THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1540 University Mathematics for Financial Studies 2016-17 Term 1
Coursework 9

Name: ________________________ Student ID: ___________________ Score: _________

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1. Evaluate $\displaystyle\iint_R ye^x\,dA$, where R is the region in the xy-plane bounded by the curves $y = x$ and $y = \sqrt{x}$.

   Solution: The two curves $y = x$ and $y = \sqrt{x}$ intersect at $x = 0$ and $x = 1$. Hence, the bounds of x and y are

   $$0 \le x \le 1, x \le y \le \sqrt{x}.$$

   Therefore, the double integral is equal to the iterated integral:

   $$\iint_R ye^x\,dA = \int_0^1 \int_x^{\sqrt{x}} ye^x\,dy\,dx = \int_0^1 \frac{1}{2}y^2e^x\Big|_{y=x}^{y=\sqrt{x}}\,dx$$
   $$= \frac{1}{2}\int_0^1 \left(x - x^2\right)e^x\,dx = \frac{1}{2}\left[(xe^x - e^x) - \left(x^2e^x - 2xe^x + 2e^x\right)\right]\Big|_{x=0}^{x=1}$$
   $$= \frac{1}{2}\left[(e - e + 1) - (e - 2e + 2e - 2)\right] = \frac{1}{2}(3 - e).$$

2. Evaluate the following integrals:

   (a) $\displaystyle\int_0^1 \int_y^1 \frac{1}{1+x^4}\,dx\,dy$

   Solution: We first change the order of integration.
   The region of integration is:

   $$R = \{(x, y)|0 \le y \le 1, y \le x \le 1\},$$

   which may also be described as follows:

   $$R = \{(x, y)|0 \le x \le 1, 0 \le y \le x\}.$$

Hence, we have:

$$\begin{aligned}\int_0^1 \int_y^1 \frac{1}{1+x^4}\,dx\,dy &= \int_0^1 \int_0^x \frac{1}{1+x^4}\,dy\,dx \\ &= \int_0^1 \frac{y}{1+x^4}\bigg|_{y=0}^{y=x} dx \\ &= \int_0^1 \frac{x}{1+x^4}\,dx = \frac{1}{2}\int_0^1 \frac{d(x^2)}{1+(x^2)^2} \\ &= \frac{1}{2}\arctan\left(x^2\right)\bigg|_{x=0}^{x=1} \\ &= \frac{\pi}{8}\end{aligned}$$

(b) $\displaystyle\int_0^\pi \int_x^\pi \frac{x\sin y}{y}\,dy\,dx$

The region $R = \{(x,y)\in\mathbb{R}^2 : 0 \le x \le \pi, x \le y \le \pi\}$ may also be described as follows:

$$R = \{(x,y)\in\mathbb{R}^2 : 0 \le y \le \pi, 0 \le x \le y\}$$

Hence, we have:

$$\begin{aligned}\int_0^\pi \int_x^\pi \frac{x\sin y}{y}\,dy\,dx &= \int_0^\pi \int_0^y \frac{x\sin y}{y}\,dx\,dy \\ &= \frac{1}{2}\int_0^\pi \frac{\sin y}{y}x^2\bigg|_{x=0}^{x=y} dy \\ &= \frac{1}{2}\int_0^\pi y\sin y\,dy \\ &= \frac{1}{2}\left(-y\cos y + \sin y\right)\bigg|_0^\pi = \frac{\pi}{2}.\end{aligned}$$

(c) $\displaystyle\int_{-1}^0 \int_{x^2}^1 \frac{x^3e^y}{y^2}\,dy\,dx$

The region of integration is:

$$\begin{aligned}R &= \{(x,y)\in\mathbb{R}^2 : -1 \le x \le 0, x^2 \le y \le 1\} \\ &= \{(x,y)\in\mathbb{R}^2 : 0 \le y \le 1, -\sqrt{y} \le x \le 0\}.\end{aligned}$$

Hence, we have:

$$\begin{aligned}\int_{-1}^0 \int_{x^2}^1 \frac{x^3e^y}{y^2}\,dy\,dx &= \int_0^1 \int_{-\sqrt{y}}^0 \frac{x^3e^y}{y^2}\,dx\,dy \\ &= \frac{1}{4}\int_0^1 \frac{x^4e^y}{y^2}\bigg|_{x=-\sqrt{y}}^{x=0} dy \\ &= -\frac{1}{4}\int_0^1 e^y\,dy = -\frac{1}{4}(e-1).\end{aligned}$$

