4.2 Homogeneous Equations with Constant Coefficients

Consider

$$y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = 0, \qquad \text{(ODE)}$$

with $a_1, a_2, \cdots, a_n \in \mathbb{R}$.

Assume $y = e^{rt}$, then

$$(r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n) e^{rt} = 0,$$

and thus

$$r^n + a_1 r^{n-1} + \cdots + a_{n-1} r + a_n = 0. \qquad \text{(CE)}$$

Characteristic Equations

Roots: $r_1, r_2, \cdots, r_k$ (real or complex)

Multiplicites: $s_1, s_2, \cdots, s_k$

$$s_1 + s_2 + \cdots + s_k = n$$

$\Rightarrow$ Equivalent Characteristic Equation

$$(r-r_1)^{s_1}(r-r_2)^{s_2} \cdots (r-r_k)^{s_k} = 0. \qquad \text{(CE)}'$$

$\Rightarrow$ Equivalent ODE

$$\left(\frac{d}{dt}-r_1\right)^{s_1}\left(\frac{d}{dt}-r_2\right)^{s_2}\cdots\left(\frac{d}{dt}-r_k\right)^{s_k}y=0. \qquad (ODE)'$$

<u>Observation</u> If y satisfies

$$\left(\frac{d}{dt}-r_i\right)^{s_i}y=0, \qquad (ODE)_i$$

then y satisfies $(ODE)'$ and thus (ODE).

<u>Expect</u> s_i solutions of $(ODE)_i$, and thus

$$s_1+s_2+\cdots+s_k=n$$

Solutions of $(ODE)'$ and of (ODE).

- Solutions of $\left(\frac{d}{dt}-r_i\right)^{s_i}y=0$.

$$e^{-r_it}\left\{\left(\frac{d}{dt}-r_i\right)\varphi=0\right\}$$

$\Rightarrow$

$$\frac{d}{dt}\left(e^{-r_it}\varphi\right)=0$$

$$e^{-r_it}\left\{\left(\frac{d}{dt}-r_i\right)\underbrace{\left(\frac{d}{dt}-r_i\right)\varphi}_{\tilde{\varphi}}=0\right\}$$

$\Rightarrow$

$$\frac{d}{dt}\left(e^{-r_i t}\tilde{\varphi}\right)=0,$$

i.e.

$$\frac{d}{dt}\left[e^{-r_i t}\left(\frac{d}{dt}-r_i\right)\varphi\right]=0$$

$\Rightarrow$

$$\frac{d}{dt}\left(\frac{d}{dt}\left(e^{-r_i t}\varphi\right)\right)=0$$

i.e.

$$\frac{d^2}{dt^2}\left(e^{-r_i t}\varphi\right)=0.$$

$\cdots$

$$e^{-r_i t}\left\{\left(\frac{d}{dt}-r_i\right)^{s_i}y=0\right\}$$

$\Rightarrow$

$$\frac{d^{s_i}}{dt^{s_i}}\left(e^{-r_i t}y\right)=0$$

$\Rightarrow$

i.e. $e^{-r_i t}y=c_1+c_2t+\cdots+c_{s_i-1}t^{s_i-1}$

$$y=e^{r_i t}\left(c_1+c_2t+\cdots+c_{s_i-1}t^{s_i-1}\right),\quad c_1,c_2,\cdots,c_{s_i-1}\in\mathbb{R}$$

Conclusion

$$\left(\frac{d}{dt}-r_i\right)^{s_i} y=0$$

has solutions

$$e^{r_i t}\{1, t, \cdots, t^{s_i-1}\} \qquad (s_i \text{ solutions})$$

Remark • If r_i is a complex number, then $e^{r_i t}\{1, t, \cdots, t^{s_i-1}\}$ are complex valued functions.

• If r_i is complex, then $\overline{r_i}$, the complex conjugate of r_i is also a root of the (CE) of multiplicity s_i. In this case, we can take the real and imaginary parts of the functions

$$e^{r_i t}\{1, t, \cdots, t^{s_i-1}\}$$

to get $2s_i$ real-valued solutions.

