MATH 3270B - Ordinary Differential Equations - 2017/18
Midterm
Time allowed: 60 minutes

NAME: ______________________________ ID: ______________________________

Answer all the questions. Show your detailed steps.

1. (16 points)

   (a) Check that the following equation is exact and find the general solution.

   $$y^2e^{xy^2} - 2xy + (2xye^{xy^2} - x^2)\frac{dy}{dx} = 0.$$

   (b) Show that the following equation is not exact but becomes exact when multiplied by some nonzero integrating factor μ.

   $$2xy^2 + 2x\sin x + x^2y\frac{dy}{dx} = 0.$$

Solution:

(a) $M = y^2e^{xy^2} - 2xy, \quad N = 2xye^{xy^2} - x^2,$
$M_y = (2y + 2xy^3)e^{xy^2} - 2x = N_x \quad \Longrightarrow exact.$
Then $\Phi(x, y)$ such that $\Phi_x = M \quad \Longrightarrow \Phi = e^{xy^2} - x^2y + g(y);$
$\Phi_y = N \quad \Longrightarrow g'(y) = 0.$
Hence the general solution is $\Phi(x, y) = e^{xy^2} - x^2y = C$, where $C \in \mathbb{R} isaconstant.$

(b) $M = 2xy^2 + 2x\sin x, \quad N = x^2y,$
$M_y = 4xy, \quad N_x = 2xy, \quad M_y \neq N_x$, not exact.
We see that $\dfrac{M_y - N_x}{N} = \dfrac{2xy}{x^2y} = \dfrac{2}{x}$, depends only on x,
$\Longrightarrow$ there exists $\mu(x)$ s.t. $\mu'(x) = \dfrac{2}{x}\mu(x)$
$\Longrightarrow \mu(x) = e^{2\int \frac{dx}{x}} = e^{2\ln|x|} = x^2.$

2. (24 points)

(a) Find a *fundamental set of solutions* to the following equation on $\mathbb{R}$ (justify the solutions found form a fundamental set of solutions):

$$y'' - 2y' + y = 0.$$

(b) Find a particular solution to the following equation by using the *Method of Undetermined Coefficients*:

$$y'' - 2y' + y = te^t.$$

(c) Find a particular solution to following nonhomogeneous equation by using the method *Variation of Parameters*:

$$y'' - 2y' + y = te^{2t}.$$

Solution:

(a) Characteristic function: $r^2 - 2r + 1 = 0, \ \ r_1 = r_2 = 1$, then $y_1 = e^t, \ y_2 = te^t$.

$$W(y_1, y_2)(t) = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = e^{2t} \neq 0, \ \forall\, t \in \mathbb{R}$$

$\Longrightarrow \{e^t, te^t\}$ a fundamental set of solutions.

(b) $g(t) = te^t$, $\alpha = 1$ is a root of (CE), $s = 2$.
$Y(t) = t^2(At + B)e^t = (At^3 + Bt^2)e^t$,
$Y'(t) = (At^3 + (3A + B)t^2 + 2Bt)e^t$,
$Y''(t) = [At^3 + (6A + B)t^2 + (6A + 4B)t + 2B]e^t$.
$\Longrightarrow Y'' - 2Y + Y = (6At + 2B)e^t = te^t$, hence $A = \dfrac{1}{6}, \ \ B = 0$.
$Y(t) = \dfrac{t^3}{6}e^t$

(c)

$$W(t) = \begin{vmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{vmatrix} = e^{2t}$$

$$W_1(t) = \begin{vmatrix} 0 & te^t \\ 1 & (t+1)e^t \end{vmatrix} = -te^t$$

$$W_2(t) = \begin{vmatrix} e^t & 0 \\ e^t & 1 \end{vmatrix} = e^t$$

$$
\begin{aligned}
Y(t) =& y_1(t) \int \frac{W_1(t)}{W(t)} g(t) dt + y_2(t) \int \frac{W_2(t)}{W(t)} g(t) dt \\
=& e^t \int \frac{-te^t}{e^{2t}} te^{2t} dt + te^t \int \frac{e^t}{e^{2t}} te^{2t} dt \\
=& -e^t \int t^2 e^t dt + te^t \int te^t dt \\
=& (t-2)e^{2t}
\end{aligned}
$$

