THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1540 University Mathematics for Financial Studies 2016-17 Term 1
Coursework 7

Name: ________________________ Student ID: ___________________ Score: __________

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1. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial^2 f}{\partial x^2}, \frac{\partial f}{\partial x \partial y}$ for the following functions:

   (a) $f(x, y) = \cos(xy^2)\sqrt{x}$.

$$\frac{\partial f}{\partial x} = -\sin(xy^2)y^2\sqrt{x} + \frac{1}{2}\cos(xy^2)\frac{1}{\sqrt{x}}$$
$$\frac{\partial f}{\partial y} = -2\sin(xy^2)x^{\frac{3}{2}}y$$
$$\frac{\partial^2 f}{\partial x^2} = -\cos(xy^2)y^4\sqrt{x} - \sin(xy^2)y^2\frac{1}{\sqrt{x}} - \frac{1}{4}\cos(xy^2)x^{-\frac{3}{2}}$$
$$\frac{\partial^2 f}{\partial x \partial y} = -2\cos(xy^2)y^3x^{\frac{3}{2}} - 3\sin(xy^2)y\sqrt{x}$$

(b) $f(x,y) = \dfrac{e^{x-y}}{x^2+y}$.

$$\frac{\partial f}{\partial x} = \frac{e^{x-y}(x^2+y-2x)}{(x^2+y)^2}$$
$$\frac{\partial f}{\partial y} = \frac{e^{x-y}(-x^2-y-1)}{(x^2+y)^2}$$
$$\frac{\partial^2 f}{\partial x^2} = \frac{(x^2+y)^2\left(e^{x-y}(2x-2)+e^{x-y}(x^2+y-2x)\right)-e^{x-y}(x^2+y-2x)\cdot 2(x^2+y)\cdot 2x}{(x^2+y)^4}$$
$$\frac{\partial^2 f}{\partial x \partial y} = \frac{(x^2+y)^2\left(e^{x-y}(-2x)+e^{x-y}(-x^2-y-1)\right)-e^{x-y}(-x^2-y-1)\cdot 2(x^2+y)\cdot 2x}{(x^2+y)^4}$$

(c) $f(x,y) = x^y, \quad x > 0.$

$$\frac{\partial f}{\partial x} = yx^{y-1}$$
$$\frac{\partial f}{\partial y} = (\ln x)x^y$$
$$\frac{\partial^2 f}{\partial x^2} = y(y-1)x^{y-2}$$
$$\frac{\partial^2 f}{\partial x \partial y} = x^{y-1} + (\ln x)yx^{y-1}$$

2. Let $f(x, y, z) = yz\, 5^{xy}$. Find f_{xy} and f_{yz}. Then show that:

$$\frac{\partial}{\partial z} f_{xy} = \frac{\partial}{\partial x} f_{yz}.$$

Proof:

$$\begin{aligned}
f_x &= y^2 z 5^{xy} \ln 5 \\
f_{xy} &= 2yz5^{xy} \ln 5 + xy^2 5^{xy} (\ln 5)^2 \\
\frac{\partial f_{xy}}{\partial z} &= 2y5^{xy} \ln 5 + xy^2 5^{xy} (\ln 5)^2 \\
f_y &= z5^{xy} + xyz5^{xy} \ln 5 \\
f_{yz} &= 5^{xy} + xy5^{xy} \ln 5 \\
\frac{\partial f_{yz}}{\partial x} &= y5^{xy} \ln 5 + y5^{xy} \ln 5 + xy^2 5^{xy} (\ln 5)^2 \\
&= 2y5^{xy} \ln 5 + xy^2 5^{xy} (\ln 5)^2
\end{aligned}$$

Hence,

$$\frac{\partial}{\partial z} f_{xy} = \frac{\partial}{\partial x} f_{yz}.$$

3. Let:

$$f(x,y) = \begin{cases} xy^2 & \text{if } y \geq 0, \\ -y^3 & \text{if } y < 0. \end{cases}$$

Find $f_{xy}(0,0)$ and $f_{yy}(0,0)$, if they exist.

