THE CHINESE UNIVERSITY OF HONG KONG
MATH 1540 Homework Set 3
Due time 6:30 pm Oct 31, 2016

1. Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ which contains the line:

$$\vec{l}(t) = \langle 2t-1, 3, t\rangle, \quad t \in \mathbb{R},$$

and the point $(2,-1,1)$.

Find an equation of the form $ax+by+cz=d$ which describes $\mathcal{P}$.

Solution

First, the directional vector of the line $\vec{l}$, which can be taken to be $\vec{v} = \langle 2,0,1\rangle$, is perpendicular to any normal vector $\vec{n}$ of the plane $\mathcal{P}$.

Let $P = (2,-1,1)$. Let Q be any point on the line, say $Q = (-1,3,0)$, which corresponds to $t=0$. The vector $\overrightarrow{QP} = \langle 2-(-1), -1-3, 1-0\rangle$ is parallel to the plane $\mathcal{P}$, hence it is also perpendicular to $\vec{n}$.

Therefore, we may take $\vec{n}$ to be

$$\begin{aligned}\vec{n} &= \vec{v} \times \overrightarrow{QP} \\ &= \langle 2,0,1\rangle \times \langle 3,-4,1\rangle \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 0 & 1 \\ 3 & -4 & 1 \end{vmatrix} = \langle 4,1,-8\rangle\end{aligned}$$

Also, point $P = (2,-1,1)$ lies on $\mathcal{P}$. Therefore, an equation which describes $\mathcal{P}$ is

$$(4)(x-2) + (1)(y-(-1)) + (-8)(z-1) = 0$$

that is

$$4x + y - 8z = -1$$

2. Consider the two planes $\mathcal{P}_1$ and $\mathcal{P}_2$ in $\mathbb{R}^3$, described respectively by the equations:

$$x - y + 2z = 5,$$

$$2x + 7y = 1.$$

   (a) Find a normal vector of length 1 of each of the two planes.

   (b) Find a vector parameterization of the line which is the intersection of the two planes.

<u>Solution</u>

(a) $\vec{n}_1 = \dfrac{\langle 1, -1, 2\rangle}{|\langle 1, -1, 2\rangle|} = \dfrac{1}{\sqrt{6}}\langle 1, -1, 2\rangle$

$\vec{n}_1 = \dfrac{\langle 2, 7, 0\rangle}{|\langle 2, 7, 0\rangle|} = \dfrac{1}{\sqrt{53}}\langle 2, 7, 0\rangle$

(b) As any directional vector $\vec{v}$ of the line $\vec{l}$ is perpendicular to both $\vec{n}_1$ and $\vec{n}_2$, we can take $\vec{v}$ to be:

$$\begin{aligned}\vec{v} &= \langle 1, -1, 2\rangle \times \langle 2, 7, 0\rangle \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 2 \\ 2 & 7 & 0 \end{vmatrix} = \langle -14, 4, 9\rangle\end{aligned}$$

Also, by finding a solution of the set of equations

$$x - y + 2z = 5,$$

$$2x + 7y = 1.$$

we can find a point on the line $\vec{l}$ (For example, $P = (-3, 1, 9/2)$)

Therefore, a vector parameterization of the line $\vec{l}$ is

$$\vec{l} = \langle -14, 4, 9\rangle t + \langle -3, 1, 9/2\rangle$$

3. Let L be the line in $\mathbb{R}^3$ described by the vector-valued function:

$$\vec{l}(t) = \langle 1, -1, 7\rangle t + \langle 2, 0, 5\rangle, \quad t \in \mathbb{R}.$$

Let $\mathcal{P}$ be the plane in $\mathbb{R}^3$ corresponding to the equation:

$$4x - 3y - z = 3.$$

Let $\mathcal{P}'$ be a plane which contains the origin, and whose intersection with $\mathcal{P}$ is the line L. Find an equation of the form $ax + by + cz = d$ which describes $\mathcal{P}'$.

Solution

The directional vector of $\vec{l}$, $\vec{v} = \langle 1, -1, 7\rangle$, is parallel to $\mathcal{P}'$. Also, $\vec{v}_0 = \langle 2-0, 0-0, 5-0\rangle = \langle 2, 0, 5\rangle$ is parallel to $\mathcal{P}'$. Therefore, a normal vector of $\mathcal{P}'$ is

$$\begin{aligned} \vec{n} &= \langle 1, -1, 7\rangle \times \langle 2, 0, 5\rangle \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -1 & 7 \\ 2 & 0 & 5 \end{vmatrix} = \langle -5, 9, 2\rangle \end{aligned}$$

Also, as $(0, 0, 0)$ lies on $\mathcal{P}'$, an equation of $\mathcal{P}'$ is

$$-5x + 9y + 2z = 0.$$

(This is somewhat of a trick question, since the solution may be obtained without using any information about the plane $\mathcal{P}$.)

