THE CHINESE UNIVERSITY OF HONG KONG
MATH 1540 Homework Set 2
Due time 6:30 pm Oct 13, 2016

1. Find the determinants of the following matrices:

   (a)
   $$\begin{pmatrix} 10 & -1 \\ 1 & -2 \end{pmatrix}$$

   (b)
   $$\begin{pmatrix} 6 & -3 & 3 \\ 0 & 2 & 7 \\ -9 & 5 & 4 \end{pmatrix}$$

   (c)
   $$\begin{pmatrix} 20 & 7 & 13 & 5 \\ 0 & 6 & -8 & 5 \\ 12 & 1 & 15 & 5 \\ 0 & 0 & 6 & 11 \end{pmatrix}$$

   Solution:

   (a)
   $$\begin{aligned} \det \begin{pmatrix} 10 & -1 \\ 1 & -2 \end{pmatrix} &= (10)(-2) - (1)(-1) \\ &= -19 \end{aligned}$$

   (b) Perform cofactor expansion on first column,
   $$\begin{aligned} &\det \begin{pmatrix} 6 & -3 & 3 \\ 0 & 2 & 7 \\ -9 & 5 & 4 \end{pmatrix} \\ &= 6 \det \begin{pmatrix} 2 & 7 \\ 5 & 4 \end{pmatrix} - 0 \det \begin{pmatrix} -3 & 3 \\ 5 & 4 \end{pmatrix} + (-9) \det \begin{pmatrix} -3 & 3 \\ 2 & 7 \end{pmatrix} \\ &= (6)(-27) + (-9)(-27) \\ &= 81 \end{aligned}$$

(c) Perform cofactor expansion on fourth row,

$$
\begin{aligned}
&\det\begin{pmatrix} 20 & 7 & 13 & 5 \\ 0 & 6 & -8 & 5 \\ 12 & 1 & 15 & 5 \\ 0 & 0 & 6 & 11 \end{pmatrix} \\
&= -6\det\begin{pmatrix} 20 & 7 & 5 \\ 0 & 6 & 5 \\ 12 & 1 & 5 \end{pmatrix} + 11\det\begin{pmatrix} 20 & 7 & 13 \\ 0 & 6 & -8 \\ 12 & 1 & 15 \end{pmatrix} \\
&= (-6)\left(20\det\begin{pmatrix} 6 & 5 \\ 1 & 5 \end{pmatrix} + 12\det\begin{pmatrix} 7 & 5 \\ 6 & 5 \end{pmatrix}\right) \\
&+ 11\left(20\det\begin{pmatrix} 6 & -8 \\ 1 & 15 \end{pmatrix} + 12\det\begin{pmatrix} 7 & 13 \\ 6 & -8 \end{pmatrix}\right) \\
&= (-6)(20\times 25 + 12\times 5) + (11)(20\times 98 + 12\times(-134)) \\
&= 512
\end{aligned}
$$

2. (a) Let A be an $n \times n$ square matrix, λ a real number. Show that there exists a nonzero $\vec{v} \in \mathbb{R}^n$ such that:

$$A\vec{v} = \lambda\vec{v}$$

if and only only if $\det(A - \lambda I) = 0$.
(Here, I is the $n \times n$ identity matrix.)

(b) Find all $\lambda \in \mathbb{R}$ such that:

$$\begin{pmatrix} 1 & 0 & -3 \\ 0 & -2 & 0 \\ -3 & 0 & 1 \end{pmatrix} \vec{x} = \lambda\vec{x}$$

has a nonzero solution $\vec{x} \in \mathbb{R}^n$.
(Such λ's are called *eigenvalues* of the matrix A.)

Solution:

(a) There exists a nonzero $\vec{v} \in \mathbb{R}^n$ such that

$$A\vec{v} = \lambda\vec{v}$$

$\Leftrightarrow A\vec{v} = \lambda\vec{v} = \lambda I\vec{v}$
$\Leftrightarrow (A - \lambda I)\vec{v} = 0$ for nonzero $\vec{v}$
$\Leftrightarrow (A - \lambda I)\vec{u} = 0$ has a nontrivial solution
$\Leftrightarrow \det(A - \lambda I) = 0$.

(b) From (a), we need to find all $\lambda \in \mathbb{R}$ such that $\det(A - \lambda I) = 0$. By solving the equation,

$$\begin{aligned} \det\begin{pmatrix} 1-\lambda & 0 & -3 \\ 0 & -2-\lambda & 0 \\ -3 & 0 & 1-\lambda \end{pmatrix} &= 0 \\ (1-\lambda)^2(-2-\lambda) - 9(-2-\lambda) &= 0 \\ (4-\lambda)(-2-\lambda)^2 &= 0 \\ \lambda &= 4 \text{ or } -2 \end{aligned}$$

3. Determine if each of the following matrices is singular (i.e. non-invertible), either by row reduction or by computing its determinant.

