THE CHINESE UNIVERSITY OF HONG KONG
MATH 1540 Homework Set 1
Due time 6:30 pm Sep 29, 2016

1. (a) Let:

$$A = \begin{pmatrix} 4 & 10 \\ -7 & 8 \\ 6 & -1 \end{pmatrix}, \quad B = \begin{pmatrix} 3 & -7 \\ 10 & 1 \end{pmatrix}, \quad C = \begin{pmatrix} 9 & -2 \\ -3 & 1 \end{pmatrix}.$$

Verify that:

$$A(B+C) = AB + AC.$$

(b) From the definition of matrix addition and multiplication:

$$(A+B)_{ij} = A_{ij} + B_{ij}, \quad (AB)_{ij} = \sum_{k=1}^{n} A_{ik} B_{kj},$$

show that, for any $m \times n$ matrix A, and $n \times l$ matrices B, C, we have:

$$A(B+C) = AB + AC.$$

Solution:

(a) By direct calculation,

$$B + C = \begin{pmatrix} 12 & -9 \\ 7 & 2 \end{pmatrix}$$

$$A(B+C) = \begin{pmatrix} 118 & -16 \\ -28 & 79 \\ 65 & -56 \end{pmatrix}$$

$$AB = \begin{pmatrix} 112 & -18 \\ 59 & 57 \\ 8 & -43 \end{pmatrix}$$

$$AC = \begin{pmatrix} 6 & 2 \\ -87 & 22 \\ 57 & -13 \end{pmatrix}$$

$$AB + AC = \begin{pmatrix} 118 & -16 \\ -28 & 79 \\ 65 & -56 \end{pmatrix} = A(B+C)$$

(b) For $1 \leq i \leq m, 1 \leq k \leq l$, we have

$$\begin{aligned}(A(B+C))_{ik} &= \sum_{j=1}^{n} A_{ij}(B+C)_{jk} \\ &= \sum_{j=1}^{n}(A_{ij}B_{jk} + A_{ij}C_{jk}) \\ &= \sum_{j=1}^{n} A_{ij}B_{jk} + \sum_{j=1}^{n} A_{ij}B_{jk} \\ &= AB_{ik} + AC_{ik} \\ &= (AB + AC)_{ik}\end{aligned}$$

Therefore, we have $A(B+C) = AB + AC$.

2. Show that, given two $m \times n$ matrices A and B, the condition $A\vec{v} = B\vec{v}$ for all $\vec{v} \in \mathbb{R}^n$ implies that $A = B$, i.e.:

$$A_{ij} = B_{ij}, \quad 1 \leq i \leq m, 1 \leq j \leq n.$$

Solution:

We fix $1 \leq j \leq n$, and take $\vec{v} = \vec{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{pmatrix}$ (the column vector with the j-th entry equal to one, other entries equal to zero), then

$$A\vec{v} = \begin{pmatrix} A_{1j} \\ \vdots \\ A_{mj} \end{pmatrix} = \begin{pmatrix} B_{1j} \\ \vdots \\ B_{mj} \end{pmatrix} = B\vec{v}$$

By comparing each entries of the two vectors, we have $A_{ij} = B_{ij}\ for\ 1 \leq i \leq m$. Since there is no restriction on $1 \leq j \leq n$ (or in other words, j is arbitrary), we have $A_{ij} = B_{ij}\ for\ 1 \leq i \leq m,\ 1 \leq j \leq n$.

3. (a) Let:

$$A = \begin{pmatrix} 1 & -1 & 3 & -5 \\ 2 & 0 & -1 & 3 \\ 7 & 9 & -4 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 4 \\ 1 & 5 \\ -3 & 0 \\ 0 & -6 \end{pmatrix},$$

$$C = \begin{pmatrix} -2 & 0 & 3 & 1 \\ 5 & -7 & 0 & 4 \end{pmatrix}.$$

Verify that $(AB)C = A(BC)$.