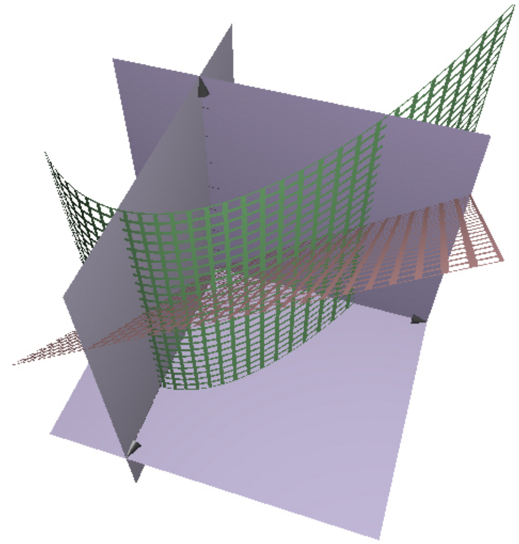
3. (a) Find the volume of the solid in the first octant of $\mathbb{R}^3$ bounded by the planes: $x = 0$, $y = 0$, $z = 0$, and $2x + y + 5z = 4$.
(We define the first octant of $\mathbb{R}^3$ to be the region: $\{(x, y, z) \in \mathbb{R}^3 : x, y, z \geq 0\}$.)
Solution: The volume of the solid is equal to the triple integral $\iiint\limits_S dV$, where S is the solid described.
We have:

$$S = \left\{(x, y, z) \middle| 0 \leq x \leq 2, 0 \leq y \leq 4 - 2x, 0 \leq z \leq \frac{4 - 2x - y}{5}\right\}.$$

Therefore,

$$\begin{aligned}
\text{Volume} &= \int_0^2 \int_0^{4-2x} \int_0^{\frac{4-2x-y}{5}} dz\, dy\, dx \\
&= \int_0^2 \int_0^{4-2x} \frac{4 - 2x - y}{5}\, dy\, dx \\
&= \int_0^2 \left[\left(\frac{4 - 2x}{5}\right) y - \frac{1}{10} y^2\right] \Bigg|_{y=0}^{y=4-2x} dy\, dx \\
&= \int_0^2 \left[\frac{(4 - 2x)^2}{5} - \frac{1}{10}(4 - 2x)^2\right] dy\, dx \\
&= \int_0^2 \frac{1}{10}(4 - 2x)^2\, dy\, dx \\
&= -\frac{1}{60}(4 - 2x)^3 \Bigg|_{x=0}^{x=2} \\
&= \frac{4^3}{60} = \frac{16}{15}.
\end{aligned}$$

(b) Find the volume of the solid in the first octant of $\mathbb{R}^3$ bounded by the surfaces $z = x + y$, $y^2 + x = 1$, and the xy-, xz- and yz-planes.



In the first octant, the surface $y^2 + x = 1$ intersects the yz-plane ($x = 0$) at $y = 1$. Hence, the solid is the set:

$$\{(x, y, z) \in \mathbb{R}^3 : 0 \le y \le 1, 0 \le x \le 1 - y^2, 0 \le z \le x + y\}$$

The volume of the solid is therefore:

$$\begin{aligned}
\int_0^1 \int_0^{1-y^2} \int_0^{x+y} 1\,dz\,dx\,dy &= \int_0^1 \int_0^{1-y^2} (x + y)\,dx\,dy \\
&= \int_0^1 (\frac{1}{2}x^2 + yx)\Big|_{x=0}^{x=1-y^2}\,dy \\
&= \int_0^1 \left(\frac{1}{2}(1 - y^2)^2 + (y - y^3)\right)\,dy \\
&= \int_0^1 \left(\frac{1}{2}(1 - 2y^2 + y^4) + (y - y^3)\right)\,dy \\
&= \left[\frac{1}{2}\left(y - \frac{2}{3}y^3 + \frac{1}{5}y^5\right) + \frac{1}{2}y^2 - \frac{1}{4}y^4\right]\Big|_{y=0}^{y=1} \\
&= \frac{1}{2}\left(1 - \frac{2}{3} + \frac{1}{5}\right) + \frac{1}{2} - \frac{1}{4}.
\end{aligned}$$

