}(T,\Phi)}{a_1}\Big)\phi_n(x)^{1/a_1}\prod_{j=1}^{k-1}\frac{\phi^{(j)}(\tau_j x_{|\ell_{j+1}(n)})^{1/A_{j+1}}}{\phi^{(j)}(\tau_j x_{|\ell_j(n)})^{1/A_j}}.
\end{aligned}$$

This finishes the proof of the lemma. □

Recall that the weighted entropy of $\mu \in \mathcal{M}(X,T)$ has been defined in Section 4 as $h_\mu^{\mathbf{a}}(T) = \sum_{i=1}^{k} a_i h_{\mu\circ\tau_{i-1}^{-1}}(T_i)$. The following Ledrappier-Young type formula was proved by Kenyon and Peres in [28, Lemma 3.1] under a slight different setting.

Proposition 5.3. *Suppose that $\mu \in \mathcal{M}(X,T)$ is ergodic. Then we have*

$$\dim_H \mu = h_\mu^{\mathbf{a}}(T).$$

5.1. **Multifractal analysis of asymptotically additive potentials.** Recall that the generic set $\mathcal{G}(\mu)$ of a measure $\mu \in \mathcal{M}(X,T)$ has been defined in (1.11), and that an equivalent definition invoking asymptotically additive potentials is given in Remark 2.2. We have the following high dimensional extension of Theorem 1.7.

Theorem 5.4. *Let $\mu \in \mathcal{M}(X,T)$. We have $\mathcal{G}(\mu) \neq \emptyset$ and $\dim_H \mathcal{G}(\mu) = h_\mu^{\mathbf{a}}(T)$.*

The proof of Theorem 5.4 will be given in Sect. 5.4. Next we consider level sets associated with Birkhoff averages of asymptotically additive potentials on X.

For $\mathbf{\Phi} = (\Phi_1, \ldots, \Phi_d) \in \mathcal{C}_{asa}(X,T)^d$, where $\Phi_i = (\log \phi_{n,i})_{n=1}^{\infty} =: (\Phi_{n,i})_{n=1}^{\infty}$, and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{R}^d$, define

$$E_{\mathbf{\Phi}}(\alpha) = \Big\{ x \in X : \lim_{n\to\infty} \frac{\Phi_{n,i}(x)}{n} = \alpha_i \text{ for } 1 \le i \le d \Big\}. \tag{5.1}$$

Denote $\mathbf{\Phi}_n(x) = (\Phi_{n,1}(x), \ldots, \Phi_{n,d}(x))$. Then the set in the right hand side of (5.1) can be simply written as $\Big\{ x \in X : \lim_{n\to\infty} \frac{\mathbf{\Phi}_n(x)}{n} = \alpha \Big\}$. For $\mu \in \mathcal{M}(X,T)$, write $\mathbf{\Phi}_*(\mu) = ((\Phi_1)_*(\mu), \ldots, (\Phi_d)_*(\mu))$ and define $L_{\mathbf{\Phi}} = \big\{ \mathbf{\Phi}_*(\mu) : \mu \in \mathcal{M}(X,T) \big\}$.

Let $\{\mathbf{\Phi}^{(j)}\}_{1 \le j \le r}$ be a family of elements of $\mathcal{C}_{asa}(X,T)^d$. Let $\mathbf{c} = (c_1, \ldots, c_r)$ be a real vector with positive entries. For $\alpha \in \mathbb{R}^d$, define

$$E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha) = \Big\{ x \in X : \lim_{n\to\infty} \sum_{j=1}^{r} \frac{\mathbf{\Phi}^{(j)}_{\lfloor c_j n \rfloor}}{\lfloor c_j n \rfloor}(x) = \alpha \Big\},$$

where $\lfloor y \rfloor$ stands for the integer part of $y \in \mathbb{R}$. It is clear that $E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha) = E_{\{\mathbf{\Phi}^{(j)}\},\lambda\mathbf{c}}(\alpha)$ for any $\lambda > 0$, and in particular, $E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha) = E_{\mathbf{\Phi}}(\alpha)$ if $r = 1$. It is remarkable that the Hausdorff dimension of the set $E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha)$ does not depend on $\mathbf{c}$ when $r \ge 2$, as shown in the following result, of which the proof will be given in Sect. 5.5.

Theorem 5.5. *Let* $\mathbf{\Phi} = \sum_{j=1}^{r} \mathbf{\Phi}^{(j)}$.

(1) *For* $\alpha \in \mathbb{R}^d$, *the following assertions are equivalent.*
  (i) $\alpha \in L_{\mathbf{\Phi}}$;
  (ii) $E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha) \neq \emptyset$;
  (iii) $\inf \big\{ P^{\mathbf{a}}(T, \mathbf{q}\cdot\mathbf{\Phi}) - \alpha\cdot\mathbf{q} :\ \mathbf{q} \in \mathbb{R}^d \big\} \ge 0$;
  (iv) $\inf \big\{ P^{\mathbf{a}}(T, \mathbf{q}\cdot\mathbf{\Phi}) - \alpha\cdot\mathbf{q} :\ \mathbf{q} \in \mathbb{R}^d \big\} > -\infty$;

  Furthermore for $\alpha \in L_{\mathbf{\Phi}}$, *we have*

$$\begin{aligned} \dim_H E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha) &= \max \big\{ h^{\mathbf{a}}_{\mu}(T) :\ \mu \in \mathcal{M}(X,T),\ \mathbf{\Phi}_*(\mu) = \alpha \big\} \\ &= \inf \Big\{ P^{\mathbf{a}}(T, \mathbf{q}\cdot\mathbf{\Phi}) - \alpha\cdot\mathbf{q} :\ \mathbf{q} \in \mathbb{R}^d \Big\}. \end{aligned}$$

(2) *Suppose that* $L_{\mathbf{\Phi}}$ *is not a singleton. Then the set* $X \setminus \bigcup_{\alpha \in L_{\mathbf{\Phi}}} E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha)$ *is of full Hausdorff dimension.*

Remark 5.6. *If we take* $r = 1$ *and* $\mathbf{\Phi} = 0$, *we find that the Hausdorff dimension of* $(X, d_{\mathbf{a}})$ *is* $P^{\mathbf{a}}(T, 0)$. *This extends the result of* [28] *which holds for special choices of* $\mathbf{a}$.

Example 5.7. Generally, the level sets $E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha)$ depend on $\mathbf{c}$. For example, let $X = \{0,1\}^{\mathbb{N}}$, and let $g \in C(X)$ be given by $g(x) = x_1$ for $x = (x_i)_{i=1}^{\infty} \in X$. Set $\Phi^{(1)} = (S_n g)_{n=1}^{\infty}$ and $\Phi^{(2)} = (-S_n g)_{n=1}^{\infty}$. Then $E_{\{\mathbf{\Phi}^{(j)}\}_{j=1}^2,(1,1)}(0) = X$, however $E_{\{\mathbf{\Phi}^{(j)}\}_{j=1}^2,(1,2)}(0) \neq X$ (it is easy to check that $x = 0^1 1^2 0^4 1^8 \cdots 0^{2^{2n}} 1^{2^{2n+1}} \cdots \notin E_{\{\mathbf{\Phi}^{(j)}\}_{j=1}^2,(1,2)}(0)$).

5.2. **Application to the multifractal analysis of a-weighted weak Gibbs measures.** As we have seen in Theorem 4.2, **a**-weighted Gibbs measures are naturally associated with almost additive potentials satisfying the bounded distortion property; this extends the classical Gibbs measures. Now we show that the notion of weak Gibbs measure associated with a continuous potential defined on X in the classical thermodynamic formalism [29] also has a natural extension in the **a**-weighted thermodynamical formalism.

Definition 5.8. Let $\Phi \in \mathcal{C}_{asa}(X,T)$. A fully supported Borel probability measure μ (not necessarily to be shift invariant) on X is called an **a**-*weighted weak Gibbs measure associated with* Φ if

$$\mu(I) \approx_n \exp\Big(\frac{-nP}{A_k}\Big)\phi_n^{\mathbf{a}}(I), \quad I \in \mathcal{A}^n, \tag{5.2}$$

where $P = P^{\mathbf{a}}(T_1,\Phi)$, $A_k = a_1 + \cdots + a_k$, $\Phi^{\mathbf{a}} = (\log \phi_n^{\mathbf{a}}) \in \mathcal{C}_{asa}(X_1,T_1)$ is defined as in (4.2), and $\approx_n$ means that there exists a sequence of positive numbers $(\kappa_n)_{n=1}^{\infty}$ with $\lim_{n\to\infty}(1/n)\log\kappa_n = 0$, such that the ratio between the left and right hand sides of $\approx_n$ lies in (κ_n^{-1},κ_n).

Remark 5.9. *It is not hard to see that if μ satisfies (5.2), then for $i = 2,\dots,k$,*

$$\mu_i(\tau_{i-1}I) \approx_n \exp\Big(\frac{-nP}{A_k}\Big)\phi^{(i-1)}(\tau_{i-1}I)^{1/A_i}\prod_{j=i}^{k-1}\phi^{(j)}(\tau_j I)^{1/A_{j+1}-1/A_j}, \quad I \in \mathcal{A}^n, \tag{5.3}$$

where $\mu_i = \mu\circ\tau_{i-1}^{-1}$, and $\phi^{(j)}$, $j = 0,\dots k-1$, are defined as in (4.1)*. Furthermore, μ satisfies* (5.2) *if and only if*

$$\phi_n(x)\exp(-nP) \approx_n \prod_{i=1}^{k}\mu_i(\tau_{i-1}x_{|n})^{a_i}, \qquad x \in X,\ n \ge 1,$$

The following result, which will be proved in Sect. 5.6, shows the existence of **a**-weighted weak Gibbs measure for any asymptotically additive potential on X.

Theorem 5.10. *Let $\Phi = (\log\phi_n)_{n=1}^{\infty} \in \mathcal{C}_{asa}(X,T)$. Then there exists at least an* **a**-*weighted weak Gibbs measure μ associated with Φ.*

Furthermore, for each $1 \le i \le k$, the potential $\mathbf{\Psi}_\mu^{(i)} := \big(\log\mu_i(\tau_{i-1}(x_{|n}))\big)_{n=1}^{\infty}$ belongs to $\mathcal{C}_{asa}(X,T)$, and for every point $x = (x_i)_{i=1}^{\infty} \in X$ and $B = B(x,e^{-n/a_1})$, we have

$$\log\mu(B) = \mathbf{\Psi}_{\mu,n}^{(1)}(x) + \sum_{i=2}^{k}\mathbf{\Psi}_{\mu,\ell_i(n)}^{(i)}(x) - \mathbf{\Psi}_{\mu,\ell_{i-1}(n)}^{(i)}(x) + c(x,n), \tag{5.4}$$

where $(c(x,n))_{n\ge1}$ is a sequence satisfying $\lim_{n\to\infty}c(x,n)/n = 0$. If moreover, $\Phi \in \mathcal{C}_{aa}(X,T)$ and satisfies the bounded distortion property, then $c(x,n)$ can be taken bounded independently of x and n, and (5.4) *takes the form*

$$\mu(B) \approx \prod_{i=1}^{k}\mu_i(\tau_{i-1}(I_i)), \tag{5.5}$$

where $I_i = x_{\ell_{i-1}(n)+1}\cdots x_{\ell_i(n)}$.