(b) (*Optional*) Show that for any $m \times n$ matrix A, $n \times l$ matrix B, and $l \times r$ matrix C, we have:

$$A(BC) = (AB)C.$$

Solution:

(a) By direct calculation,

$$AB = \begin{pmatrix} -10 & 29 \\ 3 & -10 \\ 21 & 73 \end{pmatrix}$$

$$(AB)C = \begin{pmatrix} 165 & -203 & -30 & 106 \\ -56 & 70 & 9 & -37 \\ 323 & -511 & 63 & 313 \end{pmatrix}$$

$$BC = \begin{pmatrix} 20 & -28 & 0 & 16 \\ 23 & -35 & 3 & 21 \\ 6 & 0 & -9 & -3 \\ -30 & 42 & 0 & -24 \end{pmatrix}$$

$$A(BC) = \begin{pmatrix} 165 & -203 & -30 & 106 \\ -56 & 70 & 9 & -37 \\ 323 & -511 & 63 & 313 \end{pmatrix}$$

(b) For $1 \leq i \leq m, 1 \leq j \leq n, 1 \leq k \leq l, 1 \leq s \leq r$,

$$\begin{aligned}
(BC)_{js} &= \sum_{k=1}^{l} B_{jk}C_{ks} \\
(A(BC))_{is} &= \sum_{j=1}^{n} A_{ij}(BC)_{js} \\
&= \sum_{j=1}^{n} A_{ij}\left(\sum_{k=1}^{l} B_{jk}C_{ks}\right) \\
&= \sum_{j=1}^{n}\sum_{k=1}^{l} A_{ij}B_{jk}C_{ks}
\end{aligned}$$

and

$$
\begin{aligned}
(AB)_{ik} &= \sum_{j=1}^{n} A_{ij} B_{jk} \\
((AB)C)_{is} &= \sum_{k=1}^{l} (AB)_{ik} C_{ks} \\
&= \sum_{k=1}^{l} \left(\sum_{j=1}^{n} A_{ij} B_{jk} \right) C_{ks} \\
&= \sum_{j=1}^{n} \sum_{k=1}^{l} A_{ij} B_{jk} C_{ks} \\
&= (A(BC))_{is}
\end{aligned}
$$

Therefore, $A(BC) = (AB)C$.

4. Solve the following system of linear equations by performing Gaussian elimination on the associated augmented matrix:

$$
\begin{aligned}
x_1 - x_2 + 5x_3 + 7x_4 &= -23 \\
2x_1 + 4x_3 - 4x_4 &= -16 \\
3x_2 - 2x_4 &= 0 \\
5x_1 - x_4 &= 10
\end{aligned}
$$

Solution:
First, let's write the augmented matrix.

$$
\left(\begin{array}{cccc|c}
1 & -1 & 5 & 7 & -23 \\
2 & 0 & 4 & -4 & -16 \\
0 & 3 & 0 & -2 & 0 \\
5 & 0 & 0 & -1 & 10
\end{array} \right)
$$

For the row reductions (for the sake of space, I compress several operations into one arrow),

$$
\left(\begin{array}{cccc|c} 1 & -1 & 5 & 7 & -23 \\ 2 & 0 & 4 & -4 & -16 \\ 0 & 3 & 0 & -2 & 0 \\ 5 & 0 & 0 & -1 & 10 \end{array}\right)
\xrightarrow[\substack{r_2 - 2\times r_1 \to r_2 \\ r_4 - 5\times r_1 \to r_4}]{}
\left(\begin{array}{cccc|c} 1 & -1 & 5 & 7 & -23 \\ 0 & 2 & -6 & -18 & 30 \\ 0 & 3 & 0 & -2 & 0 \\ 0 & 5 & -25 & -36 & 125 \end{array}\right)
$$

$$
\xrightarrow[r_2 \times 1/2 \to r_2]{}
\left(\begin{array}{cccc|c} 1 & -1 & 5 & 7 & -23 \\ 0 & 1 & -3 & -9 & 15 \\ 0 & 3 & 0 & -2 & 0 \\ 0 & 5 & -25 & -36 & 125 \end{array}\right)
$$

$$
\xrightarrow[\substack{r_1 + r_2 \to r_1 \\ r_3 - 3\times r_2 \to r_3 \\ r_4 - 5\times r_2 \to r_4}]{}
\left(\begin{array}{cccc|c} 1 & 0 & 2 & -2 & -8 \\ 0 & 1 & -3 & -9 & 15 \\ 0 & 0 & 9 & 25 & -45 \\ 0 & 0 & -10 & 9 & 50 \end{array}\right)
$$

$$
\xrightarrow[r_3 \times 1/9 \to r_3]{}
\left(\begin{array}{cccc|c} 1 & 0 & 2 & -2 & -8 \\ 0 & 1 & -3 & -9 & 15 \\ 0 & 0 & 1 & 25/9 & -5 \\ 0 & 0 & -10 & 9 & 50 \end{array}\right)
$$