3. (16 points)

Given a nonzero solution $y_1 = t$ to the equation

$$t^2y'' - ty' + y = 0.$$

(a) Use *reduction of order* to find another nonzero solution y_2, such that $\{y_1, y_2\}$ is a *fundamental set of solutions* of the above equation. (Justify why the set $\{y_1, y_2\}$ obtained is a fundamental set of solutions.)

(b) Use *reduction of order* to find the general solution to the following nonhomogeneous equations, by making use of the above given nonzero solution y_1 to the corresponding homogeneous equation.

$$t^2y'' - ty' + y = t^2, \quad t > 0.$$

Solution:

(a) $y_2 = u(t)y_1(t) = tu,\ y_2' = u + tu',\ y_2'' = 2u' + tu''.$

$t^2y_2'' - ty_2' + y_2 = t^2(2u' + tu'') - t(u + tu') + tu = t^2u' + t^3u'' = 0$

$\Longrightarrow u'' + \dfrac{1}{t}u' = 0$

$\Longrightarrow u' = e^{-\int \frac{1}{t}dt} = e^{-\ln|t|} = \dfrac{1}{|t|} = \dfrac{1}{t},\ t > 0.$

$\Longrightarrow u = \int \dfrac{dt}{t} = \ln t,$

$\Longrightarrow y_2 = tu = t\ln t,\ t > 0.$

(b) $y = y_1u = tu,$

$t^2(2u' + tu'') - t(tu' + u) + tu = t^2 \implies u'' + \dfrac{u'}{t} = \dfrac{1}{t},$

$u' = e^{-\int \frac{dt}{t}}(\int e^{\int \frac{dt}{t}} \dfrac{1}{t}dt + c_1) = \dfrac{1}{t}(t + c_1) = 1 + \dfrac{c_1}{t},\ c_1 \in \mathbb{R}.$

$u = \int (1 + \dfrac{c_1}{t})dt + c_2 = t + c_1\ln t + c_2,\ c_1,\ c_2 \in \mathbb{R}.$

$\Longrightarrow y = y_1u = t^2 + c_1t\ln t + c_2t,\ c_1,\ c_2 \in \mathbb{R}.$

4. (30 points)

(a) Find a *fundamental set of solutions* to the following equation:

$$y''' - 3y'' + 4y' - 2y = 0.$$

(Hint: you may use the factorization $r^3 - 3r^2 + 4r - 2 = (r-1)(r^2 - 2r + 2)$.)

(b) Use the *Method of Undetermined Coefficients* to find a particular solution to the following equation:

$$y''' - 3y'' + 4y' - 2y = te^{2t}.$$

(c) Find a particular solution to the following equation by using *Variation of Parameters*:

$$y''' - 3y'' + 4y' - 2y = \frac{e^t}{\cos t}, \quad t \in (-\frac{\pi}{2}, \frac{\pi}{2}).$$

(Hint: you may use $\int \frac{dt}{\cos t} = \frac{1}{2} \ln \left| \frac{1+\sin t}{1-\sin t} \right| + C$.)

Solution:

(a) (CE) $r^3 - 3r^2 + 4r - 2 = (r-1)(r^2 - 2r + 2) = 0$,
$r_1 = 1,\ r_2 = 1 + i,\ r_3 = 1 - i \implies y_1 = e^t, y_2 = e^t \cos t, y_3 = e^t \sin t$.
Hence $\{e^t,\ e^t \cos t,\ e^t \sin t\}$ forms a fundamental set of solutions.