Remark 5.11. (1) We recover the usual weak Gibbs measures when $\mathbf{a}=(1,0\dots,0)$ and Φ is the sequence of Birkhoff sums associated with a continuous potential over X [49, 29].

(2) By using (5.2) and (5.3), from any $(1,0,\dots,0)$-weighted weak Gibbs measure μ one can build an asymptotically additive potential of which μ is an **a**-weighted weak Gibbs measure.

We have the following result on the multifractal analysis of **a**-weighted weak Gibbs measures.

Theorem 5.12. *Let μ be an **a**-weighted weak Gibbs measure associated with some asymptotically additive potential. For $\alpha\in\mathbb{R}_+$ we define*

$$E_\mu(\alpha)=\Big\{x\in X:\ \lim_{r\to 0^+}\frac{\log\mu(B(x,r))}{\log r}=\alpha\Big\}.$$

Let $\mathbf{\Psi}_\mu=\sum_{i=1}^k a_i\mathbf{\Psi}_\mu^{(i)}$. Let $L_\mu=L_{-\mathbf{\Psi}_\mu}=\{-(\mathbf{\Psi}_\mu)_*(\lambda):\lambda\in\mathcal{M}(X,T)\}$. Then, for all $\alpha\ge 0$, $E_\mu(\alpha)\neq\emptyset$ if and only if $\alpha\in L_\mu$. For $\alpha\in L_\mu$, we have

$$\begin{aligned}\dim_H E_\mu(\alpha)&=\max\big\{h_\mu^{\mathbf{a}}(T):\ \lambda\in\mathcal{M}(X,T),\ (\mathbf{\Psi}_\mu)_*(\lambda)=-\alpha\big\}\\&=\inf\{P^{\mathbf{a}}(T,q\mathbf{\Psi}_\mu)+\alpha q:\ q\in\mathbb{R}\}.\end{aligned}$$

Proof. This result is just a corollary of Theorem 5.5. Indeed, thanks to Theorem 5.12(2) we can write

$$\begin{aligned}\frac{\log\mu(B(x,e^{-n/a_1}))}{-n/a_1}&=-a_1\frac{\mathbf{\Psi}_{\mu,n}^{(1)}(x)}{n}-a_1\sum_{i=2}^k\frac{\mathbf{\Psi}^{(i)}_{\mu,\ell_i(n)}(x)}{n}-\frac{\mathbf{\Psi}^{(i)}_{\mu,\ell_{i-1}(n)}(x)}{n}+o(1)\\&=-a_1\frac{\mathbf{\Psi}_{\mu,n}^{(1)}(x)}{n}-a_1\sum_{i=2}^k\frac{b_i\mathbf{\Psi}^{(i)}_{\mu,\lfloor b_in\rfloor}(x)}{\lfloor b_in\rfloor}-\frac{b_{i-1}\mathbf{\Psi}^{(i)}_{\mu,\lfloor b_{i-1}n\rfloor}(x)}{\lfloor b_{i-1}n\rfloor}+o(1),\end{aligned}$$

with $b_i=(a_1+\cdots+a_i)/a_1$. Thus, any set $E_\mu(\alpha)$ takes the form $E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha)$, with $\sum_{j=1}^r\mathbf{\Phi}^{(j)}=-\mathbf{\Psi}$. □

More geometric applications. A parallelepiped is a subset of X of the form

$$R(I_1,\dots,I_k)=\bigcap_{i=1}^k\tau_{i-1}^{-1}(I_i),\ \text{with } I_i\in\bigcup_{n\ge 0}\mathcal{A}_i^n.$$

If we fix $0\le\lambda_1\le\cdots\le\lambda_k$ and set

$$R_n(\lambda_1,\dots,\lambda_k,x)=R\Big(x_{|\lfloor\lambda_1 n\rfloor},\dots,\tau_{i-1}(x_{|\lfloor\lambda_i n\rfloor}),\cdots,\tau_{k-1}(x_{|\lfloor\lambda_k n\rfloor})\Big),$$

then

$$\log\mu(R_n(\lambda_1,\dots,\lambda_k,x))=\sum_{i=1}^k\mathbf{\Psi}^{(i)}_{\mu,\lfloor\lambda_i n\rfloor}(x)-\mathbf{\Psi}^{(i)}_{\mu,\lfloor\lambda_{i-1}n\rfloor}(x)+o(n),$$

with the convention $\lambda_0 = 0$. Consequently, Theorem 5.5 makes it also possible to compute the Hausdorff dimension of the sets

$$\bigcap_{m=1}^{M} \left\{ x \in X : \lim_{n\to\infty} \frac{\log \mu\big(R_n(\lambda_1^{(m)}, \dots, \lambda_k^{(m)}, x)\big)}{-n} = \beta_m \right\},$$

where $\beta \in \mathbb{R}_+^M$ and each $(\lambda_i^{(m)})_{1\le i\le m}$ satisfies $0 \le \lambda_1^{(m)} \le \cdots \le \lambda_k^{(m)}$.

5.3. **Moran measures.** Recall that the lower Hausdorff dimension of a Borel positive measure ν on X is defined as $\underline{\dim}_H(\mu) = \inf\{\dim_H E : \nu(E) > 0\}$. Equivalently, $\underline{\dim}_H(\mu) = \operatorname{ess\,inf}_\nu \liminf_{r\to 0^+} \frac{\log(\nu(B(x,r)))}{\log(r)}$ (cf. [16]). Recall also Remark 2.2. The main result in this subsection is the following.

Theorem 5.13. *Let $(\mu_p)_{p\ge1} \subset \mathcal{M}(X,T)$ be a sequence of invariant quasi-Bernoulli measures. Suppose that $(\mu_p)_{p\ge1}$ converges in the weak-star topology to a measure μ and, moreover, $(h_{\mu_p\circ\tau_{i-1}^{-1}}(T_i))_{p\ge1}$ converges to a limit h_i for all $1 \le i \le k$. Then there exists a probability measure ν of lower Hausdorff dimension larger than or equal to $\sum_{i=1}^k a_i h_i$ such that $\nu(\mathcal{G}(\mu)) > 0$. Consequently, $\dim_H \mathcal{G}(\mu) \ge \sum_{i=1}^k a_i h_i$.*

Proof. For each $p \ge 1$ and $1 \le i \le k$ let us define $\mu_{p,i} = \mu_p \circ \tau_{i-1}^{-1}$ and $\Psi_i^{(p)} := \Psi_i^{\mu_p} = \big(\log \mu_{p,i}(\tau_{i-1}(x_{|n}))\big)_{n=1}^{\infty}$. Notice that each $\Psi_{n,i}^{(p)} := \log \mu_{p,i}(\tau_{i-1}(\cdot_{|n}))$ is locally constant over n-cylinders, and $h_{p,i} := h_{\mu_{p,i}}(T_i) = -(\Psi_i^{(p)})_*(\mu_p)$. Recall that as a part of our assumptions we have $\lim_{p\to\infty} h_{p,i} = h_i$ for each $1 \le i \le k$.

Let $\widetilde{\mathcal{C}}$ be a countable set of additive potentials satisfying the bounded distortion property and such that for each $\Phi \in \mathcal{C}_{asa}(X,T)$ we can find a sequence $(\Phi^{(m)})_{m\ge1} \subset \widetilde{\mathcal{C}}$ such that $\lim_{m\to\infty} \limsup_{n\to\infty} \|\Phi_n^{(m)} - \Phi_n\|_\infty/n = 0$; the existence of such a set follows from Lemma 2.1(iii) and the separability of $C(X)$. For each $m, p \ge 1$ let $\alpha_{m,p} = \Phi_*^{(m)}(\mu_p)$. Since $\Phi_*^{(m)}(\cdot)$ is continuous over $\mathcal{M}(X,T)$ (cf. Lemma 2.1(ii)), and $\lim_{p\to\infty} \mu_p = \mu$, we have $\lim_{p\to\infty} \alpha_{m,p} = \Phi_*^{(m)}(\mu) := \alpha_m$.

For each $m \ge 1$, we denote as c_m the constant associated with $\Phi^{(m)}$ in (1.5).

The following proposition is a direct consequence of Kingman's sub-additive ergodic theorem applied for every $p \ge 1$ to each element of the families $\widetilde{\mathcal{C}}$ and $\{\Psi_i^{(p)} : 1 \le i \le k\}$ and the ergodic measure μ_p.

Proposition 5.14. *For $p, N \in \mathbb{N}$ and $\varepsilon > 0$, let*

$$\mathcal{G}_1(p,N,\varepsilon) = \bigcap_{n\ge N} \bigcap_{i=1}^{k} \Big\{ x \in X : \Big| \frac{\Psi_{n,i}^{(p)}(x)}{-n} - h_{p,i} \Big| \le \varepsilon \Big\},$$

$$\mathcal{G}_2(p,N,\varepsilon) = \bigcap_{n\ge N} \bigcap_{m=1}^{p} \Big\{ x \in X : \Big| \frac{\Phi_n^{(m)}(x)}{n} - \alpha_{m,p} \Big| \le \varepsilon \Big\},$$

and

$$\mathcal{G}(p,N,\varepsilon)=\mathcal{G}_1(p,N,\varepsilon)\cap\mathcal{G}_2(p,N,\varepsilon).$$

Then for all $p\in\mathbb{N}$ and $\varepsilon_p>0$, there exists an integer $N_p\geq 1$ such that

$$\mu_p(\mathcal{G}(p,N_p,\varepsilon_p))\geq 1-2^{-p}.$$

Let $(\varepsilon_p)_{p\geq 1}$ be a decreasing sequence converging to 0. With the notations in the previous proposition, for each p we choose any $N'_p\geq N_p$. A precise choice of the integers N'_p will be given later. Let $\mathcal{F}_p$ denote the σ-algebra generated by $\Big\{[I]:I\in\mathcal{A}_1^{N'_p}\Big\}$. We define

$$\mathcal{G}_p=\Big\{I\in\mathcal{A}_1^{N'_p}:[I]\cap\mathcal{G}(p,N_p,\varepsilon_p)\neq\emptyset\Big\}.$$

Then we denote by $\widetilde{\mu}_p$ the restriction of μ_p to $\mathcal{F}_p$ and define

$$\nu_p=\otimes_{l=1}^{p}\widetilde{\mu}_l \text{ on } \Big(X,\otimes_{l=1}^{p}\mathcal{F}_p\Big),\quad p\geq 1,$$
$$\nu=\otimes_{p=1}^{\infty}\widetilde{\mu}_p \text{ on } \Big(X,\otimes_{p=1}^{\infty}\mathcal{F}_p\Big),$$

and

$$\mathcal{G}:=\otimes_{p\geq 1}\mathcal{G}_p=\{I_1I_2\cdots I_p\cdots\in X_1:\forall\ p\geq 1,\ I_p\in\mathcal{G}_p)\}.$$

By construction, we have

$$\nu(\mathcal{G})=\prod_{p\geq 1}\widetilde{\mu}_p(\mathcal{G}_p)\geq\prod_{p\geq 1}\mu_p(\mathcal{G}(p,N_p,\varepsilon_p))=\prod_{p\geq 1}(1-2^{-p})>0.$$

To conclude, it is enough to show that we can choose the sequence $(N'_p)_{p\geq 1}$ such that

$$\mathcal{G}\subset\mathcal{G}(\mu) \text{ and} \tag{5.6}$$

$$\liminf_{n\to\infty}\frac{\log\nu(B(x,e^{-n/a_1}))}{-n/a_1}\geq\sum_{i=1}^{k}a_ih_i\quad\text{for all } x\in\mathcal{G}. \tag{5.7}$$

Then, $\nu:=\dfrac{\nu_{|\mathcal{G}}}{\nu(\mathcal{G})}$ is desired.