Solutions of (ODE)

root	Solutions
real r_i, with multiplicity s_i	$e^{r_i t}\{1, t, \dots, t^{s_i-1}\}$ s_i solutions
Complex r_i with multiplicity s_i, $r_i=\lambda+i\mu$, $\mu\neq 0$	$e^{\lambda t}\cos\mu t\,\{1, t, \dots, t^{s_i-1}\}$ $e^{\lambda t}\sin\mu t\,\{1, t, \dots, t^{s_i-1}\}$

$2s_i$ solutions related to r_i and $\bar{r}_i$

All the above solutions, total n solutions, form a fundamental set of solutions of (ODE) on $\mathbb{R}$.

Ex1 Find a Fundamental Set of Solutions

(a) $y^{(4)}-y=0$, on $\mathbb{R}$

(b) $y^{(4)}+2y''+y=0$ on $\mathbb{R}$

(c) $y^{(4)}+y^{(3)}-7y''-y'+6y=0$, on $\mathbb{R}$

Sol. (a) Characteristic equation

$$r^4 - 1 = 0$$

Roots $r_1 = 1$, $r_2 = -1$, $r_3 = i$, $r_4 = -i$.

Solutions

$y_1 = e^{t}$ (related to r_1)

$y_2 = e^{-t}$ (related to r_2)

$y_3 = \cos t$, $y_4 = \sin t$ (related to r_3, r_4)

$\Longrightarrow$ A FSS $\{e^{t}, e^{-t}, \cos t, \sin t\}$.

(b) Characteristic equation

$r^4 + 2r^2 + 1 = 0$, i.e. $(r^2+1)^2 = 0$.

Note that $r^2 + 1 = (r+i)(r-i)$. $\Longrightarrow$

$$(r+i)^2 (r-i)^2 = 0$$

Roots $r_1 = i$, $r_2 = -i$. Multiplicities are both 2.

Solutions (related to r_1, r_2)

$\cos t$, $t\cos t$

$\sin t$, $t\sin t$

$\Rightarrow$ A FSS

$\{\cos t, \sin t, t\cos t, t\sin t\}$.

(c) Characteristic equations

$r^4+r^3-7r^2-r+6=0$

$r^3(r+1)+(-7r+6)(r+1)=0$

$(r+1)(r^3-7r+6)=0$

$(r+1)(r-1)(r+3)(r-2)=0$

$\Rightarrow$ $r=-3,-1,1,2$.

$\Rightarrow$ FSS $\{e^{-3t}, e^{-t}, e^{t}, e^{2t}\}$. #

4.3 Method of Undetermined Coefficients

Method of undetermined coefficients also applies to nth order nonhomogeneous linear ODE

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \cdots + a_{n-1} y' + a_n y = g(t),$$

when $g(t)$ is the sums or products of

$e^{\alpha t}$, $\cos(\beta t)$, $\sin(\beta t)$, $P_m(t)$ (polynomial).

<u>Note</u>. Compared with the second order case, the only difference is that the multiplicity s may be > 2.

Method of Undetermined Coefficients

<u>Case I</u> $g(t) = P_n(t) e^{\alpha t}$, $\alpha \in \mathbb{R}$.

$Y(t) = t^s e^{\alpha t} Q_n(t)$ ← expect solution

$s :=$ the multiplicity of α as the root of the characteristic equation

(in particular, if α is not a root, then s

<u>Case II</u> $(\beta \neq 0)$. $g(t) = P_n(t)\cos(\beta t)e^{\alpha t}$ or $P_n(t)\sin(\beta t)e^{\alpha t}$

$$Y(t) = t^s\left(Q_n(t)\cos(\beta t) + R_n(t)\sin(\beta t)\right)e^{\alpha t}$$

$s :=$ the multiplicity of $\alpha + i\beta$ as a root of the characteristic equation

(In particular, if $\alpha + i\beta$ is not a root, then $s = 0$)

<u>Remark</u> If $g(t)$ is the linear combination of some g_i's belonging to either Case I or Case II, we just need do the decomposition to g:

$$g = g_1 + \cdots + g_k$$
$$\downarrow \qquad \downarrow$$
$$Y = Y_1 + \cdots + Y_k$$

<u>Ex 1</u>. Find a particular solution of

$$y^{(4)} + 2y^{(2)} + y = 4\cos t - \sin t.$$

Sol. Characteristic Eqn of the Homogeneous Eqn

$r^4+2r^2+1=0$, $(r^2+1)^2=0$,

$(r+i)^2(r-i)^2=0$.

