Prop 8 (Structure Th of Open Sets)

Proof. Let G be a nonempty open set, and $x \in G$. Let I_x denote the union of all intervals containing x and contained in G. Then, by 2050 techniques, you can show (Ex. below) that I_x is also an interval and is open. Do this for all $x \in G$ and note that G = union of all I_x. Will show that

(#) $\mathcal{C} \triangleq \{ I_x : x \in G \}$ is a disjoint family,

that this, if I_{x_1} and I_{x_2} has a nonempty intersection then $I_{x_1} = I_{x_2}$. Indeed, if $x \in I_{x_1} \cap I_{x_2}$ then $I_{x_1} \cup I_{x_2}$ is also an interval (Ex 2) and so $I_{x_1} \cup I_{x_2} \subseteq I_{x_1}$ (by definition of I_{x_1}). Therefore $I_{x_2} \subseteq I_{x_1}$ and similarly $I_{x_1} \subseteq I_{x_2}$; (#) is shown. By "axiom of choice", $\exists G_0 \subseteq G$ such that $\mathcal{C} = \{ I_x : x \in G_0 \}$ such that $I_x \cap I_y = \emptyset$ whenever x, y are distinct elements of G_0. Pick a rational $r_x \in I_x$ for each $x \in G_0$. Then the map $x \mapsto r_x$ is injective (one-to-one) from G_0 into $\mathbb{Q}$ so G_0 is countable, that $\mathcal{C}$ is a countable collection of open intervals (and G is the union of these intervals)

Ex1 (From 2050). Let I be a nonempty subset of $\mathbb{R}$. Then I is an interval iff it has the property that

(*) $x \in I$ whenever $\exists\, y, z \in I$ s.t. $y < x < z$.

Ex2. If I_{x_1}, I_{x_2} are intervals with nonempty intersection then $I_{x_1} \cup I_{x_2}$ is also an interval.

Sol. Let a_i, b_i be their ends resp. $(i = 1, 2)$, with $\pm\infty$ not excluded; let $c \in I_{x_1} \cap I_{x_2}$, and $a := \min\{a_1, a_2\}$, $b := \max\{b_1, b_2\}$. Then $(a, c) \subseteq I_{x_1}$ or I_{x_2} (depending on $a = a_1$ or a_2) and so $(a, c) \subseteq I_{x_1} \cup I_{x_2}$; similarly $(c, b) \subseteq I_{x_1} \cup I_{x_2}$. Consequently $(a, b) \subseteq I_{x_1} \cup I_{x_2}$ and so equal to an interval with ends a, b (if $x < a$ or $x > b$ then $x \notin I_{x_1} \cup I_{x_2}$)

Ex 3. The union of an collection of intervals with nonempty intersection is an interval.

Sol. Modify the solution given above or make use of the results of Ex 1 & Ex 2.

Proposition 9. Any open cover $\mathscr{C}$ of $A \subseteq \mathbb{R}$ has a finite subcover: Let $A \subseteq \mathbb{R}$ and $\mathscr{C}$ be a collection of open sets with union $\mathcal{U} \left(= \bigcup \{C : \in \mathscr{C}\} \right)$ such that $\mathcal{U} \supseteq A$ then $\exists$ countably ~~finitely~~ many C_1 such that $A \subseteq \bigcup_{n \in \mathbb{N}} C_n$.

Sol. For each $x \in A$, pick $C_x \in \mathscr{C}$ such that $x \in C_x$ and pick an open interval I_x with rational ends such that $x \in I_x \subseteq C_x$. Then $\{I_x : x \in A\}$ is countable (because the family of all open intervals with rational ends is countable) such that $\bigcup_{x \in A} I_x \supseteq A$. Writing $\{I_x : x \in A\} = \{I_{x_n} : n \in \mathbb{N}\}$, we see ~~have~~ that $\bigcup_{n \in \mathbb{N}} C_{x_n} \supseteq A$.

Th (Heine-Borel). Each closed and bounded subset F is compact: if $\mathscr{C}$ is an open cover of F then it has a finite subcover, i.e. $\exists$ finitely many $C_1, C_2, \dots C_n$ from $\mathscr{C}$ such that $F \subset \bigcup_{i=1}^{n} C_i$.

Proof. Special Case: $F = [a,b]$ with reals $a < b$.

Define $c_0 := \sup X$, where

$X := \{x \in [a,b] : [a,x]$ can be covered by a finitely many members of $\mathcal{C}\}$.

Then $a < c_0 \leq b$, and so $\exists\, C_0 \in \mathcal{C}$ s.t. $c_0 \in C_0$ and $\exists$ a_0, b_0 such that $a < a_0 < c_0 < b_0$ and $(a_0, b_0) \subseteq C_0$. By definition of $c_0 = \sup X$, it follows that $\exists\, x_0 \in X$ s.t. $a_0 < x_0$ $(\leq c_0 < b_0)$ and we can take then $C_1, C_2, \cdots C_n \in \mathcal{C}$ such that $[a, x_0] \subseteq \bigcup_{i=1}^{n} C_i$ and so

$$[a, c_0] \subseteq \bigcup_{i=1}^{n} C_i \cup (a_0, b_0) \subseteq \bigcup_{i=0}^{n} C_i \quad (*)$$

It remains to show that $b \leq c_0$ (i.e. $b = c_0$). Should $b > c_0$, one can then pick x such that $\min\{b, b_0\} > x > c_0$, so $x \in [a,b]$ and $x \in \bigcup_{i=0}^{n} C_i$ (by $(*)$), that is $x \in X$, contradicting the fact that $x > c_0 = \sup X$

General Case follows from the special case by considering $\mathcal{C} \cup \{\mathbb{R} \setminus F\} \cap [a,b]$ where $F \subseteq [a,b]$ closed.

Remark. The converse holds too, that is the Heine-Borel th provides the charaterization for compactness (in $\mathbb{R}$).