By definition:
For $y \geq 0$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h} = \lim_{h\to 0} \frac{hy^2 - 0}{h} = y^2$$

for $y < 0$

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h} = \lim_{h\to 0} \frac{-y^3 - (-y^3)}{h} = 0$$

Hence,

$$\lim_{h\to 0^+} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0^+} \frac{h^2}{h} = 0$$
$$\lim_{h\to 0^-} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0^-} \frac{0}{h} = 0$$

So,

$$f_{xy}(0,0) = \lim_{h\to 0^+} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0^-} \frac{f_x(0,h) - f_x(0,0)}{h} = 0.$$

For $y > 0$,

$$f_y(0,y) = \frac{d}{dy}(0 \cdot y^2) = 0.$$

For $y = 0$,

$$\lim_{h\to 0^+} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0^+} \frac{0 \cdot h^2 - 0}{h} = 0$$
$$\lim_{h\to 0^-} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0^-} \frac{-h^3 - 0}{h} = 0$$

Hence,

$$f_y(0,0) = \lim_{h\to 0^+} \frac{f(0,h) - f(0,0)}{h} = \lim_{h\to 0^-} \frac{f(0,h) - f(0,0)}{h} = 0.$$

For $y < 0$,

$$f_y(0,y) = \frac{d}{dy}(-y^3) = -3y^2.$$

Hence,

$$\lim_{h\to 0^+} \frac{f_y(0,h) - f_y(0,0)}{h} = \lim_{h\to 0^+} \frac{0-0}{h} = 0$$
$$\lim_{h\to 0^-} \frac{f_y(0,h) - f_y(0,0)}{h} = \lim_{h\to 0^-} \frac{-3h^2-0}{h} = 0$$

Hence,

$$f_{yy}(0,0) = \lim_{h\to 0^+} \frac{f_y(0,h) - f_y(0,0)}{h} = \lim_{h\to 0^-} \frac{f_y(0,h) - f_y(0,0)}{h} = 0.$$

4. Let $F(u, v) = e^{2u+5v}$. Suppose u and v are themselves functions in two variables x, y, with:

$$u(0,0) = 3 \qquad v(0,0) = -2$$
$$u_x(0,0) = -1 \qquad v_x(0,0) = 0$$
$$u_y(0,0) = 7 \qquad v_y(0,0) = 4.$$

Find $\left.\dfrac{\partial F}{\partial x}\right|_{(x,y)=(0,0)}$ and $\left.\dfrac{\partial F}{\partial y}\right|_{(x,y)=(0,0)}$.

By the Chain Rule, we have

$$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x}$$

At $(x, y) = (0, 0)$, we know that

$$\left.\frac{\partial F}{\partial u}\right|_{(x,y)=(0,0)} = 2e^{2u+5v} = 2e^{-4}, \left.\frac{\partial F}{\partial v}\right|_{(x,y)=(0,0)} = 5e^{2u+5v} = 5e^{-4}$$

$$\left.\frac{\partial u}{\partial x}\right|_{(x,y)=(0,0)} = -1, \left.\frac{\partial v}{\partial x}\right|_{(x,y)=(0,0)} = 0$$

Therefore,

$$\frac{\partial F}{\partial x} = (2e^{-4})(-1) + (5e^{-4})(0) = -2e^{-4}$$

Similarly,

$$\frac{\partial F}{\partial y} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial y}$$

At $(x, y) = (0, 0)$, we know that

$$\left.\frac{\partial u}{\partial y}\right|_{(x,y)=(0,0)} = 7, \left.\frac{\partial v}{\partial y}\right|_{(x,y)=(0,0)} = 4$$

Therefore,

$$\frac{\partial F}{\partial x} = (2e^{-4})(7) + (5e^{-4})(4) = 34e^{-4}$$