$\Longrightarrow$ roots multiplicity

roots	multiplicity
i	2
$-i$	2

$g(t)=4\cos t-\sin t$

$Y(t)=t^s(A\cos t+B\sin t)$

s = multiplicity of i as the root

$=2$

$\Longrightarrow Y(t)=(A\cos t+B\sin t)t^2$.

Substituting $Y(t)$ into eqn $\Longrightarrow$

$-8A\cos t-8B\sin t=4\cos t-\sin t$

$\Longrightarrow A=-\frac{1}{2}, \quad B=\frac{1}{8}$

$$\Rightarrow Y(t) = \left(-\frac{1}{2}\cos t + \frac{1}{8}\sin t\right)t^2.$$

<u>Ex2</u>. Find a particular solution to

$$y''' - 9y' = t^2 + 3\sin t + e^{3t}.$$

<u>Sol</u>. Characteristic Eqn of the Homogeneous Eqn

$$r^3 - 9r = 0, \quad r(r-3)(r+3) = 0.$$

roots: $0, 3, -3$.

$$g(t) = t^2 + 3\sin t + e^{3t}.$$

$$\begin{aligned} Y(t) = {} & t^{s_1} Q_2(t) \\ & + t^{s_2}(B_1\cos t + B_2 \sin t) \\ & + t^{s_3} C e^{3t} \end{aligned}$$

s_1 = multiplicity of $0 = 1$,

s_2 = multiplicity of $i = 0$,

s_3 = multiplicity of $3 = 1$

$$\Rightarrow Y(t) = t\left(A_2 t^2 + A_1 t + A_0\right)$$
$$+ \left(B_1 \cos t + B_2 \sin t\right)$$
$$+ C t e^{3t}$$

Substitute $Y(t)$ into the eqn $\Rightarrow$ the solution

Ex 3 Find the general solution

$$y^{(4)} - y = t \sin t$$

Sol. We need:

(i) find the Fundamental Set of Solutions $\{\phi_1, \phi_2, \phi_3, \phi_4\}$ of the homogeneous eqn

(ii) find a particular solution to the nohomogeneous eqn.

(i) Fundamental Set of Solutions.

The characteristic equation is

$$r^4 - 1 = 0, \quad (r^2+1)(r-1)(r+1) = 0,$$

i.e. $(r+i)(r-i)(r-1)(r+1) = 0.$

The roots are $i, -i, 1, -1$.

$\Rightarrow$ Solutions for homogeneous eqn:

related to 1, e^t

related to -1, e^{-t}

related to $\pm i$, $\cos t$, $\sin t$

$\Rightarrow$ Fundamental Set of Solutions

$\{e^t, e^{-t}, \cos t, \sin t\}$.

(ii) A particular Solution to nonhomogeneous eqn.

$g(t) = t\sin t$

$Y(t) = t^s(P_1(t)\cos t + Q_1(t)\sin t)$,

s = multiplicity of i as the root

$= 1$

$\Rightarrow Y(t) = t[(At+B)\cos t + (Ct+D)\sin t]$.

Substituting Y into the equation

$(-8Ct - 12A - 4D)\cos t + (8At + 4B - 12C)\sin t$

$= t\sin t$

$\Rightarrow$

$$\begin{cases} 8C = 12A + 4D = 0, \\ 8A = 1 \\ 4B - 12C = 0 \end{cases} \Rightarrow \begin{cases} A = \frac{1}{8} \\ B = C = 0 \\ D = -\frac{1}{24} \end{cases}$$

$$\Rightarrow Y(t) = \frac{t^2}{8}\cos t - \frac{t}{24}\sin t$$

Therefore, the general solution is

$$y = Y(t) + c_1 e^{t} + c_2 e^{-t} + c_3 \cos t + c_4 \sin t$$

$$= \frac{1}{8}t^2 \cos t - \frac{1}{24} t \sin t + c_1 e^{t} + c_2 e^{t}$$

$$+ c_3 \cos t + c_4 \sin t, \quad c_1, c_2, c_3, c_4 \in \mathbb{R}$$

4.4 The Method of Variation of Parameters

$$y^{(n)} + p_1(t) y^{(n-1)} + \cdots + p_{n-1}(t) y' + p_n(t) y = g(t). \qquad (NH)$$

