Week 12
Definite Integrals

Motivation

Given a continuous function over a closed interval. We want to approximate the area of the region bounded by the graph of the function and the x-axis.
One way to do so is by viewing the region roughly as a union of sequence of rectangles, and then adding up the areas of these rectangles.

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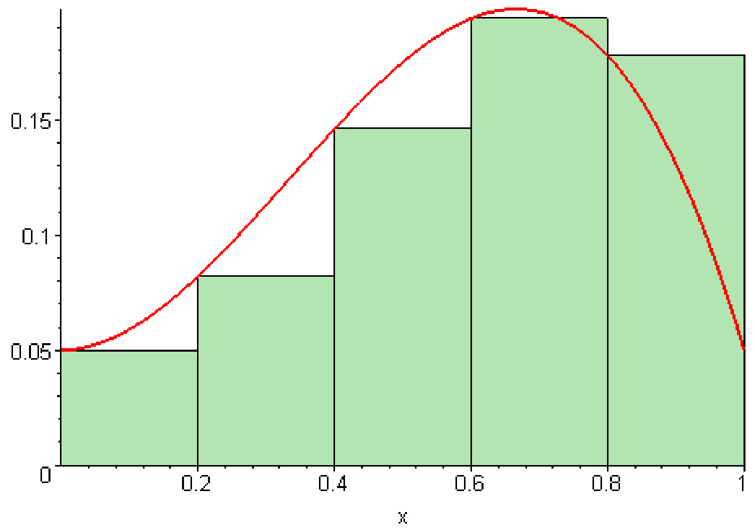

5 rectangles.

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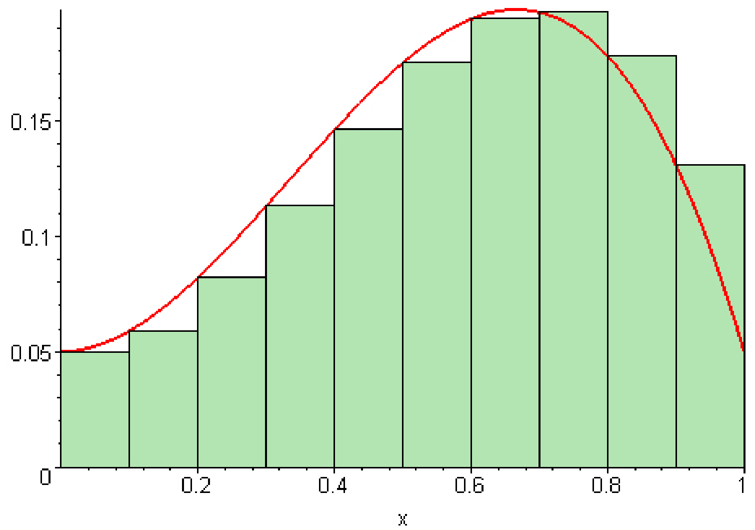

10 rectangles.

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Intuitively, we see that the more (and smaller) rectangles are used, the more closely their union approximates the region in question.

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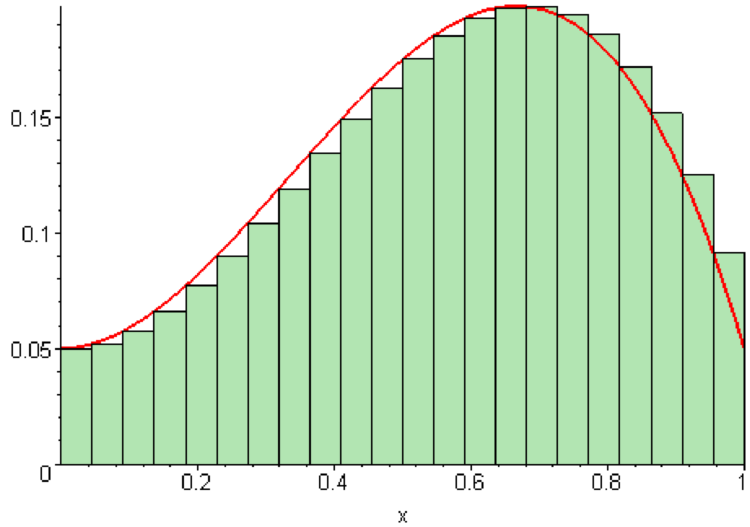

Definition.

Let n be a positive integer.

Let $f : [a, b] \longrightarrow \mathbb{R}$ be a continuous function on a closed interval.

Let:

$$\Delta x = \frac{b-a}{n}.$$

The **Left Riemann Sum** of f over $[a, b]$ associated with n subintervals of equal lengths is:

$$\begin{aligned} LS_n(f) &= \sum_{k=0}^{n-1} f(a + k\Delta x)\Delta x \\ &= (f(a) + f(a + \Delta x) + f(a + 2\Delta x) + \ldots + f(a + (n-1)\Delta x))\,\Delta x \end{aligned}$$

Each summand may be thought of as the area of the rectangle whose base is the subinterval $[a + k\Delta x, a + (k+1)\Delta x]$, and whose height is the value of f at the left endpoint of the subinterval.

Definition.