Let us establish (5.6) and (5.7).

Proof of (5.6). We choose $N'_1=N_1$ and require that the sequence $(N'_p)_{p\geq 1}$ satisfies

$$M_p:=(p+1)\max_{1\leq m\leq p+1}\log(c_m)+\max_{1\leq m\leq p+1}\max_{1\leq l\leq N_{p+1}}\|\Phi_l^{(m)}\|_\infty=o\Big(\sum_{l=1}^{p}N'_l\Big) \tag{5.8}$$

as $p\to\infty$. Then, for every $p\geq 1$ let

$$L_p=\sum_{i=1}^{p}N'_i.$$

Due to the density of $\widetilde{\mathcal{C}}$, it is enough to prove that for each $m\geq 1$ and $x\in\mathcal{G}$ we have

$$\lim_{n\to\infty}\frac{\Phi_n^{(m)}(x)}{n}=\Phi_*^{(m)}(\mu)\ (:=\alpha_m). \tag{5.9}$$

Fix $m \geq 1$ and $x \in \mathcal{G}$. For $n \geq N_1$, let $t(n) = \max\{p : L_p \leq n\}$. For all $n > L_{m+1}$, write

$$\Phi_n^{(m)}(x) = \Phi_{L_m}^{(m)}(x) + \sum_{p=m+1}^{t(n)} \Phi_{N'_p}^{(m)}(\sigma^{L_{p-1}}x) + \Phi_{n-L_{t(n)}}^{(m)}(\sigma^{L_{t(n)}}x).$$

By construction, for $m+1 \leq p \leq t(n)$ we have $\sigma^{L_{p-1}}x_{|N'_p} \in \mathcal{G}_p$. Consequently, there exists $x' \in \mathcal{G}(p, N_p, \varepsilon_p)$ such that $x'_{|N'_p} = \sigma^{L_{p-1}}x_{|N'_p}$ and thus

$$|\Phi_{N'_p}^{(m)}(\sigma^{L_{p-1}}x) - N'_p\alpha_{m,p}| \leq |\Phi_{N'_p}^{(m)}(\sigma^{L_{p-1}}x) - \Phi_{N'_p}^{(m)}(x')| + |\Phi_{N'_p}^{(m)}(x') - N'_p\alpha_{m,p}| \leq \log(c_m) + N'_p\varepsilon_p.$$

This yields

$$\Big|\alpha_m\Big(\sum_{p=m+1}^{t(n)} N'_p\Big) - \Big(\sum_{p=m+1}^{t(n)} \Phi_{N'_p}^{(m)}(\sigma^{L_{p-1}}x)\Big)\Big| \leq t(n)\log(c_m) + \sum_{p=m+1}^{t(n)} N'_p(|\alpha_{m,p} - \alpha_m| + \varepsilon_p).$$

Also, if $n - L_{t(n)} \leq N_{t(n)+1}$, we have $|\Phi_{n-L_{t(n)}}^{(m)}(\sigma^{L_{t(n)}}x)| \leq \max_{1\leq l\leq N_{t(n)+1}} \|\Phi_l^{(m)}\|_\infty$, and if $n - L_{t(n)} > N_{t(n)+1}$, then $[\sigma^{L_{t(n)}}x_{|n-L_{t(n)}}] \cap \mathcal{G}(t(n)+1, N_{t(n)+1}, \varepsilon_{t(n)+1}) \neq \emptyset$. By the same argument as above we get

$$\Big|\alpha_m(n - L_{t(n)}) - \Phi_{n-L_{t(n)}}^{(m)}(\sigma^{L_{t(n)}}x)\Big| \leq \log(c_m) + (n - L_{t(n)})(|\alpha_{m,t(n)+1} - \alpha_m| + \varepsilon_{t(n)+1}).$$

It follows that

$$\begin{aligned} |\Phi_n^{(m)}(x) - n\alpha_m| &\leq |\Phi_{L_m}^{(m)}(x)| + M_{t(n)} \\ &\quad + \Big(\sum_{p=m+1}^{t(n)} N'_p(|\alpha_{m,p} - \alpha_m| + \varepsilon_p)\Big) + (n - L_{t(n)})(|\alpha_{m,t(n)+1} - \alpha_m| + \varepsilon_{t(n)+1}). \end{aligned}$$

Due to our choice for $(N'_p)_{p\geq1}$ and the fact that both $|\alpha_{m,p} - \alpha_m|$ and ε_p tend to 0 as p tends to ∞, as well as $M_{t(n)} = o(n)$, we obtain (5.9). This proves (5.6).

Proof of (5.7). For each $p \geq 1$, since μ_p is quasi-Bernoulli, we can fix $\kappa_p > 1$ such that (1.6) holds for $\mu = \mu_p$ and with the constant sequence $c = \kappa_p$.

We need additional properties for $(N'_p)_{p\geq1}$.

The first one is that

$$N'_{p+1} \geq \frac{a_1 + \cdots + a_k}{a_1}\Big(\sum_{i=1}^{p} N'_p\Big) = \frac{a_1 + \cdots + a_k}{a_1} L_p.$$

The second one is

$$(5.10) \qquad \sum_{l=1}^{p+2} \log(\kappa_p) + \max_{1\leq i\leq k}\max_{j\in\{p+1,p+2\}} (h_i N_j + \max_{1\leq n\leq N_j} \|\Psi_{n,i}^{(j)}\|_\infty) = o(L_p) \text{ as } p \to \infty.$$

Fix $x = (x_i)_{i=1}^\infty \in \mathcal{G}$ and $n \geq N'_1$. For $i = 1, \ldots, k$, we use U_i to denote the word $x_{\ell_{i-1}(n)+1} \cdots x_{\ell_i(n)}$. Then by Lemma 5.1,

$$B(x, e^{-n/a_1}) = \{y \in X : \ \forall\ 1 \leq i \leq k,\ \tau_{i-1}(y) \in \tau_{i-1}(U_1 \ldots U_i)\}.$$

Write $B = B(x, e^{-n/a_1})$ for simplicity. Since $N'_{p+1} \geq (a_1 + \cdots + a_k)L_p/a_1$, there are only two cases to be distinguished: either $L_{t(n)} \leq n < \ell_k(n) < L_{t(n)+1}$ or $L_{t(n)} \leq n < L_{t(n)+1} = L_{t(\ell_k(n))} \leq \ell_k(n)$. We deal with the second case and leave the easier first case to the reader.

Let i_0 be the unique $2 \leq i \leq k$ such that $\ell_{i-1}(n) < L_{t(n)+1} \leq \ell_i(n)$. Let

$$\mathcal{C}_n(B) = \left\{ (J_1, \ldots, J_k) \in \prod_{i=1}^{k} \mathcal{A}_1^{\ell_i(n)-\ell_{i-1}(n)} : \ \forall\ 1 \leq i \leq k,\ \tau_{i-1}(J_i) = \tau_{i-1}(U_i) \right\}.$$

We have

$$\nu(B) = \sum_{(J_1,\ldots,J_k)\in\mathcal{C}_n(B)} \nu(J_1 \cdots J_k). \tag{5.11}$$

Write $J_1 (= U_1) = \widetilde{J}_1 \widehat{J}_1$ with $\widetilde{J}_1 \in \mathcal{A}_1^{L_{t(n)}}$ and $\widehat{J}_1 \in \mathcal{A}_1^{n-L_{t(n)}}$, and write $J_{i_0} = \widetilde{J}_{i_0} \widehat{J}_{i_0}$, with $\widetilde{J}_{i_0} \in \mathcal{A}_1^{L_{t(n)+1}-\ell_{i_0-1}(n)}$ and $\widehat{J}_{i_0} \in \mathcal{A}_1^{\ell_{i_0}(n)-L_{t(n)+1}}$. This yields, by definition of $\nu_{t(n)}$ and ν,

$$\nu(B) = \sum_{(J_1,\ldots,J_k)\in\mathcal{C}_n(B)} \nu_{t(n)}(\widetilde{J}_1) \cdot \mu_{t(n)+1}(\widehat{J}_1 J_2 \cdots J_{i_0-1} \widetilde{J}_{i_0}) \cdot \mu_{t(n)+2}(\widehat{J}_{i_0} J_{i_0+1} \cdots J_k).$$

Now, by using the quasi-Bernoulli properties of $\mu_{t(n)+1}$ and $\mu_{t(n)+2}$ we get

$$\nu(B) \approx \sum_{(J_1,\ldots,J_k)\in\mathcal{C}_n(B)} \nu_{t(n)}(\widetilde{J}_1) \cdot \mu_{t(n)+1}(\widehat{J}_1) \cdot \prod_{i=2}^{i_0-1} \mu_{t(n)+1}(J_i)$$
$$\cdot\, \mu_{t(n)+1}(\widetilde{J}_{i_0}) \cdot \mu_{t(n)+2}(\widehat{J}_{i_0}) \cdot \prod_{i=i_0+1}^{k} \mu_{t(n)+2}(J_i),$$

where $\approx$ means the expressions on its left and right hand sides differ from each other by a multiplicative constant belonging to $[\max(\kappa_{t(n)+1}, \kappa_{t(n)+2})^{-k}, \max(\kappa_{t(n)+1}, \kappa_{t(n)+2})^{k}]$.

