THE CHINESE UNIVERSITY OF HONG KONG
MATH 1540 Homework Set 4
Due time 6:30 pm Nov 14, 2016

1. Show that:

   (a) $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{x+y-3} = 0.$

   (b) $\lim_{(x,y)\to(1,1)} \dfrac{x^2-y^2}{x-y} = 2.$

   (c) $\lim_{(x,y)\to(1,-1)} \dfrac{x^2-xy-2y^2}{x+y} = 3.$

   (d) $\lim_{(x,y)\to(0,0)} \dfrac{x^2-y^2}{xy}$ does not exist.

   (e) $\lim_{(x,y)\to(0,0)} \dfrac{y}{x^3+y}$ does not exist.

   (f) $\lim_{(x,y)\to(0,0)} \dfrac{x^6}{x^4+y^2} = 0.$

   (Hint: Consider using Sandwich Theorem.)

   Solution:

   (a) $\lim_{(x,y)\to(0,0)} \sin\left(\dfrac{x^2-y^2}{x+y-3}\right) = \sin\left(\dfrac{(0)^2-(0)^2}{(0)+(0)-3}\right) = 0$

   (b) $\lim_{(x,y)\to(1,1)} \dfrac{x^2-y^2}{x-y} = \lim_{(x,y)\to(1,1)} (x+y) = (1)+(1) = 2$

   (c) $\lim_{(x,y)\to(1,-1)} \dfrac{x^2-xy-2y^2}{x+y} = \lim_{(x,y)\to(1,-1)} \dfrac{(x+y)(x-2y)}{x+y}$
   $= \lim_{(x,y)\to(1,-1)} (x-2y) = (1)-2(-1) = 3$

   (d) If the limit exists, then when $(x,y)\to(0,0)$ along any path, $\dfrac{x^2-y^2}{xy}$ should converge to the same value.
   First, we consider the line $(x,y) = (t,2t)$, $t\in\mathbb{R}$. Along this line, the approach of (x,y) towards $(0,0)$ corresponds to the approach of t towards 0. Hence, as (x,y) approach $(0,0)$ along this line, the expression $\dfrac{x^2-y^2}{xy}$ approaches the limit:

   $$\lim_{t\to 0} \frac{t^2-(2t)^2}{t\cdot 2t} = -\frac{3}{2}.$$

   Now, consider the line $(x,y) = (t,-2t)$, $t\in\mathbb{R}$. As (x,y) approaches $(0,0)$ along this line, the expression $\dfrac{x^2-y^2}{xy}$ approaches the limit:

   $$\lim_{t\to 0} \frac{t^2-(-2t)^2}{t(-2t)} = \frac{3}{2}.$$

Since $\dfrac{x^2-y^2}{xy}$ approaches different values as (x,y) approaches $(0,0)$ along different paths, the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{x^2-y^2}{xy}$ does not exist.

(e) First, we consider the line $(x,y) = (t,0)$, $t \in \mathbb{R}$. The approach of (x,y) towards $(0,0)$ corresponds to the approach of t towards 0. Hence, the limit of $\dfrac{y}{x^3+y}$ as (x,y) approaches $(0,0)$ along this line is:

$$\lim_{t\to 0} \frac{0}{t^3+0} = 0.$$

On the other hand, as (x,y) approaches $(0,0)$ along the line $(x,y) = (0,t)$, $t \in \mathbb{R}$, the expression $\dfrac{y}{x^3+y}$ approaches the limit:

$$\lim_{t\to 0} \frac{t}{0^3+t} = 1 \neq 0.$$

We conclude that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y}{x^3+y}$ does not exist.

(f) Observe that, for $(x,y) \neq 0$, we have:

$$\frac{x^6}{x^4+y^2} = \frac{0}{y^2} = 0 \quad \text{if} \quad x = 0,$$

$$0 \leq \frac{x^6}{x^4+y^2} \leq \frac{x^6}{x^4} = x^2 \quad \text{if} \quad x \neq 0.$$

Since $\displaystyle\lim_{(x,y)\to(0,0)} x^2 = 0$, by the Sandwich Theorem we conclude that:

$$\lim_{(x,y)\to(0,0)} \frac{x^6}{x^4+y^2} = 0.$$

2. (a) Let $f(x,y) = 5x^7 - 2xy^3 + 6$. Show that $f_x(-1,1) = 33$, $f_y(2,2) = -48$.
$\dfrac{\partial f}{\partial x} = 35x^6 - 2y^3, \quad \dfrac{\partial f}{\partial y} = -6xy^2.$
Hence, $f_x(-1,1) = 33$, $f_y(2,2) = -48$.

