THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH1540 University Mathematics for Financial Studies 2016-17 Term 1
Coursework 1

Name: ________________________ Student ID: ___________________ Score: _________

Show your work!

1. Determine if each of the following maps between vector spaces is linear. If the map is linear, give the corresponding matrix.

   (a) $f : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$,

$$f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} x+2y \\ -x-y \\ x+y \end{pmatrix}, \quad x, y \in \mathbb{R}.$$

Yes. Verify the following relations,

$$f\left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}\right) = f\left(\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}\right) + f\left(\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}\right),$$

and

$$f\left(\lambda \begin{pmatrix} x \\ y \end{pmatrix}\right) = \lambda f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Indeed,

$$\begin{aligned} f\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + f\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} &= \begin{pmatrix} x_1+2y_1 \\ -x_1-y_1 \\ x_1+y_1 \end{pmatrix} + \begin{pmatrix} x_2+2y_2 \\ -x_2-y_2 \\ x_2+y_2 \end{pmatrix} \\ &= \begin{pmatrix} (x_1+x_2)+2(y_1+y_2) \\ -(x_1+x_2)-(y_1+y_2) \\ (x_1+x_2)+(y_1+y_2) \end{pmatrix} \\ &= f\left(\begin{pmatrix} x_1+x_2 \\ y_1+y_2 \end{pmatrix}\right), \end{aligned}$$

and

$$f\left(\lambda \begin{pmatrix} x \\ y \end{pmatrix}\right) = \begin{pmatrix} \lambda x+2\lambda y \\ -\lambda x-\lambda y \\ \lambda x+\lambda y \end{pmatrix} = \lambda \begin{pmatrix} x+2y \\ -x-y \\ x+y \end{pmatrix} = \lambda f\left(\begin{pmatrix} x \\ y \end{pmatrix}\right).$$

Since:

$$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1+2\times 0 \\ -1-(-1)\times 0 \\ 1+1\times 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$$

and

$$f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0+2\times 1 \\ -1\times 0-(-1)\times 1 \\ 0+1\times 1 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix},$$

the corresponding matrix is
$$\begin{pmatrix} 1 & 2 \\ -1 & -1 \\ 1 & 1 \end{pmatrix}.$$

(b) $l : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$,
$$l\left(\begin{pmatrix} x \\ y \end{pmatrix}\right) = 2\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad x, y \in \mathbb{R}.$$

No. Since if l is linear, then
$$l\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

But by definition of l,
$$l\begin{pmatrix} 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

2. Let:
$$A = \begin{pmatrix} 1 & -3 & 7 \\ 2 & 0 & -4 \end{pmatrix}, \quad B = \begin{pmatrix} 2 & 0 & -7 \\ 0 & 5 & -4 \\ 3 & 0 & 5 \end{pmatrix}, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 6 & -1 & 7 \\ -1 & 2 & 6 \end{pmatrix}$$

(a) Compute:
   i. $B + C$.
   ii. AB.
   iii. AC.

(b) Show that $A(B + C) = AB + AC$.

(a) i. We first calculate $B + C$
$$B + C = \begin{pmatrix} 3 & 0 & -7 \\ 6 & 4 & 3 \\ 2 & 2 & 11 \end{pmatrix}$$

ii. Then we calculate AB
$$AB = \begin{pmatrix} 1 & -3 & 7 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 2 & 0 & -7 \\ 0 & 5 & -4 \\ 3 & 0 & 5 \end{pmatrix} = \begin{pmatrix} 23 & -15 & 40 \\ -8 & 0 & -34 \end{pmatrix}$$

iii. Then we calculate AC
$$AC = \begin{pmatrix} 1 & -3 & 7 \\ 2 & 0 & -4 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 6 & -1 & 7 \\ -1 & 2 & 6 \end{pmatrix} = \begin{pmatrix} -24 & 17 & 21 \\ 6 & -8 & -24 \end{pmatrix}$$