Accordingly, write $U_1 = \widetilde{U}_1 \widehat{U}_1$ with $\widetilde{U}_1 \in \mathcal{A}_1^{L_{t(n)}}$ and $\widehat{U}_1 \in \mathcal{A}_1^{n-L_{t(n)}}$, and write $U_{i_0} = \widetilde{U}_{i_0} \widehat{U}_{i_0}$, with $\widetilde{U}_{i_0} \in \mathcal{A}_1^{L_{t(n)+1}-\ell_{i_0-1}(n)}$ and $\widehat{U}_{i_0} \in \mathcal{A}_1^{\ell_{i_0}(n)-L_{t(n)+1}}$. Remembering the definition of $\mathcal{C}_n(B)$ we get

$$\nu(B) \approx \prod_{i=1}^{6} T_i,$$

where

$$T_1 = \nu_{t(n)}(\widetilde{U}_1),\ T_2 = \mu_{t(n)+1}(\widehat{U}_1),$$
$$T_3 = \prod_{i=2}^{i_0-1} \mu_{t(n)+1,i}(\tau_{i-1}(U_i)),\ T_4 = \mu_{t(n)+1,i_0}(\tau_{i_0-1}(\widetilde{U}_{i_0})),$$
$$T_5 = \mu_{t(n)+2,i_0}(\tau_{i_0-1}(\widehat{U}_{i_0})),\ T_6 = \prod_{i=i_0+1}^{k} \mu_{t(n)+2,i}(\tau_{i-1}(U_i)).$$

Let us write $\widetilde{U}_1 = K_1 \cdots K_{t(n)}$ with $K_p \in \mathcal{A}_1^{N'_p}$, for $1 \le p \le t(n)$. By construction

$$T_1 = \prod_{p=1}^{t(n)} \mu_p(K_p).$$

Now, we notice that $x_{|\ell_k(n)} = K_1 \cdots K_{t(n)} \widehat{U}_1 U_2 \cdots U_{i_0-1} \widetilde{U}_{i_0} \widehat{U}_{i_0} U_{i_0+1} \cdots U_k$. Since $x \in \mathcal{G}$, we have K_p belongs to $\mathcal{G}_p$ for $1 \le p \le t(n)$. This yields

$$\begin{aligned} \Big|\log T_1 + h_1\Big(\sum_{p=1}^{t(n)} N'_p\Big)\Big| &= \Big|\sum_{p=1}^{t(n)} \log \mu_p(K_p) + h_1\Big(\sum_{p=1}^{t(n)} N'_p\Big)\Big| \\ &\le R_1 := \sum_{p=1}^{t(n)} N'_p(|h_1 - h_{p,1}| + \varepsilon_p). \end{aligned}$$

To control T_2, we notice that if $n - L_{t(n)} \le N_{t(n)+1}$ then $\widehat{U}_1 \in \bigcup_{l=1}^{N_{t(n)+1}} \mathcal{A}_1^l$, hence $|\log(T_2)| \le \max_{1\le l\le N_{t(n)+1}} \|\Psi_{l,1}^{(t(n)+1)}\|_\infty$, and $h_1(n - L_{t(n)}) \le h_1 N_{t(n)+1}$. If $N_{t(n)+1} < n - L_{t(n)} \le N'_{t(n)+1}$, since $[\widehat{U}_1] = [x_{L(n)+1} \cdots x_n]$, we have $[\widehat{U}_1] \cap \mathcal{G}(t(n)+1, N_{t(n)+1}, \varepsilon_{t(n)+1}) \neq \emptyset$, and since the mapping $\Psi_{n-L_{t(n)},1}^{(t(n)+1)}$ is constant over $[\widehat{U}_1]$ we obtain

$$\begin{aligned} |\log T_2 + h_1(n - L_{t(n)})| &\le |\log T_2 + h_{t(n)+1,1}(n - L_{t(n)})| + (n - L_{t(n)})|h_1 - h_{t(n)+1,1}| \\ &\le (n - L_{t(n)})(|h_1 - h_{t(n)+1,1}| + \varepsilon_{t(n)+1}). \end{aligned}$$

In all cases,

$$\begin{aligned} |\log T_2 + h_1(n - L_{t(n)})| &\le R_2 := n(|h_1 - h_{t(n)+1,1}| + \varepsilon_{t(n)+1}) \\ &\qquad + h_1 N_{t(n)+1} + \max_{1\le l\le N_{t(n)+1}} \|\Psi_{l,1}^{(t(n)+1)}\|_\infty. \end{aligned}$$

To control T_3 we proceed as follows. Fix $2 \le i \le i_0 - 1$. Let $\overline{U}_i = [x_{L_{t(n)}+1} \cdots x_{\ell_{i-1}(n)}]$. By using the quasi-Bernoulli property of $\mu_{t(n)+1}$, which holds with the constant $\kappa_{t(n)+1}$, we can get

$$\Big|\log \mu_{t(n)+1}(U_i) - \big(\log \mu_{t(n)+1}(\overline{U}_i U_i) - \log \mu_{t(n)+1}(\overline{U}_i)\big)\Big| \le \log(\kappa_{t(n)+1}). \tag{5.12}$$

Let $N \in \{\ell_{i-1}(n) - L_{t(n)}, \ell_i(n) - L_{t(n)}\}$, and set $U = \overline{U}_i$ if $N = \ell_{i-1}(n) - L_{t(n)}$ and $U = \overline{U}_i U_i$ otherwise. If $N \le N_{t(n)+1}$, we have $|\log \mu_{t(n)+1,i}(\tau_{i-1}(U))| \le \max_{1\le l\le N_{t(n)+1}} \|\Psi_{l,i}^{(t(n)+1)}\|_\infty$, and $h_i N \le h_i N_{t(n)+1}$. If $N_{t(n)+1} < N \le N'_{t(n)+1}$, since $[U] = [x_{L(n)+1} \cdots x_{L(n)+N}]$, we have $[U] \cap \mathcal{G}(t(n)+1, N_{t(n)+1}, \varepsilon_{t(n)+1}) \neq \emptyset$, and since the mapping $\Psi_{N,i}^{(t(n)+1)}$ is constant over $[U]$ we obtain

$$\begin{aligned} &|\log \mu_{t(n)+1,i}(\tau_{i-1}(U)) + h_i N| \\ &\quad \le |\log \mu_{t(n)+1,i}(\tau_{i-1}(U)) + h_{t(n)+1,i} N| + N|h_i - h_{t(n)+1,i}| \\ &\quad \le N(|h_i - h_{t(n)+1,i}| + \varepsilon_{t(n)+1}), \end{aligned}$$

hence (using that $N \le \ell_i(n)$)

$$\big|\log \mu_{t(n)+1,i}(\tau_{i-1}(\overline{U}_i U_i) - \log \mu_{t(n)+1,i}(\tau_{i-1}(\overline{U}_i)) + h_i(\ell_i(n) - \ell_{i-1}(n))\big| \le r_i := 2\big(\ell_i(n)(|h_i - h_{t(n)+1,i}| + \varepsilon_{t(n)+1}) + h_i N_{t(n)+1} + \max_{1\le l\le N_{t(n)+1}} \|\Psi_{l,i}^{(t(n)+1)}\|_\infty\big).$$

Combining this with (5.12) we get

$$\Big|\log T_3 + \sum_{i=2}^{i_0-1} h_i(\ell_i(n) - \ell_{i-1}(n))\Big| \le R_3 := \sum_{i=2}^{i_0-1} \big(r_i + \log(\kappa_{t(n)+1})\big).$$

By using the same arguments as for T_2 and T_3 we obtain

$$\begin{cases} |\log T_4 + h_{i_0}(L_{t(n)+1} - \ell_{i_0-1}(n))| & \le R_4, \\ |\log T_5 + h_{i_0}(\ell_{i_0}(n) - L_{t(n)+1})| & \le R_5, \\ \Big|\log T_6 + \displaystyle\sum_{i=i_0+1}^{k} h_i(\ell_i(n) - \ell_{i-1}(n))\Big| & \le R_6, \end{cases}$$

with

$$\begin{aligned} R_4 &= 2\big(L_{t(n)+1}(|h_{i_0} - h_{t(n)+1,i_0}| + \varepsilon_{t(n)+1}) + h_{i_0+1} N_{t(n)+1} \\ &\quad + \max_{1\le l\le N_{t(n)+1}} \|\Psi_{l,i_0}^{(t(n)+1)}\|_\infty + \log(\kappa_{t(n)+1})\big); \\ R_5 &= \ell_{i_0}(n)(|h_{i_0} - h_{t(n)+2,i_0}| + \varepsilon_{t(n)+2}) + h_{i_0} N_{t(n)+2} + \max_{1\le l\le N_{t(n)+2}} \|\Psi_{l,i_0}^{(t(n)+2)}\|_\infty; \\ R_6 &= 2\sum_{i=i_0+1}^{k} \Big(\ell_i(n)(|h_i - h_{t(n)+2,i}| + \varepsilon_{t(n)+2}) + h_i N_{t(n)+2} \\ &\quad + \max_{1\le l\le N_{t(n)+2}} \|\Psi_{l,i}^{(t(n)+2)}\|_\infty + \log(\kappa_{t(n)+2})\Big). \end{aligned}$$

All the previous estimates yield, by construction of $(\varepsilon_p)_{p\ge1}$, $(N'_p)_{p\ge1}$ and the convergence of $h_{p,i}$ to h_i as $p \to \infty$,

$$\begin{aligned} \left|\log(\nu(B)) + \sum_{i=1}^{k} h_i(\ell_i(n) - \ell_{i-1}(n))\right| &\le k\sum_{p=1}^{t(n)+2} \log(\kappa_p) + \sum_{i=1}^{6} R_i \\ &= o(L_{t(n)} + \ell_k(n)) = o(n). \end{aligned}$$

Since $\lim_{n\to\infty} \frac{a_1}{n}\sum_{i=1}^{k} h_i(\ell_i(n) - \ell_{i-1}(n)) = \sum_{i=1}^{k} a_i h_i$, we get $\lim_{n\to\infty} \frac{\log(\nu(B(x, e^{-n/a_1})))}{-n/a_1} = \sum_{i=1}^{k} a_i h_i$. This finishes the proof of Theorem 5.13. □

5.4. **Proof of Theorem 5.4.** By Theorem 4.5, there exists a sequence of invariant quasi-Bernoulli measures $(\mu_p)_{p\ge1}$ converging to μ in the weak-star topology, such that $h_{\mu_p\circ\tau_{i-1}^{-1}}(T_i)$ converges to $h_{\mu\circ\tau_{i-1}^{-1}}(T_i)$ for each $1 \le i \le k$, as $p \to \infty$ (use the same argument as in Remark 1.6). Then, the lower bound for $\dim_H \mathcal{G}(\mu)$ is a direct consequence of Theorem 5.13. For the upper bound, we notice that $\mathcal{G}(\mu) \subset \bigcap_{\Phi\in\mathcal{C}(X,T)} E_\Phi(\Phi_*(\mu))$, where