(b) $g(t) = te^t$, $\alpha = 2$ is not a root, $s = 0$.
$Y(t) = (At + B)e^{2t}$; $Y'(t) = (2At + A + 2B)e^{2t}$
$Y''(t) = (4At + 4A + 4B)e^{2t}$; $Y'''(t) = (8At + 12A + 8B)e^{2t}$.
$Y''' - 3Y'' + 4Y' - 2Y = (2At + 4A + 2B)e^{2t} = te^{2t}$,
hence $2A = 1,\ 4A + 2B = 0 \implies A = \frac{1}{2},\ B = -1$.
$\implies Y(t) = (\frac{t}{2} - 1)e^{2t}$.

(c)

$$\begin{aligned}
W(t) = &\begin{vmatrix} e^t & e^t \cos t & e^t \sin t \\ e^t & e^t(\cos t - \sin t) & e^t(\sin t + \cos t) \\ e^t & -2e^t \sin t & 2e^t \cos t \end{vmatrix} \\
=&e^{3t} \begin{vmatrix} 1 & \cos t & \sin t \\ 1 & \cos t - \sin t & \sin t + \cos t \\ 1 & -2\sin t & 2\cos t \end{vmatrix} \\
=&e^{3t} \begin{vmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -2\sin t - \cos t & 2\cos t - \sin t \end{vmatrix} \\
=&e^{3t}(\sin t(\sin t - 2\cos t) + \cos t(2\sin t + \cos t)) = e^{3t}.
\end{aligned}$$

$$W_1(t) = \begin{vmatrix} 0 & e^t \cos t & e^t \sin t \\ 0 & e^t(\cos t - \sin t) & e^t(\sin t + \cos t) \\ 1 & -2e^t \sin t 2e^t \cos t & \end{vmatrix} = e^{2t}$$

$$\mathrm{W}_2(t) = \begin{vmatrix} e^t & 0 & e^t \sin t \\ e^t & 0 & e^t(\sin t + \cos t) \\ e^t & 1 & 2e^t \cos t \end{vmatrix} = -\cos t e^{2t}$$

$$\mathrm{W}_3(t) = \begin{vmatrix} e^t & e^t \cos t & 0 \\ e^t & e^t(\cos t - \sin t) & 0 \\ e^t & -2e^t \sin t & 1 \end{vmatrix} = -\sin t e^{2t}$$

Hence,

$$\begin{aligned} Y(t) =& y_1 \int \frac{W_1(s)}{W(s)} g(s)ds + y_2 \int \frac{W_2(s)}{W(s)} g(s)ds + y_3 \int \frac{W_3(s)}{W(s)} g(s)ds \\ =& \frac{1}{2} e^t \ln |\frac{1 + \sin t}{1 - \sin t}| + e^t \sin t \ln(\cos x) - te^t \cos t. \end{aligned}$$

5. (14 points)

Consider the following third order linear equation

$$y''' + p(t)y'' + q(t)y' + r(t)y = 0, \quad t \in \mathbb{R},$$

where $p(t), q(t), r(t)$ are given continuous functions on $\mathbb{R}$.

(a) Can the function $y = t^2 \sin t$ be a solution to the above equation on the whole real line $\mathbb{R}$? If yes, construct such $p(t), q(t), r(t)$. If no, explain why.

(b) Can the set $\{t, t^2, \sin t\}$ be a fundamental set of solutions to the above equation on the whole real line $\mathbb{R}$? If yes, construct such $p(t), q(t), r(t)$. If no, explain why.

bf Solution:

(a) No.

$$y(0) = 0;\ y'(t) = 2t\sin t + t^2 \cos t,\ then y'(0) = 0; y''(t) = 2\sin t + 4t\cos t - t^2 \sin t,\ then y''(0) =$$

$y \equiv 0$ is the unique solution to the problem

$$\begin{cases} y''' + p(t)y'' + q(t)y' + r(t)y = 0, \\ y(0) = y'(0) = y''(0) = 0 \end{cases}$$

(b)

$$W(t) = \begin{vmatrix} t & t^2 & \sin t \\ 1 & 2t & \cos t \\ 0 & 2 & -\sin t \end{vmatrix}, W(0) = 0.$$