Given a Fundamental Set of Solutions

$\{y_1, y_2, \cdots, y_n\}$ of the Homogeneous Equation.

Expect a Particular Solution of (NH) of the form

$$y(t) = u_1(t) y_1 + u_2(t) y_2 + \cdots + u_n(t) y_n. \qquad (0)$$

Then

$$y' = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'$$
$$+ u_1' y_1 + u_2' y_2 + \cdots + u_n' y_n.$$

Set

$$y_1 u_1' + y_2 u_2' + \cdots + y_n u_n' = 0. \qquad (1)'$$

Then

$$y' = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n' \qquad (1)$$

Step 1

$$y'' = u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n''$$
$$+ u_1' y_1' + u_2' y_2' + \cdots + u_n' y_n'$$

Set

$$y_1' u_1' + y_2' u_2' + \cdots + y_n' u_n' = 0 \qquad (2)'$$

Then

Step2 $$y'' = u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n'' \qquad (2)$$

⋯ Continue such calculations n-1 steps, then

$$\begin{cases} y_1^{(k)} u_1' + y_2^{(k)} u_2' + \cdots + y_n^{(k)} u_n' = 0 & (k+1)' \\ y^{(k+1)} = u_1 y_1^{(k+1)} + u_2 y_2^{(k+1)} + \cdots + u_n y_n^{(k+1)} & (k+1) \end{cases}$$

$$k = 0, 1, \cdots, n-2.$$

When $k = n-2$, $(k+1) \Longrightarrow$

$$y^{(n-1)} = u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + \cdots + u_n y_n^{(n-1)}.$$

Then

$$y^{(n)} = u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)}$$
$$+ u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)}. \qquad (n)$$

$(k+1)'$, $k = 0, 1, \cdots, n-2$

$\Longrightarrow$

$$\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad (*)$$

$(n-1)$ equations! Still need one more equation to uniquely solve $u_1', \cdots, u_n'$.

$(k+1)'$, $k = 0, 1, \cdots, n-2$, and (n)

$\Longrightarrow$

$$y' = u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'$$

$$y'' = u_1 y_1'' + u_2 y_2'' + \cdots + u_n y_n''$$

$$\vdots$$

$$y^{(n-1)} = u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + \cdots + u_n y_n^{(n-1)}$$

$$y^{(n)} = u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)} + u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)}$$

Substituting ~~for~~ for y (given by (0)), y', ..., $y^{(n)}$ ~~in~~ in (NH) yields

$$g(t) = u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)}$$

$$+ u_1 y_1^{(n)} + u_2 y_2^{(n)} + \cdots + u_n y_n^{(n)}$$

$$+ p_1(t)\left(u_1 y_1^{(n-1)} + u_2 y_2^{(n-1)} + \cdots + u_n y_n^{(n-1)}\right)$$

$$+ \cdots$$

$$+ p_{n-1}(t)\left(u_1 y_1' + u_2 y_2' + \cdots + u_n y_n'\right)$$

$$+ p_n(t)\left(u_1 y_1 + u_2 y_2 + \cdots + u_n y_n\right)$$

$$= u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)}$$

$$+ u_1\left(y_1^{(n)} + p_1(t) y_1^{(n-1)} + \cdots + p_{n-1}(t) y_1' + p_n(t) y_1\right)$$

$$+ u_2\left(y_2^{(n)} + p_1(t) y_2^{(n-1)} + \cdots + p_{n-1}(t) y_2' + p_n(t) y_2\right)$$

$$+ \cdots$$

$$+ u_n\left(y_n^{(n)} + p_1(t) y_n^{(n-1)} + \cdots + p_{n-1}(t) y_n' + p_n(t) y_n\right)$$

$$= u_1' y_1^{(n-1)} + u_2' y_2^{(n-1)} + \cdots + u_n' y_n^{(n-1)}$$

$\Longrightarrow$

$$y_1^{(n-1)} u_1 + y_2^{(n-1)} u_2 + \cdots + y_n^{(n-1)} u_n = g(t) \quad (**)$$