$\Phi \in \mathcal{C}(X,T)$ means $\Phi = (S_n\varphi)_{n=1}^\infty$ for some $\varphi \in C(X)$. Thus, by using Lemma 5.16 whose proof is independent of the present one, we obtain

$$\begin{aligned}\dim_H \mathcal{G}(\mu) &\le \inf_{\Phi\in\mathcal{C}(X,T)} \dim_H E_\Phi(\Phi_*(\mu)) \\ &\le \inf_{\Phi\in\mathcal{C}(X,T)} \inf_{q\in\mathbb{R}} P^{\mathbf{a}}(T,q\Phi) - q\Phi_*(\mu) \\ &= \inf_{q\in\mathbb{R}} \inf_{\Phi\in\mathcal{C}(X,T)} P^{\mathbf{a}}(T,q\Phi) - q\Phi_*(\mu) \\ &= \inf_{\Phi\in\mathcal{C}(X,T)} P^{\mathbf{a}}(T,\Phi) - \Phi_*(\mu).\end{aligned}$$

Now we note that, on the one hand, the **a**-weighted topological pressure is the Legendre-Fenchel transform of the **a**-weighted entropy defined on the compact convex set $\mathcal{M}(X,T)$ of $C(X)^*$ endowed with the weak-star topology, and on the other hand, the **a**-weighted entropy is upper semi-continuous. Hence we have $\inf_{\Phi\in\mathcal{C}(X,T)} P^{\mathbf{a}}(T,\Phi) - \Phi_*(\mu) = h_\mu^{\mathbf{a}}(T)$ by mimicking the proof of Theorem 3.12 in [43]. This yields the conclusion. $\square$

5.5. **Proof of Theorem 5.5.** We first prove Theorem 5.5(1). For $\alpha \in L_{\mathbf{\Phi}}$ let

$$f_{\mathbf{\Phi}}(\alpha) = \max\{h_\mu^{\mathbf{a}}(T):\ \mu \in \mathcal{M}(X,T),\ \mathbf{\Phi}_*(\mu) = \alpha\}.$$

Since the mapping $\mu \in \mathcal{M}(X,T) \mapsto \sum_{i=1}^k a_i h_{\mu\circ\tau_{i-1}^{-1}}(T_i)$ is upper semi-continuous and affine, the equality $f_{\mathbf{\Phi}}(\alpha) = \inf\left\{P^{\mathbf{a}}(T,\mathbf{q}\cdot\mathbf{\Phi}) - \alpha\cdot\mathbf{q}:\ \mathbf{q}\in\mathbb{R}^d\right\}$ for $\alpha \in L_{\mathbf{\Phi}}$ is obtained by exactly the same arguments as those used to prove Theorem 5.2(iii) in [22]; one just replaces the usual entropy by the **a**-weighted one. Similarly, the proof of the equivalence between (i), (iii) and (iv) follow the same lines as that of Theorem 5.2 (ii) in [22].

Consequently, to conclude it only remains to show that

$$E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha) \ne \emptyset \text{ and } \dim_H E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha) \ge f_{\mathbf{\Phi}}(\alpha) \text{ if } \alpha \in L_{\mathbf{\Phi}}; \tag{5.13}$$

$$\dim_H E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha) \le \inf\left\{P^{\mathbf{a}}(T,\mathbf{q}\cdot\mathbf{\Phi}) - \alpha\cdot\mathbf{q}:\ \mathbf{q}\in\mathbb{R}^d\right\} \text{ if } E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha) \ne \emptyset, \tag{5.14}$$

since these properties clearly yield the equivalence of (i) and (ii), as well as the value of $\dim_H E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha)$.

Assertion (5.13) is an immediate consequence of Theorem 5.4 and the following lemma.

Lemma 5.15. *Let $\alpha = (\alpha_1,\dots,\alpha_d) \in L_{\mathbf{\Phi}}$ and $\mu \in \mathcal{M}(X,T)$ such that $\mathbf{\Phi}_*(\mu) = \alpha$. We have $\mathcal{G}(\mu) \subset E_{\{\Phi^{(j)}\},\mathbf{c}}(\alpha)$.*

Proof of Lemma 5.15. By definition of $\mathbf{\Phi}$, we have $\alpha_i = \sum_{j=1}^r (\Phi_i^{(j)})_*(\mu)$ for each $1 \le i \le d$. Moreover, by the definition of $\mathcal{G}(\mu)$, we have $\mathcal{G}(\mu) \subset E_{\Phi_i^{(j)}}((\Phi_i^{(j)})_*(\mu))$ for each $1 \le j \le r$ and $1 \le i \le d$, hence for each $x \in \mathcal{G}(\mu)$ we have $\lim_{n\to\infty} \sum_{j=1}^r \frac{\Phi^{(j)}_{\lfloor c_j n\rfloor,i}(x)}{\lfloor c_j n\rfloor} = \alpha_i$ for each $1 \le i \le d$. This yields $\mathcal{G}(\mu) \subset E_{\{\Phi^{(j)}\},\mathbf{c}}(\alpha)$. $\square$

Now we establish (5.14). Define the following sequence of functions

$$\mathbf{\Phi}_{\mathbf{c},n} = n\sum_{j=1}^{r}\frac{\mathbf{\Phi}^{(j)}_{\lfloor c_j n\rfloor}}{\lfloor c_j n\rfloor}. \tag{5.15}$$

We have the following lemma, which yields (5.14).

Lemma 5.16. *Fix $\alpha\in\mathbb{R}^d$ and suppose that $E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha)\neq\emptyset$. For every $\varepsilon>0$ and $\mathbf{q}\in\mathbb{R}^d$, we have $\dim_H E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha,\varepsilon)\le P^{\mathbf{a}}(T,\mathbf{q}\cdot\mathbf{\Phi})-\alpha\cdot\mathbf{q}+(4|\mathbf{q}|+a_1)\varepsilon$, where $E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha,\varepsilon)=\{x\in X:\limsup_{n\to\infty}|\mathbf{\Phi}_{\mathbf{c},n}(x)/n-\alpha|\le\varepsilon\}$. Consequently, if $E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha)\neq\emptyset$, then $\dim_H E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha)\le\inf_{\mathbf{q}\in\mathbb{R}^d}P^{\mathbf{a}}(T,\mathbf{q}\cdot\mathbf{\Phi})-\alpha\cdot\mathbf{q}$, i.e., (5.14) holds.*

Proof of Lemma 5.16. Since $E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha)=E_{\{\mathbf{\Phi}^{(j)}\},\lambda\mathbf{c}}(\alpha)$ for all $\lambda>0$, without loss of generality we assume that $c_j>1$ for all j.

Fix $\varepsilon>0$ and $\mathbf{q}\in\mathbb{R}^d$. For each $1\le j\le r$, choose $\widetilde{\mathbf{\Phi}}^{(j)}\in\mathcal{C}_{aa}(X,T)^d$ such that each of its components satisfies the bounded distortion property and

$$\sup_{1\le i\le d}\limsup_{n\to\infty}\|\widetilde{\mathbf{\Phi}}^{(j)}_{i,n}-\mathbf{\Phi}^{(j)}_{i,n}\|_\infty/n\le\varepsilon/r.$$

Then we define $\widetilde{\mathbf{\Phi}}=\sum_{j=1}^r\widetilde{\mathbf{\Phi}}^{(j)}$ and the sequence of functions

$$\widetilde{\mathbf{\Phi}}_{\mathbf{c},n}=n\sum_{j=1}^{r}\frac{\widetilde{\mathbf{\Phi}}^{(j)}_{\lfloor c_j n\rfloor}}{\lfloor c_j n\rfloor}\quad(n\ge 1).$$

Endow the space $\mathbb{R}^d$ with the norm $|(z_1,\ldots,z_d)|=\max_{1\le i\le d}|z_i|$. By construction we have $\limsup_{n\to\infty}\|\widetilde{\mathbf{\Phi}}_{\mathbf{c},n}-\mathbf{\Phi}_{\mathbf{c},n}\|_\infty/n\le\varepsilon$ so

$$E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha,\varepsilon)\subset E_{\{\widetilde{\mathbf{\Phi}}^{(j)}\},\mathbf{c}}(\alpha,2\varepsilon)=\{x\in X:\limsup_{n\to\infty}|\widetilde{\mathbf{\Phi}}_{\mathbf{c},n}(x)/n-\alpha|\le 2\varepsilon\}.$$

The definition of the $\mathbf{a}$-weighted topological pressure implies

$$|P^{\mathbf{a}}(T,\mathbf{q}\cdot\widetilde{\mathbf{\Phi}})-P^{\mathbf{a}}(T,\mathbf{q}\cdot\mathbf{\Phi})|\le|\mathbf{q}|\varepsilon. \tag{5.16}$$

Let us denote by $\mu_{\mathbf{q}}$ the unique $\mathbf{a}$-weighted equilibrium state of $\mathbf{q}\cdot\widetilde{\mathbf{\Phi}}$ (see Theorem 4.2). The following key property holds.

Lemma 5.17. *For all $x\in X$, we have $\limsup_{n\to\infty}f_n(x)^{1/n}\ge 1$, where*

$$f_n(x)=\frac{\mu_{\mathbf{q}}(B(x,e^{-n/a_1}))}{\exp\big((\mathbf{q}\cdot\widetilde{\mathbf{\Phi}}_{\mathbf{c},n}(x)-nP^{\mathbf{a}}(T,\mathbf{q}\cdot\widetilde{\mathbf{\Phi}}))/a_1\big)}.$$

It is worth mentioning that the idea of considering the asymptotic behavior of such a function f_n at *each* point of X goes back to [34] for the upper bound estimate of $\dim_H X$ when $k=2$. The proof of Lemma 5.17 will be given later. To finish the proof of Lemma 5.16, we need the following classical lemma.

Lemma 5.18 ([7], Ch. 14)**.** *Let E be a non-empty subset of a compact metric space (Y, d) endowed with an ultrametric distance. Let ν be a positive Borel measure on Y. Then* $\dim_H E \leq \sup_{x\in E} \liminf_{r\to 0^+} \dfrac{\log \nu(B(x,r))}{\log(r)}$.