   (a)
   $$\begin{pmatrix} 1 & 0 & -4 \\ 7 & 4 & 6 \\ 3 & -5 & -2 \end{pmatrix}$$

   (b)
   $$\begin{pmatrix} 0 & 5 & 7 & 3 \\ 6 & -3 & 0 & 0 \\ 8 & 3 & -7 & -7 \\ -5 & -5 & 2 & -6 \end{pmatrix}$$

   (c)
   $$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -2 & -2 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & -2 & 1 & 3 \end{pmatrix}$$

   Solution

   (a) By row reduction,

   $$\begin{aligned} \begin{pmatrix} 1 & 0 & -4 \\ 7 & 4 & 6 \\ 3 & -5 & -2 \end{pmatrix} &\xrightarrow[r_3 - 3\times r_1 \to r_3]{r_2 - 7\times r_1 \to r_2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 4 & 34 \\ 0 & -5 & 10 \end{pmatrix} \\ &\xrightarrow[r_3 + 5\times r_2 \to r_3]{r_2 \div 4 \to r_2} \begin{pmatrix} 1 & 0 & -4 \\ 0 & 1 & 17/2 \\ 0 & 0 & 105/2 \end{pmatrix} \end{aligned}$$

   As the last row is not all zeroes, the matrix is not singular.

(b) By row reduction,

$$\begin{pmatrix} 0 & 5 & 7 & 3 \\ 6 & -3 & 0 & 0 \\ 8 & 3 & -7 & -7 \\ -5 & -5 & 2 & -6 \end{pmatrix} \xrightarrow[r_1 \div 6 \to r_1]{r_2 \leftrightarrow r_1} \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 5 & 7 & 3 \\ 8 & 3 & -7 & -7 \\ -5 & -5 & 2 & -6 \end{pmatrix}$$

$$\xrightarrow[r_4 + 5 \times r_1 \to r_4]{r_3 - 8 \times r_1 \to r_3} \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 5 & 7 & 3 \\ 0 & 7 & -7 & -7 \\ 0 & -15/2 & 2 & -6 \end{pmatrix}$$

$$\xrightarrow[r_2 \div 7 \to r_2]{r_3 \leftrightarrow r_2} \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 5 & 7 & 3 \\ 0 & -15/2 & 2 & -6 \end{pmatrix}$$

$$\xrightarrow[r_4 + 15/2 \times r_2 \to r_4]{r_3 - 5 \times r_2 \to r_3} \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 12 & 8 \\ 0 & 0 & -11/2 & -27/2 \end{pmatrix}$$

$$\xrightarrow[r_4 + 11/2 \times r_3 \to r_4]{r_3 \div 12 \to r_3} \begin{pmatrix} 1 & -1/2 & 0 & 0 \\ 0 & 1 & -1 & -1 \\ 0 & 0 & 1 & 2/3 \\ 0 & 0 & 0 & -59/6 \end{pmatrix}$$

As the last row is not all zeroes, the matrix is not singular.

(c) By row reduction,

$$\begin{pmatrix} 1 & -1 & -1 & 1 \\ 1 & -2 & -2 & 3 \\ -1 & 1 & 1 & -1 \\ 3 & -2 & 1 & 3 \end{pmatrix} \xrightarrow[\substack{r_3 + r_1 \to r_3 \\ r_4 - 3 \times r_1 \to r_4}]{r_2 - r_1 \to r_2} \begin{pmatrix} 1 & -1 & -1 & 1 \\ 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 4 & 0 \end{pmatrix}$$

As the second row is of all zeroes, the matrix is singular.

4. Let:

$$A = \begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & -1 \\ 4 & 3 & 2 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 16 \\ 8 \\ 15 \end{pmatrix}.$$

Given that C is invertible, solve the following matrix equation:

$$A\vec{x} = \vec{b}$$

using:

(a) Cramer's Rule.

(b) Row reduction of the augmented matrix $\left(A \mid \vec{b} \right)$.

(c) $\vec{x} = A^{-1}\vec{b}$, where A^{-1} is obtained by performing row reduction on $\left(\, A \mid I \, \right)$.