(b)

$$A(B+C) = \begin{pmatrix} 1 & -3 & 7 \\ 2 & 0 & -4 \end{pmatrix} \begin{pmatrix} 3 & 0 & -7 \\ 6 & 4 & 3 \\ 2 & 2 & 11 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 61 \\ -2 & -8 & -58 \end{pmatrix}$$

$$AB + AC = \begin{pmatrix} 23 & -15 & 40 \\ -8 & 0 & -34 \end{pmatrix} + \begin{pmatrix} -24 & 17 & 21 \\ 6 & -8 & -24 \end{pmatrix} = \begin{pmatrix} -1 & 2 & 61 \\ -2 & -8 & -58 \end{pmatrix}$$

Hence, $A(B+C) = AB + AC$

3. Find all solutions to the following systems of linear equations:

(a)

$$\begin{aligned} -2x_1 + x_2 + x_3 &= -2 \\ 3x_1 - 2x_2 &= 7 \\ x_1 - 2x_2 + 5x_3 &= 15 \end{aligned}$$

Solution: Perform row reduction on the augmented matrix:

$$\begin{pmatrix} -2 & 1 & 1 & -2 \\ 3 & -2 & 0 & 7 \\ 1 & -2 & 5 & 15 \end{pmatrix}$$

$\rightarrow$

$$\begin{pmatrix} 1 & -2 & 5 & 15 \\ 0 & 1 & -15/4 & -19/2 \\ 0 & 0 & 1 & 2 \end{pmatrix}$$

Hence:

$$\begin{aligned} x_3 &= 2 \\ x_2 - (15/4)x_3 &= -19/2 \\ x_1 - 2x_2 + 5x_3 &= 15 \end{aligned}$$

So, the solution is:

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 2 \end{pmatrix}$$

(b)

$$\begin{aligned} -x + 2z + 13w &= -2 \\ x + 2y - z + w &= 5 \end{aligned}$$

Solution: Row reduction on the augmented matrix:

$$\begin{pmatrix} -1 & 0 & 2 & 13 & -2 \\ 1 & 2 & -1 & 1 & 5 \end{pmatrix}$$

$\rightarrow$

$$\begin{pmatrix} 1 & 0 & -2 & -13 & 2 \\ 0 & 2 & 1 & 14 & 3 \end{pmatrix}$$

This corresponds to a linear system with 4 unknowns, and 2 pivots. So, correspondingly there are $4-2=2$ free variables $z=s$, $w=t$, $s,t\in\mathbb{R}$. The solutions are:

$$x = 2+2s+13t, \quad y = \frac{1}{2}(3-s-14t), \quad z = s, \quad w = t \quad s,t\in\mathbb{R},$$

or:

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 2 \\ -1/2 \\ 1 \\ 0 \end{pmatrix} s + \begin{pmatrix} 13 \\ -7 \\ 0 \\ 1 \end{pmatrix} t + \begin{pmatrix} 2 \\ 3/2 \\ 0 \\ 0 \end{pmatrix}.$$

4. Let A be an $m\times n$ matrix, $\vec{b}\in\mathbb{R}^m$, and C an *invertible* $m\times m$ matrix. Show that $\vec{x}\in\mathbb{R}^n$ satisfies:

$$A\vec{x} = \vec{b}$$

if and only if it satisfies:

$$(CA)\vec{x} = C\vec{b}.$$

Proof of the "if" part: Suppose $\vec{x}$ satisfies $(CA)\vec{x}=C\vec{b}$. Since C is invertible, the matrix C^{-1} exists. Multiplying by C^{-1} on both sides of the equation $(CA)\vec{x}=C\vec{b}$, we have:

$$\underbrace{C^{-1}(CA)}_{=(C^{-1}C)A=A}\ \vec{x} = \underbrace{C^{-1}(C\vec{b})}_{=(C^{-1}C)\vec{b}=\vec{b}}.$$

Hence, $\vec{x}$ also satisfies $A\vec{x}=\vec{b}$.

Proof of the "only if" part: Suppose $\vec{x}$ satisfies $A\vec{x}=\vec{b}$. Multiplying by C on both sides of the equation, we have:

$$(CA)\vec{x} = C(A\vec{x}) = C\vec{b}.$$

Hence, $\vec{x}$ also satisfies $(CA)\vec{x}=C\vec{b}$.