Now, if $x \in E_{\{\widetilde{\mathbf{\Phi}}^{(j)}\},\mathbf{c}}(\alpha, 2\varepsilon)$ then, due to Lemma 5.17, for infinitely many n we have simultaneously $f_n(x) \geq \exp(-n\varepsilon)$, and $\exp(\mathbf{q}\cdot \widetilde{\mathbf{\Phi}}_{\mathbf{c},n}(x)) \geq \exp(n\alpha\cdot\mathbf{q}) - 3|\mathbf{q}|\varepsilon n$. Consequently,

$$\liminf_{n\to\infty} \frac{\log \mu_{\mathbf{q}}(B(x, e^{-n/a_1}))}{-n/a_1} \leq P^{\mathbf{a}}(T, \mathbf{q}\cdot\widetilde{\mathbf{\Phi}}) - \alpha\cdot\mathbf{q} + (3|\mathbf{q}| + a_1)\varepsilon.$$

Now, Lemma 5.18 and (5.16) yield

$$\begin{aligned}\dim_H E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha,\varepsilon) &\leq \dim_H E_{\{\widetilde{\mathbf{\Phi}}^{(j)}\},\mathbf{c}}(\alpha, 2\varepsilon) \leq P^{\mathbf{a}}(T,\mathbf{q}\cdot\widetilde{\mathbf{\Phi}}) - \alpha\cdot\mathbf{q} + (3|\mathbf{q}|+a_1)\varepsilon \\ &\leq P^{\mathbf{a}}(T, \mathbf{q}\cdot\mathbf{\Phi}) - \alpha\cdot\mathbf{q} + (4|\mathbf{q}| + a_1)\varepsilon.\end{aligned}$$

Letting $\varepsilon \to 0$, we obtain $\dim_H E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha) \leq P^{\mathbf{a}}(T, \mathbf{q}\cdot\mathbf{\Phi}) - \alpha\cdot\mathbf{q}$. Since $\mathbf{q}\in\mathbb{R}^d$ is arbitrarily given, we have

$$\dim_H E_{\{\Phi^{(j)}\},\mathbf{c}}(\alpha) \leq \inf_{\mathbf{q}\in\mathbb{R}^d} P^{\mathbf{a}}(T,\mathbf{q}\cdot\mathbf{\Phi}) - \alpha\cdot\mathbf{q}.$$

This finishes the proof of Lemma 5.16. □

Before we prove Lemma 5.17, we give some auxiliary lemmas.

Lemma 5.19 ([28], Lemma 4.1)**.** *Let $m \geq 1$ be an integer. For $1 \leq j \leq m$ let $f_j : \mathbb{N} \to \mathbb{R}$, $\beta_j > 0$ and $\lambda_j > 0$. If $\sup_{n\geq 1} |f_j(n+1) - f_j(n)| < \infty$ for each j, then*

$$\limsup_{t\to\infty} \frac{1}{t} \sum_{j=1}^{r} \Big(\beta_j f_j\Big(\Big\lfloor \frac{t}{\lambda_j}\Big\rfloor\Big) - f_j\Big(\Big\lfloor\frac{\beta_j t}{\lambda_j}\Big\rfloor\Big)\Big) \geq 0.$$

The following lemma is essentially the same as the above one.

Lemma 5.20. *Let $m \geq 1$ be an integer. Consider $(\beta_p)_{1\leq j\leq m}$ and $(\gamma_p)_{1\leq j\leq m}$ two positive vectors, as well as $v_1, \ldots, v_m$, m bounded sequences such that $v_j(n+1) - v_j(n) = O(n^{-1})$ for each $1 \leq j \leq m$. Then $\limsup_{n\to\infty} \sum_{j=1}^m v_j(\lfloor\beta_j n\rfloor) - v_j(\lfloor\gamma_j n\rfloor) \geq 0$.*

Proof of Lemma 5.17. Let us denote $\mathbf{q}\cdot\widetilde{\mathbf{\Phi}}_{\mathbf{c}}$, $\mathbf{q}\cdot\widetilde{\mathbf{\Phi}}$ and $\mathbf{q}\cdot\widetilde{\mathbf{\Phi}}^{(j)}$ by $\widetilde{\Phi}_c$, $\widetilde{\Phi}$ and $\widetilde{\Phi}^{(j)}$ respectively. Next write $\widetilde{\Phi}_{\mathbf{c}}$ under the following form:

$$\widetilde{\Phi}_{\mathbf{c},n} = \widetilde{\Phi}_n + n\sum_{j=1}^{r} \frac{\widetilde{\Phi}^{(j)}_{\lfloor c_j n\rfloor}}{\lfloor c_j n\rfloor} - \frac{\widetilde{\Phi}^{(j)}_n}{n}.$$

Let $x\in X$, $n\geq 1$ and let $B = B(x, e^{-n/a_1})$. By Lemma 5.2, we have

$$\mu_{\mathbf{q}}(B) \;\approx\; \exp\Big(\frac{-nP^{\mathbf{a}}(T,\widetilde{\Phi})}{a_1}\Big)\exp(\widetilde{\Phi}_n(x)/a_1)\prod_{j=1}^{k-1} \frac{\widetilde{\phi}^{(j)}(\tau_j(x_{|\ell_{j+1}(n)}))^{1/A_{j+1}}}{\widetilde{\phi}^{(j)}(\tau_j(x_{|\ell_j(n)}))^{1/A_j}}.$$

Combining this with the definition of $f_n(x)$ yields

$$f_n(x) \approx \exp((\widetilde{\Phi}_n - \widetilde{\Phi}_{\mathbf{c},n}))(x)/a_1) \prod_{j=1}^{k-1} \frac{\widetilde{\phi}^{(j)}(\tau_j(x_{|\ell_{j+1}(n)}))^{1/A_{j+1}}}{\widetilde{\phi}^{(j)}(\tau_j(x_{|\ell_j(n)}))^{1/A_j}}. \tag{5.17}$$

Now, for $n \geq 1$, let us define

$$\begin{cases} u^{(j)}(n) = -\widetilde{\Phi}_n^{(j)}(x)/(a_1 n) & \text{for } 1 \leq j \leq r, \\ \widetilde{u}^{(j)}(n) = \log \widetilde{\phi}^{(j)}(\tau_j(x_{|n}))/(a_1 n) & \text{for } 1 \leq j \leq k-1. \end{cases}$$

We notice that since the almost additive potentials $\widetilde{\Phi}$ and $\widetilde{\mathbf{\Phi}}^{(j)}$ satisfy the bounded distortion property, for any $v \in \{u^{(j)}, \widetilde{u}^{(j)} : 1 \leq j \leq r,\ 1 \leq i \leq k-1\}$ the sequence $(v(n))_{n\geq 1}$ is bounded and $v(n+1) - v(n) = O(n^{-1})$. Then, by using (5.17) we can get

$$\frac{\log f_n(x)}{n} = \sum_{j=1}^{r} (u^{(j)}(\lfloor c_j n \rfloor) - u^{(j)}(n)) + \sum_{j=1}^{k-1} (\widetilde{u}^{(i)}(\lfloor \widetilde{c}_i n \rfloor) - \widetilde{u}^{(i)}(\lfloor \widetilde{c}_{i-1} n \rfloor)) + O\Big(\frac{1}{n}\Big).$$

Then, the fact that $\limsup_{n\to\infty} \frac{\log f_n(x)}{n} \geq 0$ comes from Lemma 5.20. This finishes the proof of Lemma 5.17. □

Now we come to the proof of Theorem 5.5(2). It is based on the following lemma and a modification of the Moran construction achieved in the proof of Theorem 5.13. The proof of the lemma is postponed to the end of the section.

Lemma 5.21. *Assume that $L_{\mathbf{\Phi}}$ is not a singleton. Then for all $\varepsilon > 0$, there are two invariant quasi-Bernoulli measures ν_1 and ν_2 on X with $\mathbf{\Phi}_*(\nu_1) \neq \mathbf{\Phi}_*(\nu_2)$, and a non-negative vector $(h_i)_{1\leq i\leq k}$ such that $\sum_{i=1}^k a_i h_i \geq \dim_H X - \varepsilon$ and $h_{\nu_l \circ \tau_{i-1}^{-1}}(T_i) \geq h_i$ for each $l \in \{1, 2\}$ and $1 \leq i \leq k$.*

Let $\delta = |\mathbf{\Phi}_*(\nu_1) - \mathbf{\Phi}_*(\nu_2)|$. For each $1 \leq j \leq r$ and $1 \leq i \leq d$, let $g_i^{(j)}$ be a Hölder potential such that $\limsup_{n\to\infty} \|\Phi_{i,n}^{(j)} - S_n g_i^{(j)}\|_\infty / n \leq \delta/8r$. For each $1 \leq j \leq r$ let $\mathbf{G}^{(j)} = ((S_n g_i^{(j)})_{n=1}^\infty)_{1\leq i\leq d}$, and define $\mathbf{G} = \sum_{j=1}^r \mathbf{G}^{(j)}$. By construction, we have $\limsup_{n\to\infty} \|\mathbf{\Phi}_{\mathbf{c},n} - \mathbf{G}_{\mathbf{c},n}\|_\infty / n \leq \delta/8$ (recall that $\mathbf{\Phi}_{\mathbf{c},n}$ is defined as in (5.15), and we define $\mathbf{G}_{\mathbf{c},n}$ similarly). Moreover, for each $l \in \{1, 2\}$ we have $|\mathbf{\Phi}_*(\nu_l) - \mathbf{G}_*(\nu_l)| \leq \delta/8$, hence $|\mathbf{G}_*(\nu_1) - \mathbf{G}_*(\nu_2)| \geq 3\delta/4$. Thus, the set

$$D_{\mathbf{G}} = \bigcap_{l=1}^{2} \{x \in X : \liminf_{n\to\infty} |\mathbf{G}_{\mathbf{c},n}(x) - n\mathbf{G}_*(\nu_l)|/n \leq \delta/4\}$$

is included in the set of divergent points $X \setminus \bigcup_{\alpha \in L_{\mathbf{\Phi}}} E_{\{\mathbf{\Phi}^{(j)}\},\mathbf{c}}(\alpha)$, and the conclusion will follow if we prove that

$$\dim_H D_{\mathbf{G}} \geq \dim_H X - \varepsilon.$$

Now we briefly explain how to modify the Moran construction done in the proof of Theorem 5.13. At first, without loss of generality, we suppose that the c_j's are greater than 1. Also, we include the potentials $(S_n g_i^{(j)})_{n=1}^\infty)$ in the family $\widetilde{C}$. Then, the only changes are that for each $p \geq 1$, one takes $\mu_{2p-1} = \nu_1$ and $\mu_{2p} = \nu_2$ and to the controls (5.8) and (5.10) one adds $L_{p-1} = o(\sqrt{N'_p})$. Then, for $p \geq 1$, let $n_p = L_{p-1} + \sqrt{N'_p}$. For p large enough, for each $1 \leq j \leq r$ we have $\lfloor c_j n_p \rfloor \in [L_{p-1} + \sqrt{N'_p}, L_p]$, so that for each $x \in \mathcal{G}$, $1 \leq j \leq r$ and $1 \leq i \leq d$ we have $\lim_{p\to\infty} S_{\lfloor c_j n_{2p-1}\rfloor} g_i^{(j)}(x)/\lfloor c_j n_{2p-1}\rfloor = \nu_1(g_i^{(j)})$ and $\lim_{p\to\infty} S_{\lfloor c_j n_{2p}\rfloor} g_i^{(j)}(x)/\lfloor c_j n_{2p}\rfloor = \nu_2(g_i^{(j)})$. Consequently, for each $x \in \mathcal{G}$, we have $\lim_{p\to\infty} \mathbf{G}_{\mathbf{c},n_{2p-1}}(x)/n_{2p-1} = \mathbf{G}_*(\nu_1)$ and $\lim_{p\to\infty} \mathbf{G}_{\mathbf{c},n_{2p}}(x)/n_{2p} = \mathbf{G}_*(\nu_2)$, so $\mathcal{G} \subset D_{\mathbf{G}}$. Moreover, the simultaneous controls from below of the entropies $h_{\nu_l\circ\tau_{i-1}^{-1}}(T_i)$ by the same h_i yield, for every $x \in \mathcal{G}$, $\liminf_{n\to\infty} \frac{\log \nu(B(x,e^{-n/a_1}))}{-n/a_1} \geq \sum_{i=1} a_i h_i \geq \dim_H X - \varepsilon$. □

Proof of Lemma 5.21. Let $g \in C(X)$ be the zero function. Let ν_1 be the $\mathbf{a}$-weighted equilibrium state of g. Then by Theorem 4.2 and Remark 5.6, ν_1 is quasi-Bernoulli, and $h^{\mathbf{a}}_{\nu_1}(T) := \sum_{i=1}^k h_{\nu_1\circ\tau_{i-1}^{-1}}(T_i) = P^{\mathbf{a}}(T,0) = \dim_H X$.