$$
\xrightarrow[r_4 + 10\times r_3 \to r_4]{}
\left(\begin{array}{cccc|c} 1 & 0 & 2 & -2 & -8 \\ 0 & 1 & -3 & -9 & 15 \\ 0 & 0 & 1 & 25/9 & -5 \\ 0 & 0 & 0 & 331/9 & 0 \end{array}\right)
$$

$$
\xrightarrow[r_4 \times 9/331 \to r_4]{}
\left(\begin{array}{cccc|c} 1 & 0 & 2 & -2 & -8 \\ 0 & 1 & -3 & -9 & 15 \\ 0 & 0 & 1 & 25/9 & -5 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)
$$

$$
\xrightarrow[r_4 \times 9/331 \to r_4]{}
\left(\begin{array}{cccc|c} 1 & 0 & 2 & -2 & -8 \\ 0 & 1 & -3 & -9 & 15 \\ 0 & 0 & 1 & 25/9 & -5 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)
$$

$$
\xrightarrow[\substack{r_1 + 2\times r_4 \to r_1 \\ r_2 + 9\times r_4 \to r_2 \\ r_4 - 25/9\times r_4 \to r_3}]{}
\left(\begin{array}{cccc|c} 1 & 0 & 2 & 0 & -8 \\ 0 & 1 & -3 & 0 & 15 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)
$$

$$
\xrightarrow[\substack{r_1 - 2\times r_3 \to r_1 \\ r_2 + 3\times r_3 \to r_2}]{}
\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right)
$$

Therefore, we have $(x_1, x_2, x_3, x_4) = (2, 0, -5, 0)$

5. Find all solutions $\vec{x} \in \mathbb{R}^4$ to the following matrix equation:

$$
\begin{pmatrix} 5 & 10 & -9 & -4 \\ 1 & 2 & 1 & 2 \\ -1 & -2 & 3 & 2 \end{pmatrix} \vec{x} = \begin{pmatrix} 23 \\ -1 \\ -7 \end{pmatrix}.
$$

Solution:
First, let's write the augmented matrix.

$$\left(\begin{array}{cccc|c} 5 & 10 & -9 & -4 & 23 \\ 1 & 2 & 1 & 2 & -1 \\ -1 & -2 & 3 & 2 & -7 \end{array}\right)$$

For the row reductions (for the sake of space, I compress several operations into one arrow),

$$\begin{aligned}
\left(\begin{array}{cccc|c} 5 & 10 & -9 & -4 & 23 \\ 1 & 2 & 1 & 2 & -1 \\ -1 & -2 & 3 & 2 & -7 \end{array}\right) &\xrightarrow[\text{exchange } r_1 \text{ and } r_2]{} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & -1 \\ 5 & 10 & -9 & -4 & 23 \\ -1 & -2 & 3 & 2 & -7 \end{array}\right) \\
&\xrightarrow[\substack{r_1 \times (-5) + r_2 \to r_2 \\ r_1 + r_3 \to r_3}]{} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & -1 \\ 0 & 0 & -14 & -14 & 28 \\ 0 & 0 & 4 & 4 & -8 \end{array}\right) \\
&\xrightarrow[\substack{r_2 \times (-\frac{1}{14}) \to r_2 \\ r_3 - r_2 \to r_3}]{} \left(\begin{array}{cccc|c} 1 & 2 & 1 & 2 & -1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right) \\
&\xrightarrow[r_1 - r_2 \to r_1]{} \left(\begin{array}{cccc|c} 1 & 2 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)
\end{aligned}$$

This augmented matrix corresponds to the linear system:

$$\begin{aligned} x_1 + 2x_2 + x_4 &= 1 \\ x_3 + x_4 &= -2 \end{aligned}$$

There are 4 variables and two pivots (corresponding to x_1, x_3), so we let x_2 and x_4 be free parameters u and v, respectively. Then, $x_3 = -2 - v$ and $x_1 = 1 - 2u - v$. Hence, we have $(x_1, x_2, x_3, x_4) = (1 - 2u - v, u, -2 - v, v), u, v \in \mathbb{R}$, or equivalently:

$$\vec{x} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} + u \begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + v \begin{pmatrix} -1 \\ 0 \\ -1 \\ 1 \end{pmatrix}, \quad u, v \in \mathbb{R}.$$

6. For what values of $a, b, c \in \mathbb{R}$ would the following matrix equation have a unique solution $\vec{x} \in \mathbb{R}^3$?

$$\begin{pmatrix} 2 & 0 & 4 \\ 1 & 1 & 3 \\ a & b & c \\ -1 & -5 & -7 \end{pmatrix} \vec{x} = \vec{0}$$