4. Let L_1 and L_2 be two lines in $\mathbb{R}^3$ parameterized, respectively, by the following vector-valued functions:

$$\vec{l}_1(t) = \langle t, 1+2t, -3-t\rangle, \quad t \in \mathbb{R};$$

$$\vec{l}_2(t) = \langle -1+3t, 5t, 2t\rangle, \quad t \in \mathbb{R}.$$

(a) Show that the two lines do not meet, and are not parallel to each other.

(b) Find an equation whose graph is the plane containing L_2 and parallel to L_1.

(c) Find the minimal distance between L_1 and L_2.

Solution

(a) If L_1 and L_2 have an intersection, then there exists t_1, t_2 such that $\vec{l}_1(t_1) = \vec{l}_2(t_2)$. By solving the system of linear equations

$$\begin{cases} t_1 = -1 + 3t_2 \\ 1 + 2t_1 = 5t_2 \\ -3 - t_1 = 2t_2 \end{cases} \Rightarrow \begin{cases} t_1 - 3t_2 = -1 \\ 2t_1 - 5t_2 = -1 \\ t_1 + 2t_2 = -3 \end{cases}$$

We can see that there is no solution for the system, meaning that the two lines do not meet.

Also, a directional vector of $\vec{l}_1$ is $\vec{v}_1 = \langle 1, 2, -1\rangle$ and a directional vector of $\vec{l}_2$ is $\vec{v}_2 = \langle 3, 5, 2\rangle$. Since neither directional vector is a scalar multiple of the other, the two lines are not parallel to each other.

(b) To satisfy the requirement, any normal vector $\vec{n}$ to the plane is perpendicular to both $\vec{v}_1$ and $\vec{v}_2$, which are both parallel to the plane. Therefore, we may take $\vec{n}$ to be:

$$\begin{aligned}\vec{n} &= \langle 1, 2, -1\rangle \times \langle 3, 5, 2\rangle \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 3 & 5 & 2 \end{vmatrix} = \langle 9, -5, -1\rangle\end{aligned}$$

The point $(-1, 0, 0)$ lies on L_2, hence also on the plane. Therefore, an equation of the plane is:

$$9(x+1) - 5y - z = 0,$$

or equivalently:

$$9x - 5y - z = -9.$$

(c) The distance d between L_1 and L_2 is the distance between any point on L_1 and the plane $\mathcal{P}$. Taking a point $P = (-1, 0, 0)$ on $\mathcal{P}$ and a point $Q = (0, 1, -3)$ on L_1, the distance d is the length of the projection of $\overrightarrow{PQ}$ onto any normal vector $\vec{n}$ of $\mathcal{P}$. From Part (b), we may take $\vec{n} = \langle 9, -5, -1\rangle$. Hence, we have:

$$\begin{aligned} d &= |\mathrm{Proj}_{\vec{n}}\langle 0-(-1), 1-0, -3-0\rangle| \\ &= \left| \frac{\langle 1, 1, -3\rangle \cdot \langle 9, -5, -1\rangle}{|\langle 9, -5, -1\rangle|^2} \langle 9, -5, -1\rangle \right| \\ &= \frac{7}{\sqrt{107}} \end{aligned}$$

5. Show that the distance D between a point $P = (x', y', z')$ and the plane $ax + by + cz = d$ in $\mathbb{R}^3$ is given by:

$$D = \left|\text{Proj}_{\vec{n}}\overrightarrow{P_0P}\right| = \frac{|ax' + by' + cz' - d|}{\sqrt{a^2 + b^2 + c^2}}$$

(Here, $\vec{n}$ is any normal vector of the plane, and $P_0 = (x_0, y_0, z_0)$ is any point which lies on the plane.)

Proof:
A normal vector to the plane is $\vec{n} = \langle a, b, c \rangle$.

$$\begin{aligned}
\left|\text{Proj}_{\vec{n}}\overrightarrow{P_0P}\right| &= \left|\left(\overrightarrow{P_0P} \cdot \frac{\vec{n}}{|\vec{n}|}\right)\frac{\vec{n}}{|\vec{n}|}\right| \\
&= \frac{\left|\overrightarrow{P_0P} \cdot \vec{n}\right|}{|\vec{n}|} \\
&= \frac{|a(x' - x_0) + b(y' - y_0) + c(z' - z_0)|}{\sqrt{a^2 + b^2 + c^2}} \\
&= \frac{|ax' + by' + cz' - (ax_0 + by_0 + cz_0)|}{\sqrt{a^2 + b^2 + c^2}}
\end{aligned}$$

Since the point (x_0, y_0, z_0) lies on the plane, it satisfies the equation $ax + by + cz = d$. Hence, $ax_0 + by_0 + cz_0 = d$, and we have:

$$\left|\text{Proj}_{\vec{n}}\overrightarrow{P_0P}\right| = \frac{|ax' + by' + cz' - d|}{\sqrt{a^2 + b^2 + c^2}}.$$

(Note: there was a typo in the distance formula in the original assignment. The current version is the correct one.)