Fix $\varepsilon > 0$, and for each $1 \leq i \leq k$ let $h_i = h_{\nu_1\circ\tau_{i-1}^{-1}}(T_i) - \varepsilon/(a_1 + \cdots + a_k)$. Since $L_{\mathbf{\Phi}}$ is not a singleton, we can pick $\mu \in \mathcal{M}(X,T)$ such that $\mathbf{\Phi}_*(\mu) \neq \mathbf{\Phi}_*(\nu_1)$. Take a large positive integer n so that

$$h_{\mu_2\circ\tau_{i-1}^{-1}}(T_i) \geq h_{\nu_1\circ\tau_{i-1}^{-1}}(T_i) - \varepsilon/(2a_1 + \cdots + 2a_k), \quad (1 \leq i \leq k)$$

where $\mu_2 = (1-1/n)\nu_1 + (1/n)\mu$. Note that $\mathbf{\Phi}_*(\mu_2) = (1-1/n)\mathbf{\Phi}_*(\nu_1) + (1-1/n)\mathbf{\Phi}_*(\mu) \neq \mathbf{\Phi}_*(\nu_1)$. Now by Remark 4.6, we can pick an invariant quasi Bernoulli measure ν_2 so that $h_{\nu_2\circ\tau_{i-1}^{-1}}(T_i) \geq h_{\mu_2\circ\tau_{i-1}^{-1}}(T_i) - \varepsilon/(2a_1+\cdots+2a_k)$, hence $h_{\nu_2\circ\tau_{i-1}^{-1}}(T_i) \geq h_i$ for each $1 \leq i \leq k$. By construction, the pair of measures $\{\nu_1, \nu_2\}$ is as desired. □

5.6. **Proof of Theorem 5.10.** Since $P^{\mathbf{a}}(T,\Phi)/A_k$ is by construction equal to the classical topological pressure of $\Phi^{\mathbf{a}}$, the problem reduces to proving the following assertion: Let $\Psi = (\log(\psi_n))_{n=1}^\infty \in \mathcal{C}_{asa}(X,T)$. There exists a fully supported measure ν such that

$$\nu(x_{|n}) \approx_n \exp(-nP(T,\Psi))\psi_n(x) \quad (\forall\, x \in X, \forall\, n \geq 1).$$

By Lemma 2.1(iii)), we can fix $(g_p)_{p\geq 1}$, a sequence of Hölder potentials such that $\limsup_{n\to\infty} \|(\log(\psi_n) - S_n g_p)\|_\infty/n \leq 2^{-(p+1)}$ for each $p \geq 1$. Then fix a sequence $(r_p)_{p\geq 1}$ such that for each $p \geq 1$ we have $\sup_{n\geq r_p} \|(\log(\psi_n) - S_n g_p)\|_\infty/n \leq 2^{-p}$. In particular, we have $|P_\psi - P_{g_p}| \leq 2^{-p}$, where P_ψ and P_{g_p} stand for $P(T,\Psi)$ and $P(T,g_p)$ respectively.

For each $p \geq 1$, let μ_p be a Gibbs state for g_p and $\kappa_p > 1$ a constant such that

$$\kappa_p^{-1} \leq \frac{\mu_p(x_{|n})}{\exp(-nP_{g_p}\exp(S_n g_p(x))} \leq \kappa_p \quad (\forall\, x \in X, \forall\, n \geq 1).$$

Let $(N_p)_{p\geq 1}$ be a sequence of integers such that

$$\begin{cases} N_p \geq \max(r_p, r_{p+1}), \\ (\log(\kappa_1) + \cdots + \log(\kappa_{p+1})) + L_{p-1} + M_{p+1} = o(\sqrt{N_p}), \end{cases}$$

where $L_p = \sum_{j=1}^p N_j$, and $M_p = \max\{\|g_j\|_\infty : 1 \le j \le p\}$.

For each $p \ge 1$ let $\mathcal{F}_p$ denote the σ-algebra generated by $\Big\{[I] : I \in \mathcal{A}_1^{N_p}\Big\}$. Then denote by $\widetilde{\mu}_p$ the restriction of μ_p to $\mathcal{F}_p$ and define

$$\nu = \otimes_{p=1}^\infty \widetilde{\mu}_p \text{ on } \Big(X, \otimes_{p=1}^\infty \mathcal{F}_p\Big).$$

For $n \ge N_1$ let $t(n) = \max\{p : L_p \le n\}$. For any $x \in X$ and $n \ge 1$ we have

$$\nu(x_{|n}) = \Big(\prod_{p=1}^{t(n)} \mu_p(T^{L_{p-1}} x_{|N_p})\Big) \mu_{t(n)+1}\big(T^{L_{t(n)}} x_{|n-L_{t(n)}}\big).$$

For each $1 \le p \le t(n)-1$ we have

$$\begin{aligned} & \big|\log\big(\mu_p(T^{L_{p-1}} x_{|N_p})\big) - P_\psi N_p - S_{N_p} g_{t(n)+1}(T^{L_{p-1}} x_{|N_p})\big| \\ \le\ & \log(\kappa_p) + |P_\psi - P_{g_p}| N_p + \big|S_{N_p}(g_p - g_{t(n)+1})(T^{L_{p-1}} x_{|N_p})\big| \\ \le\ & \log(\kappa_p) + 2^{-p} N_p + 2M_{t(n)+1} N_p. \end{aligned}$$

Moreover,

$$\begin{aligned} & \big|\log\big(\mu_{t(n)}(T^{L_{t(n)-1}} x_{|N_{t(n)}})\big) - P_\psi N_{t(n)} - S_{N_{t(n)}} g_{t(n)+1}(T^{L_{t(n)-1}} x_{|N_{t(n)}})\big| \\ \le\ & \log(\kappa_{t(n)}) + |P_\psi - P_{g_{t(n)}}| N_{t(n)} + \big|S_{N_{t(n)}}(g_{t(n)} - g_{t(n)+1})(T^{L_{t(n)-1}} x_{|N_{t(n)}})\big| \\ \le\ & \log(\kappa_{t(n)}) + 2^{-t(n)} N_{t(n)} + \|S_{N_{t(n)}}(g_{t(n)} - g_{t(n)+1})\|_\infty. \end{aligned}$$

Also, denoting $n - L_{t(n)}$ by R_n we have

$$\begin{aligned} & \big|\log\big(\mu_{t(n)+1}(T^{L_{t(n)}} x_{|R_n})\big) - P_\psi R_n - S_{R_n} g_{t(n)+1}(T^{L_{t(n)}} x_{|R_n})\big| \\ \le\ & \log(\kappa_{t(n)+1}) + |P_\psi - P_{g_{t(n)+1}}| R_n \le \log(\kappa_{t(n)+1}) + 2^{-t(n)+1} R_n. \end{aligned}$$

We deduce from the previous estimations that

$$\begin{aligned} & \big|\log(\nu(x_{|n})) - nP_\psi - \log(\psi_n(x))\big| \\ \le\ & \|\log(\psi_n) - S_n g_{t(n)+1}\|_\infty + \big|\log(\nu(x_{|n})) - nP_\psi - S_n g_{t(n)+1}(x)\big| \\ \le\ & \|\log(\psi_n) - S_n g_{t(n)+1}\|_\infty + \|S_{N_{t(n)}}(g_{t(n)} - g_{t(n)+1})\|_\infty \\ & +2M_{t(n)+1} L_{t(n)-1} + 2^{-t(n)+1}(n - L_{t(n)}) + \sum_{p=1}^{t(n)} 2^{-p} N_p + \sum_{p=1}^{t(n)+1} \log(\kappa_p). \end{aligned}$$

Since $n \ge N_{t(n)} \ge \max(r_{t(n)}, r_{t(n)+1})$ we have $\|\log(\psi_n) - S_n g_{t(n)+1}\|_\infty \le 2^{-t(n)+1} n$ and $\|S_{N_{t(n)}}(g_{t(n)} - g_{t(n)+1})\|_\infty \le (2^{-t(n)} + 2^{-t(n)+1}) N_{t(n)}$. So both terms are $o(n)$, uniformly in x. Moreover, by construction $2M_{t(n)+1} L_{t(n)-1} = (o(\sqrt{n}))^2 = o(n)$, $2^{-t(n)+1}(n - L_{t(n)}) + \sum_{p=1}^{t(n)} 2^{-p} N_p = o(n)$ and $\sum_{p=1}^{t(n)+1} \log(\kappa_p) = o(\sqrt{n})$ uniformly in x. Consequently,

$$\lim_{n\to\infty} \frac{1}{n} \sup_{x \in X} \big|\log(\nu(x_{|n})) - nP_\psi - \log(\psi_n(x))\big| = 0.$$