Let $f:[a,b]\longrightarrow\mathbb{R}$ be a continuous function on a closed interval. The **definite integral** $\int_a^b f(x)\,dx$ of f over $[a,b]$ is equal to the limit as n tends to infinity of the left Riemann sum defined previously. That is:

$$\begin{aligned}\int_a^b f(x)\,dx &= \lim_{n\to\infty} LS_n(f)\\ &= \lim_{n\to\infty}\frac{b-a}{n}\sum_{k=0}^{n-1} f\left(a+\frac{k(b-a)}{n}\right)\end{aligned}$$

It is an established theorem that the limit exists if f is continuous.
(In fact: One could define the definite integral in terms of the Right Riemann Sum or the Midpoint Riemann Sum. All these sums tend to same limit in the case where f is continuous.)

Our eventual goal is to show that if F is an antiderivative of a continuous function f, then:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b := F(b)-F(a).$$

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- Integration by Substitution

$$\int_a^b f(u(x))u'(x)\,dx = \int_{u(a)}^{u(b)} f(u)\,du = F(u(b))-F(u(a))$$

  if F is an antiderivative of f.
- Integration by Parts

$$\int_a^b u(x)v'(x)dx = u(x)v(x)\Big|_a^b - \int_a^b v(x)u'(x)\,dx.$$

Before we prove the main theorem, we first state a couple of preliminary results.

Definition.

For a continuous function f on $[a, b]$, we define:

$$\int_a^a f(x)\,dx = 0.$$

$$\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx.$$

Claim.

Let f be a continuous function on an interval I. For all $a, b, c \in I$, we have:

$$\int_a^b f(x)\,dx + \int_b^c f(x)\,dx = \int_a^c f(x)\,dx.$$

Claim.

Let f, g be continuous functions on $[a, b]$. If $f(x) \leq g(x)$ for all $x \in [a, b]$, then:

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx.$$

Theorem.

(**Mean Value Theorem for Integrals**) Let f be a continuous function on $[a, b]$. There exists $c \in [a, b]$ such that:

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\,dx.$$

Proof.

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Since f is continuous on $[a,b]$, by the Extreme Value Theorem it has a maximum value M and minimum value m on $[a,b]$.
In other words,

$$m \leq f(x) \leq M$$

for all $x \in [a,b]$. Hence:

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$$\underbrace{\int_a^b m\,dx}_{m(b-a)} \leq \int_a^b f(x)\,dx \leq \underbrace{\int_a^b M\,dx}_{M(b,a)}.$$

Dividing each expression by $b-a$, we have:

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$$m \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq M.$$

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Let x_1, x_2 be elements in $[a,b]$ such that $M = f(x_1)$ and $m = f(x_2)$. Since f is continuous on $[a,b]$, and $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$ is a number between $f(x_1)$ and $f(x_2)$, by the Intermediate Value Theorem there exists c between x_1 and x_2 such that:

$$f(c) = \frac{1}{b-a}\int_a^b f(x)\,dx.$$

This c lies in $[a,b]$, since x_1, x_2 lies in $[a,b]$.

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Theorem.

(**Fundamental Theorem of Calculus Part I**) Let f be a continuous function on $[a,b]$. Define a function $F:[a,b]\longrightarrow\mathbb{R}$ as follows:

$$F(x)=\int_a^x f(t)\,dt,\quad x\in\mathbb{R}.$$

Then, F is continuous on $[a,b]$ and differentiable on (a,b), with:

$$F'(x)=f(x)$$

for all $x\in(a,b)$. Equivalently:

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$$\frac{d}{dx}\int_a^x f(t)\,dt=f(x)$$

Proof.

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By definition:

$$\begin{aligned}F'(x)&=\lim_{h\to 0}\frac{F(x+h)-F(x)}{h}.\\&=\lim_{h\to 0}\frac{\int_a^{x+h} f(t)\,dt-\int_a^x f(t)\,dt}{h}.\\&=\lim_{h\to 0}\frac{\int_x^{x+h} f(t)\,dt}{h}.\end{aligned}$$

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By the Mean Value Theorem for Integrals, there exists $c_h\in[x,x+h]$ such that:

$$f(c_h)=\frac{\int_x^{x+h} f(t)\,dt}{h}.$$

Hence:

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$$F'(x)=\lim_{h\to 0}f(c_h)=f(x),$$

since for any h the number c_h lies between x and $x+h$, and f is continuous.

We leave the proof of the continuity of F on $[a,b]$ as an exercise.

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Corollary.

Let f be a continuous function. Let g and h be differentiable functions. Then:

$$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x).$$

Example.

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Evaluate:

$$\frac{d}{dx}\int_{\sin x}^{x^3+1} e^{-t^2}\,dt$$

Example.

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Evaluate:

$$\lim_{h\to 0^+}\frac{1}{\ln(1+h)}\int_2^{2+h}\sqrt{t^4+1}\,dt$$

Theorem.

(**Fundamental Theorem of Calculus Part II**) Let f be a continuous function on $[a,b]$. Let F be a continuous function on $[a,b]$ which is an antiderivative of f over (a,b). Then:

$$\int_a^b f(x)\,dx = F(b) - F(a).$$

Proof.

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By the Fundamental Theorem of Calculus Part I, we know that $G(x) = \int_a^x f(t)\,dt$ is also an antiderivative of f. By Lagrange's Mean Value Theorem and the continuity of F and G on $[a,b]$, for all $x \in [a,b]$ we have:

$$G(x) = F(x) + C$$

for some constant C.

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Since $G(a) = \displaystyle\int_a^a f(t)\,dt = 0$, we have $C = -F(a)$.

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Hence:

$$\int_a^b f(t)\,dt = G(b) = F(b) + C = F(b) - F(a).$$

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