Solution:

(a)

$$x_1 = \frac{\det\begin{pmatrix} 16 & 3 & 0 \\ 8 & 2 & -1 \\ 15 & 3 & 2 \end{pmatrix}}{\det\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & -1 \\ 4 & 3 & 2 \end{pmatrix}} = \frac{19}{-19} = -1$$

$$x_2 = \frac{\det\begin{pmatrix} -1 & 16 & 0 \\ 0 & 8 & -1 \\ 4 & 15 & 2 \end{pmatrix}}{\det\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & -1 \\ 4 & 3 & 2 \end{pmatrix}} = \frac{-95}{-19} = 5$$

$$x_3 = \frac{\det\begin{pmatrix} -1 & 3 & 16 \\ 0 & 2 & 8 \\ 4 & 3 & 15 \end{pmatrix}}{\det\begin{pmatrix} -1 & 3 & 0 \\ 0 & 2 & -1 \\ 4 & 3 & 2 \end{pmatrix}} = \frac{-38}{-19} = 2$$

(b)

$$\left(\begin{array}{ccc|c} -1 & 3 & 0 & 16 \\ 0 & 2 & -1 & 8 \\ 4 & 3 & 2 & 15 \end{array}\right) \xrightarrow[r_3 - 4 \times r_1 \to r_3]{r_1 \times (-1) \to r_1} \left(\begin{array}{ccc|c} 1 & -3 & 0 & -16 \\ 0 & 2 & -1 & 8 \\ 0 & 15 & 2 & 79 \end{array}\right)$$

$$\xrightarrow{\substack{r_2 \div 2 \to r_2 \\ r_1 + 3 \times r_2 \to r_1 \\ r_3 - 15 \times r_2 \to r_3}} \left(\begin{array}{ccc|c} 1 & 0 & -3/2 & -4 \\ 0 & 1 & -1/2 & 4 \\ 0 & 0 & 19/2 & 19 \end{array}\right)$$

$$\xrightarrow{\substack{r_3 \div 19/2 \to r_3 \\ r_1 + 3/2 \times r_3 \to r_1 \\ r_2 + 1/2 \times r_3 \to r_2}} \left(\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 2 \end{array}\right)$$

Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$$

(c)

$$\left(\begin{array}{ccc|ccc} -1 & 3 & 0 & 1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 4 & 3 & 2 & 0 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[r_3 - 4\times r_1 \to r_3]{r_1 \times (-1) \to r_1} \left(\begin{array}{ccc|ccc} 1 & -3 & 0 & -1 & 0 & 0 \\ 0 & 2 & -1 & 0 & 1 & 0 \\ 0 & 15 & 2 & 4 & 0 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{r_1 + 3\times r_2 \to r_1 \\ r_3 - 15\times r_2 \to r_3}]{r_2 \div 2 \to r_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -3/2 & -1 & 3/2 & 0 \\ 0 & 1 & -1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 19/2 & 4 & -15/2 & 1 \end{array}\right)$$

$$\xrightarrow[\substack{r_1 + 3/2\times r_3 \to r_1 \\ r_2 + 1/2\times r_3 \to r_2}]{r_3 \div 19/2 \to r_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -7/19 & 6/19 & 3/19 \\ 0 & 1 & 0 & 4/19 & 2/19 & 1/19 \\ 0 & 0 & 1 & 8/19 & -15/19 & 2/19 \end{array}\right)$$

Therefore,

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1}\vec{b} = \begin{pmatrix} -7/19 & 6/19 & 3/19 \\ 4/19 & 2/19 & 1/19 \\ 8/19 & -15/19 & 2/19 \end{pmatrix} \begin{pmatrix} 16 \\ 8 \\ 15 \end{pmatrix} = \begin{pmatrix} -1 \\ 5 \\ 2 \end{pmatrix}$$

5. Show that in general $\det(A+B) \neq \det A + \det B$.

Solution: Consider $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, then

$$\det A = \det B = 0 \quad \text{but} \quad \det(A+B) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1 \neq \det A + \det B$$

Linear Independence

Definition. We say that a set of vectors:

$$\vec{v}_1, \vec{v}_2, \dots \vec{v}_m$$

in $\mathbb{R}^n$ are **linearly independent** if the only solution to

$$x_1 v_1 + x_2 v_2 + \cdots x_m v_m = \vec{0}$$

is $x_1 = x_2 = \cdots = x_m = 0$.

This is equivalent to saying that the matrix equation:

$$\underbrace{\begin{pmatrix} | & | & & | \\ \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_m \\ | & | & & | \end{pmatrix}}_{A} \vec{x} = \vec{0}$$

has $\vec{x} = \vec{0}$ as its only solution.

Examining the augmented matrix $\left(\begin{array}{c|c} A & \vec{0} \end{array} \right)$, we see that $A\vec{x} = \vec{0}$ has a unique solution $\vec{x} = 0$ if and only if A is row equivalent to a matrix of the form:

$$\begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & \ddots & * \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

In particular, if $\vec{v}_1, \ldots, \vec{v}_m$ are linearly independent, then $m \le n$.