When $\Phi \in \mathcal{C}_{aa}(X,T)$ and satisfies the bounded distortion property, relation (5.5) is obtained by using (5.11), which holds for any positive measure ν, and then the quasi-Bernoulli property of μ. Then (5.5) yields (5.4) in this case. To get (5.4) in the general case, let $(g_p)_{p\geq 1}$ as above. For each $p \geq 1$, let μ_p be the unique **a**-weighted-equilibrium state associated with g_p. By construction, we have $\lim_{p\to\infty} \limsup_{n\to\infty} \|\mathbf{\Psi}^{(i)}_{\mu,n} - \mathbf{\Psi}^{(i)}_{\mu_p,n}\|_\infty/n = 0$ for each $1 \leq i \leq k$; in particular $\mathbf{\Psi}^{(i)}_\mu \in \mathcal{C}_{asa}(X,T)$. Fix $\varepsilon > 0$. Applying (5.11) to μ, we can find $p_\varepsilon \in \mathbb{N}_+$ and $N_\varepsilon \in N_+$ such that for $n \geq N_\varepsilon$ we have

$$\begin{cases} \|\mathbf{\Psi}^{(i)}_{\mu,n} - \mathbf{\Psi}^{(i)}_{\mu_{p_\varepsilon},n}\|_\infty \leq n\varepsilon & \forall\, 1 \leq i \leq k \\ \exp(-2\ell_k(n)\varepsilon)\mu_{p_\varepsilon}(B) \leq \mu(B) \leq \exp(2\ell_k(n)\varepsilon)\mu_{p_\varepsilon}(B) & \forall\, B \in \mathcal{B}_n \end{cases}.$$

Let $c(x,n)$ be associated with μ_{p_ε} like in (5.4) for μ_{p_ε}. We know that $c(x,n)$ is bounded independently of x and n by a constant $c(\mu_{p_\varepsilon})$. By using the validity of (5.4) for μ_{p_ε}, for every $n \geq N_\varepsilon$ large enough so that $c(\mu_{p_\varepsilon}) \leq n\varepsilon$, for every $x \in X$ and $B = B(x, e^{-n/a_1})$ we get

$$\begin{aligned} & \Big| \log \mu(B) - \mathbf{\Psi}^{(1)}_{\mu,n}(x) + \sum_{i=2}^{k} \mathbf{\Psi}^{(i)}_{\mu,\ell_i(n)}(x) - \mathbf{\Psi}^{(i)}_{\mu,\ell_{i-1}(n)}(x) \Big| \\ \leq\ & \Big| \log \mu_{p_\varepsilon}(B) - \mathbf{\Psi}^{(1)}_{\mu_{p_\varepsilon},n}(x) + \sum_{i=2}^{k} \mathbf{\Psi}^{(i)}_{\mu_{p_\varepsilon},\ell_i(n)}(x) - \mathbf{\Psi}^{(i)}_{\mu_{p_\varepsilon},\ell_{i-1}(n)}(x) \Big| \\ & + |\log \mu(B) - \log \mu_{p_\varepsilon}(B)| + 2\sum_{i=1}^{k} \|\mathbf{\Psi}^{(i)}_{\mu,\ell_i(n)} - \mathbf{\Psi}^{(i)}_{\mu_{p_\varepsilon},\ell_i(n)}\|_\infty \\ \leq\ & c(\mu_{p_\varepsilon}) + (2k+2)\ell_k(n)\varepsilon \leq (n + (2k+2)\ell_k(n))\varepsilon. \end{aligned}$$

This yields the desired result. □

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References

[1] J. Barral, M. Mensi, Gibbs measures on self-affine Sierpinski carpets and their singularity spectrum, *Ergod. Th. & Dynam. Sys.* **27** (2007), no. 5, 1419–1443.
[2] J. Barral, M. Mensi, Multifractal analysis of Birkhoff averages on "self-affine" symbolic spaces, *Nonlinearity*, **21** (2008), no. 10, 2409–2425.
[3] L. Barreira, Nonadditive thermodynamic formalism: equilibrium and Gibbs measures. *Discrete Contin. Dyn. Syst.* **16** (2006), 279–305.
[4] L. Barreira, P. Doutor, Almost additive multifractal analysis, *J. Math. Pures. Appl.*, **92** (2009), 1–17.
[5] L. Barreira and J. Schmeling, Sets of "non-typical" points have full topological entropy and full Hausdorff dimension. *Israel J. Math.* **116** (2000), 29–70.
[6] Bedford, T. *Crinkly curves, Markov partitions and box dimension in self-similar sets*, Ph.D. Thesis, University of Warwick, 1984.
[7] P. Billingsley, *Ergodic theory and information.* Wiley, New York, 1965.
[8] R. Bowen, Topological entropy for noncompact sets, *Trans. Amer. Math. Soc.*, **184** (1973), 125–136.
[9] R. Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms.* Lecture notes in Math. No. **470**, Springer-Verlag, 1975.

[10] R. Bowen, Hausdorff dimension of quasicircles. *Inst. Hautes Études Sci. Publ. Math.* No. **50** (1979), 11–25.

[11] G. Brown, G. Michon and J. Peyrière, On the multifractal analysis of measures. *J. Statist. Phys.* **66** (1992), no. 3-4, 775–790.

[12] H. Cajar, *Billingsley dimension in probability spaces.* Lecture Notes in Mathematics, **892**. Springer-Verlag, Berlin-New York, 1981.

[13] Y. L. Cao, D. J. Feng and W. Huang, The thermodynamic formalism for sub-additive potentials. *Discrete Contin. Dyn. Syst.* **20** (2008), 639–657.

[14] T. Downarowicz and J. Serafin, Fiber entropy and conditional variational principles in compact non-metrizable spaces. *Fund. Math.* **172** (2002), no. 3, 217–247.

[15] K. J. Falconer, The Hausdorff dimension of self-affine fractals. *Math. Proc. Cambridge Philos. Soc.* **103** (1988), no. 2, 339–350.

[16] A. H. Fan, Sur la dimension des mesures, *Studia Mathematica* **111** (1994), 1–17.

[17] A. H. Fan, D. J. Feng and J. Wu, Recurrence, dimension and entropy. *J. London Math. Soc.* (2) **64** (2001), no. 1, 229–244.

[18] A. H. Fan, L. Liao, J. Peyrière, Generic points in systems of specification and Banach valued Birkhoff ergodic average. *Discrete Contin. Dyn. Syst.* **21** (2008), 1103–1128.

[19] D. J. Feng, Lyapounov exponents for products of matrices and multifractal analysis. Part I: positive matrices. *Israël J. Math.* **138** (2003), 353–376.

[20] D. J. Feng, The variational principle for products of non-negative matrices. *Nonlinearity* **17** (2004) 447–457.

[21] D. J. Feng, Weighted equilibrium states for factor maps between subshifts. Preprint. arXiv:0909.4250v1

[22] D. J. Feng and W. Huang, Lyapunov spectrum of asymptotically sub-additive potentials. arXiv:0905.2680v1.

[23] D. J. Feng and K. S. Lau, The pressure function for products of non-negative matrices, *Math. Res. Lett.* **9** (2002), 363-378.

[24] D. Gatzouras and Y. Peres, The variational principle for Hausdorff dimension: a survey. Ergodic theory of Z^d actions (Warwick, 1993–1994), 113–125, London Math. Soc. Lecture Note Ser., 228, Cambridge Univ. Press, Cambridge, 1996.

[25] D. Gatzouras and Y. Peres, Invariant measures of full dimension for some expanding maps. *Ergod. Th. & Dynam. Sys.* **17** (1997), no. 1, 147–167.

[26] T. Jordan, K. Simon, Multifractal analysis for Birkhoff averages for some self-affine IFS. *Dyn. Sys.* **22** (2007), 469-483.

[27] A. Käenmäki, On natural invariant measures on generalised iterated function systems. *Ann. Acad. Sci. Fenn. Math.* **29** (2004), 419–458.

[28] R. Kenyon and Y. Peres, Measures of full dimension on affine-invariant sets, *Ergod. Th. & Dynam. Sys.* **16** (1996), 307–323.

[29] M. Kesseböhmer, Large deviation for weak Gibbs measures and multifractal spectra, *Nonlinearity* **14** (2001), 395–409.

[30] J. F. King, The singularity spectrum for general Sierpinski carpets, *Adv. Math.* **116** (1995), 1–8.

[31] S. Lalley and D. Gatzouras, Hausdorff and Box Dimensions of Certain Self-Affine Fractals, *Indiana Univ. Math. J.* **41** (1992), 533–568.

[32] F. Ledrappier and P. Walters, A relativised variational principle for continuous transformations, *J. Lond. Math. Soc.* **16** (1977), 568–576.

[33] D. Lind and B. Marcus, *An Introduction to Symbolic Dynamics and Coding*, Cambridge University Press, Cambridge, 1995

[34] C. McMullen, The Hausdorff dimension of general Sierpinsky carpets, *Nagoya Math. J.* **96** (1984), 1–9.

[35] A. Mummert, The thermodynamic formalism for almost-additive sequences. *Discrete Contin. Dyn. Syst.* **16** (2006), 435–454.

[36] L. Olsen, Self-affine multifractal Sierpinski sponges in $\mathbb{R}^d$, *Pacific J. Math.* **183** (1998), 143–199.

[37] Ya. B. Pesin, *Dimension theory in dynamical systems. Contemporary views and applications.* University of Chicago Press, 1997.

[38] Ya. B. Pesin and H. Weiss, The multifractal analysis of Gibbs measures: motivation, mathematical foundation, and examples. *Chaos* **7** (1997), 89–106.

[39] C. E. Pfister and W. G. Sullivan, On the topological entropy of saturated sets, *Ergod. Th. & Dynam. Sys.* **27** (2007), 929–956.
[40] D. A. Rand, The singularity spectrum $f(\alpha)$ for cookie-cutters, *Ergod. Th. & Dynam. Sys.* **9** (1989), 527–541.
[41] R. T. Rockafellar, R. *Convex analysis.* Princeton Mathematical Series, No. 28, Princeton University Press, Princeton, N.J. 1970.
[42] D. Ruelle, Statistical mechanics on a compact set with Z^v action satisfying expansiveness and specification. *Trans. Amer. Math. Soc.* **187** (1973), 237–251.
[43] D. Ruelle, *Thermodynamic formalism. The mathematical structures of classical equilibrium statistical mechanics.* Encyclopedia of Mathematics and its Applications, 5. Addison-Wesley Publishing Co., Reading, Mass., 1978.
[44] D. Ruelle, Repellers for real analytic maps. *Ergod. Th. & Dynam. Sys.* **2** (1982), 99–107.
[45] P. Walters, A variational principle for the pressure of continuous transformations. *Amer. J. Math.* **97** (1975), 937–971.
[46] P. Walters, *An introduction to ergodic theory.* Springer-Verlag, 1982.
[47] Y. Zhao and Y. L. Cao, On the topological pressure of random bundle transformations in sub-additive case, *J. Math. Anal. Appl.* **342** (2008), 715-725.
[48] Y. Yayama, Existence of a measurable saturated compensation function between subshifts and its applications. Preprint. arXiv:0906.4989
[49] M. Yuri, Zeta functions for certain non-hyperbolic systems and topological Markov approximations, *Ergod. Th. & Dynam. Sys.* **17** (1997), 997–1000.

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