Solution:
Consider the augmented matrix of the system.

$$\left(\begin{array}{ccc|c} 2 & 0 & 4 & 0 \\ 1 & 1 & 3 & 0 \\ a & b & c & 0 \\ -1 & -5 & -7 & 0 \end{array}\right) \xrightarrow[\substack{r_1\ interchange\ r_2 \\ r_3\ interchange\ r_4}]{} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 2 & 0 & 4 & 0 \\ -1 & -5 & -7 & 0 \\ a & b & c & 0 \end{array}\right)$$

$$\xrightarrow[\substack{r_2 - 2\times r_1 \to r_2 \\ r_3 + r_1 \to r_3}]{} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & -2 & -2 & 0 \\ 0 & -4 & -4 & 0 \\ a & b & c & 0 \end{array}\right)$$

$$\xrightarrow[r_2 \times 1/(-2) \to r_2]{} \left(\begin{array}{ccc|c} 1 & 1 & 3 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & -4 & -4 & 0 \\ a & b & c & 0 \end{array}\right)$$

$$\xrightarrow[\substack{r_1 - r_2 \to r_1 \\ r_3 + 4\times r_2 \to r_3}]{} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ a & b & c & 0 \end{array}\right)$$

$$\xrightarrow[\substack{r_4 - a\times r_1 \to r_4 \\ r_4 - b\times r_2 \to r_4}]{} \left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2a-b+c & 0 \end{array}\right)$$

In order to have a unique solution, we need $-2a-b+c \neq 0$ (or x_3 can be any real number).

7. (Optional) An (undirected) *graph* consists of two sets of data: A set of points, called *vertices*, and a set of unordered pairs of vertices, called *edges*.

   For example, the graph with vertices $\{V_1, V_2, V_3, V_4, V_5\}$ and edges

   $$\{\{V_1, V_2\}, \{V_1, V_3\}, \{V_1, V_4\}, \{V_4, V_5\}\}$$

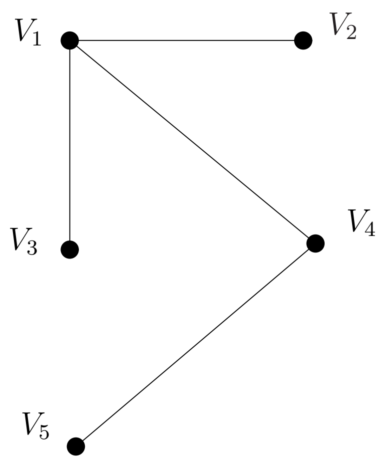
   may be visualized as follows:

   The *adjacency matrix* of a graph with n vertices is an $n \times n$ matrix $A = (A_{ij})$ defined by:

   $$A_{ij} = \begin{cases} 1 & \text{if } \{V_i, V_j\} \text{ is an edge of the graph,} \\ 0 & \text{if there is no edge connecting } V_i \text{ and } V_j. \end{cases}$$

   In the example above, the corresponding adjacency matrix is:

   $$A = \begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$



A *walk* in a graph is a sequence of edges linking one vertex to another. The number of edges in the sequence is called the *length* of the walk. In the example above, the sequence $\{V_1, V_4\}, \{V_4, V_5\}$ is a walk of length two from V_1 to V_5.

Prove the following theorem:

Theorem. Let $A = (A_{ij})$ be the adjacency matrix of a graph. Show that, for any integer $n \geq 1$, the number $(A^n)_{ij}$ (the ij-th entry of A^n) is equal to the number of walks of length k from V_i to V_j.

Solution:
We use M.I. to prove the theorem.

When $k = 1$, it is trivial by the definition of A.

Suppose the statement is true for $k = m$, then for $k = m + 1$,

We have $(A^{m+1})_{ij} = \sum_{l=1}^{m} (A^m)_{il} A_{lj}$. Observe that $(A^m)_{il}$ is equal to the number of walks of length m from V_i to V_l by the hypothesis, while A_{lj} tells us whether V_l is connected to V_j by an edge. Therefore, $(A^m)_{il} A_{lj}$ is equal to the number of walks of length $m + 1$ from V_i to V_j, with the last edge fixed to be $\{V_l, V_j\}$. Therefore, $(A^{m+1})_{ij} = \sum_{l=1}^{m} (A^m)_{il} A_{lj}$ means the total number of walks of length $m + 1$ from V_i to V_j.