Example. Determine if the following vectors are linearly independent:

$$\left\{ \vec{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \\ 0 \end{pmatrix}, \vec{v}_2 = \begin{pmatrix} 2 \\ 5 \\ 0 \\ -3 \end{pmatrix}, \vec{v}_3 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 4 \end{pmatrix}, \vec{v}_4 = \begin{pmatrix} 3 \\ 6 \\ 2 \\ 5 \end{pmatrix} \right\}.$$

Let

$$A = \begin{pmatrix} 1 & 2 & 0 & 3 \\ -1 & 5 & 1 & 6 \\ 2 & 0 & 0 & 2 \\ 0 & -3 & 4 & 5 \end{pmatrix}$$

be the matrix whose columns are the $\vec{v}_i$'s.

Applying Gaussian elimination, we see that A is row equivalent to:

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

What this tells us is that the equation:

$$x_1\vec{v}_1 + \cdots + x_4\vec{v}_4 = \vec{0}$$

is equivalent to:

$$x_1 \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} + x_2 \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} + x_3 \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} + x_4 \begin{pmatrix} 1 \\ 1 \\ 2 \\ 0 \end{pmatrix} = \vec{0},$$

which has a solution where not all x_i's are zero. For example,

$$x_1 = 1, x_2 = 1, x_3 = 2, x_4 = -1.$$

Hence, the vectors $\vec{v}_1, \ldots \vec{v}_4$ are not linearly independent.

Note also from the row reduced augmented matrix that the smaller matrix formed by $\vec{v}_1, \vec{v}_2, \vec{v}_3$ is row equivalent to:

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

which implies that $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linear independent.

---

6. Determine if each of the following sets of vectors in $\mathbb{R}^3$ are linearly independent:

   (a)
   $$\left\{ \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix} \right\}$$

   (b)
   $$\left\{ \begin{pmatrix} -1 \\ -4 \\ 3 \end{pmatrix}, \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix} \right\}$$

   (c)
   $$\left\{ \begin{pmatrix} -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 5 \\ 0 \\ -7 \end{pmatrix} \right\}$$

   Solution:

   (a)

$$\begin{pmatrix} -1 & 1 & -2 \\ -4 & 2 & 1 \\ 3 & 3 & 2 \end{pmatrix} \xrightarrow[r_3 - 3 \times r_1 \to r_3]{\substack{r_1 \times (-1) \to r_1 \\ r_2 + 4 \times r_1 \to r_2}} \begin{pmatrix} 1 & -1 & 2 \\ 0 & -2 & 9 \\ 0 & 6 & -4 \end{pmatrix}$$

$$\xrightarrow[r_3 - 6 \times r_2 \to r_3]{\substack{r_2 \div (-2) \to r_2 \\ r_1 + r_2 \to r_1}} \begin{pmatrix} 1 & 0 & -5/2 \\ 0 & 1 & -9/2 \\ 0 & 0 & 23 \end{pmatrix}$$

$$\xrightarrow[r_2 + 9/2 \times r_3 \to r_2]{\substack{r_3 \div 23 \to r_3 \\ r_1 + 5/2 \times r_3 \to r_1}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

   Therefore, the set is linearly independent.

(b) The matrix we formed here is of size 3×4. Observe that after Gaussian elimination, the last column of the matrix must be of all zeroes. Therefore, the set is not linearly independent.

Remark: For a set of m vectors in $\mathbb{R}^n$, if $m > n$, then the set is not linearly independent.

(c)

$$\begin{pmatrix} -2 & 5 \\ 1 & 0 \\ 2 & -7 \end{pmatrix} \xrightarrow{\substack{r_1 \div (-2) \to r_1 \\ r_2 - r_1 \to r_2 \\ r_3 - 2 \times r_1 \to r_3}} \begin{pmatrix} 1 & -5/2 \\ 0 & 5/2 \\ 0 & 5 \end{pmatrix}$$

$$\xrightarrow{\substack{r_2 \div 5/2 \to r_2 \\ r_1 + 5/2 \times r_2 \to r_1 \\ r_3 - 5 \times r_2 \to r_3}} \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}$$

Therefore, the set is linearly independent.

7. Show that if three vectos v_1, v_2, v_3 in $\mathbb{R}^3$ are linearly independent, then every vector $\vec{v} \in \mathbb{R}^3$ may be expressed uniquely as a linear combination of v_1, v_2, v_3. In other words, there are unique scalars $\lambda_1, \lambda_2, \lambda_3$ such that:

$$\vec{v} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \lambda_3 \vec{v}_3.$$

Solution: If $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are linearly independent, then $\begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$ is invertible (as it can be transformed to identity matrix through Gaussian elimination). Observe that

$$\vec{v} = \lambda_1 \vec{v}_1 + \lambda_2 \vec{v}_2 + \lambda \vec{v}_3 \Rightarrow \begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \end{pmatrix} = \begin{pmatrix} | \\ \vec{v} \\ | \end{pmatrix}$$

As $\begin{pmatrix} | & | & | \\ \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ | & | & | \end{pmatrix}$ is invertible, the equation has unique solution $\lambda_1, \lambda_2, \lambda_3$.