# Tutorial 12 for MATH4220

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1. **The Exterior of a Circle**

Consider the following Dirichlet problem for the exterior of a circle

$$\begin{cases} u_{xx} + u_{yy} = 0, & x^2 + y^2 > a^2 \\ u = h(\theta), & x^2 + y^2 = a^2 \\ u \text{ is bounded as} & x^2 + y^2 \to \infty \end{cases}$$

**Solution:** In polar coordinates, it suffices to solve

$$\begin{cases} r^2 u_{rr} + r u_r + u_{\theta\theta} = 0, & a < r < \infty, 0 < \theta < 2\pi \\ u = h(\theta), & 0 < \theta < 2\pi \\ u(r, 0) = u(r, 2\pi), & a < r < \infty \\ u_\theta(r, 0) = u_\theta(r, 2\pi), & a < r < \infty \\ u \text{ is bounded as} & r \to \infty \end{cases}$$

Find a seperable solution in polar coordinates, $u = R(r)\Theta(\theta)$. Thus by the equation, we have

$$\frac{r^2 R'' + rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda(\text{constant})$$

Solve the eigenvalue problem

$$\begin{cases} \Theta'' = -\lambda\Theta, & 0 < \theta < 2\pi \\ \Theta(0) = \Theta(2\pi), & \Theta'(0) = \Theta'(2\pi) \end{cases}$$

Thus the eigenvalues are $\lambda_n = n^2$ and the corresponding eigenfunctions are

$$\Theta_n = a_n \cos n\theta + b_n \sin n\theta, \quad n = 0, 1, 2, \cdots$$

It remains to solve

$$r^2 R'' + rR' - \lambda R = 0, \quad a < r < \infty$$

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When $n=0$, $r^2R''+rR'=0$, thus $R_0r=c_0+d_0\ln r$. When $n\geq 1$, $R_n(r)=c_nr^{-n}+d_nr^n$. Since $u$ is bounded as $r\to\infty$, thus $R_0(r)=c_0, R_n(r)=c_nr^{-n}$

Thus

$$\begin{aligned}u(r,\theta)=&\sum_{n=0}^{\infty}R_n(r)\Theta_n(\theta)\\=&a_0c_0+\sum_{n=1}^{\infty}c_nr^{-n}(a_n\cos n\theta+b_n\sin n\theta)\\=&\frac{A_0}{2}+\sum_{n=1}^{\infty}r^{-n}(A_n\cos n\theta+B_n\sin n\theta)\end{aligned}$$

Set $r=a$,

$$h(\theta)=\frac{A_0}{2}+\sum_{n=1}^{\infty}a^{-n}(A_n\cos n\theta+B_n\sin n\theta)$$

where

$$A_n=\frac{a^n}{\pi}\int_0^{2\pi}\cos n\theta h(\theta)d\theta,\ n=0,1,\cdots$$
$$B_n=\frac{a^n}{\pi}\int_0^{2\pi}\sin n\theta h(\theta)d\theta,\ n=1,2,\cdots$$

Actually, this series can be summed explicitly.

$$u(r,\theta)=\frac{r^2-a^2}{2\pi}\int_0^{2\pi}\frac{h(\phi)}{r^2+a^2-2ar\cos\theta-\phi}d\phi\quad\text{(in polar coordinates)}$$
$$u(\vec{x})=\frac{|\vec{x}|^2-a^2}{2\pi a}\int_{|\vec{x}|=a}\frac{u(\vec{x}')}{|\vec{x}-\vec{x}'|^2}dS(\vec{x}')\quad\text{(in rectangle coordinates)}$$

2. **The wedge**

Consider the following Dirichlet problem for a wedge

$$\begin{cases}u_{xx}+u_{yy}=0, & 0<a^2<x^2+y^2, 0<\theta<\beta\\ u=h(\theta), & x^2+y^2=a^2\\ u=0, & \theta=0,\beta\end{cases}$$

**Solution:** In polar coordinates,

$$\begin{cases}r^2u_{rr}+ru_r+u_{\theta\theta}=0, & 0<r<a, 0<\theta<\beta\\ u=h(\theta), & 0<\theta<\beta\\ u(r,0)=u(r,\beta)=0, & 0<r<a\end{cases}$$

Find a seperable solution in polar coordinates, $u=R(r)\Theta(\theta)$,

$$\frac{r^2R''+rR'}{R}=-\frac{\Theta''}{\Theta}=\lambda(\text{constant})$$

Solve the eigenvalue problem

$$\begin{cases}\Theta''=-\lambda\Theta, & 0<\theta<\beta\\ \Theta(0)=\Theta(\beta)=0\end{cases}$$

Thus the eigenvalues are $\lambda_n = (\frac{n\pi}{\beta})^2$ and the corresponding eigenfunctions are

$$\Theta_n = \sin(\frac{n\pi}{\beta}\theta), \quad n = 1, 2, \cdots$$

It remains to solve

$$r^2 R'' + rR' - \lambda R = 0, \quad a < r < b.$$

Then $n \geq 1$, $R_n(r) = c_n r^{-\frac{n\pi}{\beta}} + d_n r^{\frac{n\pi}{\beta}}$. Since $u$ is bounded when $0 < r < a$, thus $c_n = 0, R_n(r) = d_n r^{\frac{n\pi}{\beta}}$.

Thus

$$\begin{aligned} u(r,\theta) &= \sum_{n=1}^{\infty} R_n(r)\Theta_n(\theta) \\ &= \sum_{n=1}^{\infty} d_n r^{\frac{n\pi}{\beta}} \sin(\frac{n\pi}{\beta}\theta) \end{aligned}$$

Set $r = a$,

$$h(\theta) = \sum_{n=1}^{\infty} d_n a^{\frac{n\pi}{\beta}} \sin(\frac{n\pi}{\beta}\theta)$$

where

$$d_n a^{\frac{n\pi}{\beta}} = \frac{2}{\beta}\int_0^{\beta} \sin(\frac{n\pi}{\beta}\theta) h(\theta) d\theta, \ n = 1, 2, \cdots$$