4. (Optional) **Polar Coordinates.** A point $P = (x, y)$ in the xy-plane may be described by polar coordinates:

$$P = (r, \theta)_{pol},$$

where $r = \sqrt{x^2 + y^2}$ is the distance between (x, y) and $(0, 0)$; and θ is the angle between the vector $\langle x, y \rangle$ and the positive x-axis. Hence, we have:

$$x = r\cos\theta, \quad y = r\sin\theta.$$

Theorem (Polar Integration). Let R be a region in $\mathbb{R}^2$ described in polar coordinates as follows:

$$R = \{(r, \theta)_{pol} : \theta_1 \leq \theta \leq \theta_2, r_1 \leq r \leq r_2\},$$

where $\theta_i, r_i \in \mathbb{R}$. Let f be a continuous function on R. Then,

$$\iint_R f dA = \int_{\theta_1}^{\theta_2} \int_{r_1}^{r_2} f((r, \theta)_{pol})\, r\, drd\theta$$

(Notice the extra "r" in front of $drd\theta$.)

Example. Find the integral of $f(x, y) = xy^2$ over the unit disk:

$$R = \{(x, y) : x^2 + y^2 \leq 1\}.$$

First, describe R using polar coordinates:

$$R = \{(r, \theta)_{pol} : 0 \leq \theta \leq 2\pi, 0 \leq r \leq 1\}$$

Then, by the theorem we have:

$$\begin{aligned}
\iint_R f(x, y) dA &= \int_0^{2\pi} \int_0^1 f(r\cos\theta, r\sin\theta) r dr d\theta \\
&= \int_0^{2\pi} \int_0^1 (r\cos\theta)(r\sin\theta)^2 r dr d\theta \\
&= \int_0^{2\pi} \sin^2\theta\cos\theta \left[\int_0^1 r^4 dr\right] d\theta \\
&= \int_0^{2\pi} \sin^2\theta\cos\theta \left[\frac{1}{5} r^5 \Big|_0^1\right] d\theta \\
&= \frac{1}{5} \int_{\theta=0}^{\theta=2\pi} \sin^2\theta d(\sin\theta) \\
&= \frac{1}{5}\left(\frac{1}{3}\right) \sin^3\theta \Big|_0^{2\pi} = 0.
\end{aligned}$$

Use polar integration to evaluate the following:

(a) $\int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)dydx.$

Solution: The integral is equal to the double integral over the region:

$$R = \left\{ (r,\theta)_{pol} | 0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2} \right\}.$$

Hence:

$$\begin{aligned} \int_0^2 \int_0^{\sqrt{4-x^2}} (x^2+y^2)dydx &= \int_0^2 \int_0^{\frac{\pi}{2}} r^2 \cdot r d\theta dr \\ &= \int_0^2 \int_0^{\frac{\pi}{2}} r^3 d\theta dr = \int_0^2 r^3 \theta \Big|_{\theta=0}^{\theta=\pi/2} dr = \frac{\pi}{2} \int_0^2 r^3 dr = 2\pi \end{aligned}$$

(b) The volume of the solid bounded by the cylinder $x^2+y^2=1$, the plane $z=x$ (from above), and the xy-plane (from below).

Solution:
The volume of the solid is equal to the triple integral of the function $f(x,y) = x = r\cos\theta$ over the region:

$$R = \left\{ (r,\theta)_{pol} | 0 \le r \le 1, -\frac{\pi}{2} \le \theta \le \frac{\pi}{2} \right\}.$$

Hence, the volume of the solid is:

$$\begin{aligned} Volume &= \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (r\cos\theta) r \, d\theta \, dr \\ &= \int_0^1 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} r^2 \cos\theta \, d\theta \, dr \\ &= \int_0^1 r^2 \sin\theta \Big|_{\theta=-\frac{\pi}{2}}^{\theta=\frac{\pi}{2}} \, dr \\ &= \int_0^1 2r^2 \, dr \\ &= \frac{2}{3} r^3 \Big|_{r=0}^{r=1} \\ &= \frac{2}{3} \end{aligned}$$

(In the original wording of the problem, without "from above" and "from below", the volume would be twice the amount computed above, namely $4/3$).