(b) Let $f(x,y) = \sqrt{xy-y^2}$. Show that:

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(3,2)} = \frac{2}{2\sqrt{2}}.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(3,2)} = \frac{-1}{2\sqrt{2}}.$$

These identities may be shown to hold by evaluating the following partial derivatives at $(x,y) = (3,2)$:

$$\frac{\partial f}{\partial x} = \frac{y}{2\sqrt{xy-y^2}}, \quad \frac{\partial f}{\partial y} = \frac{x-2y}{2\sqrt{xy-y^2}}.$$

(c) Let $f(x,y) = \log_x y, \quad y > 0, x \neq 1$. Show that:

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(e,2)} = \frac{-\ln 2}{e}.$$

$$\left.\frac{\partial f}{\partial y}\right|_{(x,y)=(e,2)} = \frac{1}{2}.$$

We have $\log_x y = \dfrac{\ln y}{\ln x}$, then

$$\frac{\partial f}{\partial x} = \frac{-\ln y}{x(\ln x)^2}, \quad \frac{\partial f}{\partial y} = \frac{1}{y \ln x}.$$

(d) Let $f(x,y) = x^y + y^x, \quad x, y > 0$. Show that:

$$\left.\frac{\partial f}{\partial x}\right|_{(x,y)=(1,e^2)} = e^2 + 2e^2.$$

Note that $\dfrac{d}{dx}c^x = c^x \ln c$ for constant c, then

$$\frac{\partial f}{\partial x} = yx^{y-1} + y^x \ln y, \quad \frac{\partial f}{\partial y} = x^y \ln x + xy^{x-1}.$$

3. (a) Let $f(x,y,z) = xz + y^2 z + \cos(z)$.
Via explicit computation of second order partial derivatives, show that:

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y}$$

We have: $\dfrac{\partial f}{\partial y} = 2yz$, $\dfrac{\partial f}{\partial z} = x + y^2 - \sin z$, hence:

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{\partial}{\partial y}(x + y^2 - \sin z) = 2y.$$

$$\frac{\partial^2 f}{\partial z \partial y} = \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial z}(2yz) = 2y = \frac{\partial^2 f}{\partial y \partial z}.$$

(b) Let $f(x,y,z) = xyz + \sqrt{xz}$.
Via explicit computation of third order partial derivatives, show that:

$$f_{xzy} = f_{yzx}$$

We have:

$$f_x = yz + \frac{\sqrt{z}}{2\sqrt{x}},$$

$$f_{xz} = (f_x)_z = y + \frac{1}{4\sqrt{xz}},$$
$$f_{xzy} = (f_{xz})_y = 1.$$
$$f_y = xz,$$
$$f_{yz} = x,$$
$$f_{yzx} = 1 = f_{xzy}.$$

4. Let:

$$f(x,y) = \begin{cases} x^2y & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

Show that:

(a) $f_x(-1,2) = 0$.
(b) $f_x(3,-7) = -42$.
(c) $f_x(0,y) = 0$, for all $y \in \mathbb{R}$.
(d) $f_{xy}(0,0) = 0$.

(a), (b) For $x \neq 0$, we have

$$f_x(x,y) = \begin{cases} 2xy & \text{if } x > 0, \\ 0 & \text{if } x < 0, \end{cases}$$

Hence, $f_x(-1,2) = 0$ and $f_x(3,-7) = -42$.

(c) For $y \in \mathbb{R}$, by definition of $f_x(0,y)$ we have:

$$f_x(0,y) = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h}.$$

Evaluating the left and right limits separately, we have:

$$\lim_{h\to 0^-} \frac{f(h,y) - f(0,y)}{h} = \lim_{h\to 0^-} \frac{0 - 0^2y}{h} = 0,$$

$$\lim_{h\to 0^+} \frac{f(h,y) - f(0,y)}{h} = \lim_{h\to 0^+} \frac{h^2y - 0^2y}{h} = 0.$$

Hence, $f_x(0,y) = \lim_{h\to 0} \frac{f(h,y) - f(0,y)}{h} = 0$.

(d) From the work done in parts (a), (b) and (c) we see that:

$$f_x(x,y) = \begin{cases} 2xy & \text{if } x > 0, \\ 0 & \text{if } x \le 0. \end{cases}$$

By definition,

$$f_{xy}(0,0) = \lim_{h\to 0} \frac{f_x(0,h) - f_x(0,0)}{h}.$$

Evaluating the left and right limits separately, we have:

$$\lim_{h\to 0^-} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0^-} \frac{0-0}{h} = 0$$

$$\lim_{h\to 0^+} \frac{f_x(0,h) - f_x(0,0)}{h} = \lim_{h\to 0^+} \frac{0-0}{h} = 0$$

Hence, $f_{xy}(0,0) = 0$.

(Alternatively, we may compute $f_{xy}(0,0)$ as follows:

$$f_{xy}(0,0) = \frac{d}{dy}\left(f_x(0,y)\right)\Big|_{y=0} = \frac{d}{dy}(0)\Big|_{y=0} = 0.)$$

5. (Optional) Let g be a continuous function defined on $\mathbb{R}$. Let $f(x,y) = \displaystyle\int_{xy}^{y} g(t)\,dt$. Find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

By the Fundamental Theorem of Calculus, for a continuous function g and differentiable functions a, b in one variable, we have:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} g(t)\,dt = g(b(x))b'(x) - g(a(x))a'(x).$$

Therefore,

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{d}{dx}\int_{xy}^{y} g(t)\,dt \\ &= g(y)\left(\frac{\partial}{\partial x}y\right) - g(xy)\left(\frac{\partial}{\partial x}(xy)\right) = g(y)\cdot 0 - g(xy)\cdot y = -yg(xy).\end{aligned}$$

Similarly,

$$\begin{aligned}\frac{\partial f}{\partial y} &= \frac{d}{dy}\int_{xy}^{y} g(t)\,dt \\ &= g(y)\left(\frac{\partial}{\partial y}y\right) - g(xy)\left(\frac{\partial}{\partial y}(xy)\right) = g(y)\cdot 1 - g(xy)\cdot x = g(y) - xg(xy).\end{aligned}$$