$(*) \ \& \ (**) \Longrightarrow$

$$\underbrace{\begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}}_{A(t)} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_{n-1}' \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}$$

the m-th coelum

A_m: the matrix obtained by replacing the m-th colum of A by $(0,0,\cdots,0,1)^T$

$$A_m(t) = \begin{pmatrix} y_1 & \cdots & y_{m-1} & 0 & y_{m+1} & \cdots & y_n \\ y_1' & \cdots & y_{m-1}' & 0 & y_{m+1}' & \cdots & y_n' \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ y_1^{(n-2)} & \cdots & y_{m-1}^{(n-2)} & 0 & y_{m+1}^{(n-2)} & \cdots & y_n^{(n-2)} \\ y_1^{(n-1)} & \cdots & y_{m-1}^{(n-1)} & 1 & y_{m+1}^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

By Cramer's rule $\Longrightarrow$

$$u_m' = \frac{|A_m(t)|}{|A(t)|} g(t)$$

$$= \frac{W_m(t)}{W(t)} g(t), \quad m = 1, 2, \cdots, n.$$

$\Longrightarrow$

$$u_m(t) = \int_{t_0}^{t} \frac{W_m(s)}{W(s)} g(s)\,ds, \quad m = 1, 2, \cdots, n.$$

$\Longrightarrow$

$$y(t) = y_1(t) \int_{t_0}^{t} \frac{W_m(s)}{W(s)} g(s)\,ds + \cdots + y_n(t) \int_{t_0}^{t} \frac{W_n(s)}{W(s)} g(s)\,ds.$$

Conclusion: (Variation of Parameters)

$$Y(t) = \sum_{m=1}^{n} y_m(t) \int_{t_0}^{t} \frac{W_m(s)}{W(s)} g(s)\,ds$$

is a particular solution of (NH), where W is the Wronskian of $\{y_1, \cdots, y_n\}$ and W_m is obtained by replacing the m-th column of the matrix for W with $(0, 0, \cdots, 0, 1)^T$.

Ex Use the method of variation of parameters to determine the general solution

$$y''' - y' = 4t.$$

Sol. The homogeneous equation is

$$y''' - y' = 0.$$

The characteristic equation is

$$r^3 - r = 0, \text{ ie } r(r-1)(r+1) = 0$$

The roots are $0, 1, -1$.

$$y_1 = 1, \quad y_2 = e^t, \quad y_3 = e^{-t},$$

$\{1, e^t, e^{-t}\}$ is a fundamental set of solutions of the homogeneous equations.

$$W(t) = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 2$$

$$W_1(t) = \begin{vmatrix} 0 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 1 & e^t & e^{-t} \end{vmatrix} = -2$$

$$W_2(t) = \begin{vmatrix} 1 & 0 & e^{-t} \\ 0 & 0 & -e^{-t} \\ 0 & 1 & e^{-t} \end{vmatrix} = e^{-t}$$

$$W_3(t) = \begin{vmatrix} 1 & e^{t} & 0 \\ 0 & e^{t} & 0 \\ 0 & e^{t} & 1 \end{vmatrix} = e^{t}$$

A particular solution

$$Y(t) = y_1(t)\int \frac{W_1(t)}{W(t)} g(t)\,dt + y_2(t)\int \frac{W_2(t)}{W(t)} g(t)\,dt$$

$$+ y_3(t)\int \frac{W_3(t)}{W(t)} g(t)\,dt$$

$$= \int \frac{-2}{2} 4t\,dt + e^{t}\int \frac{e^{-t}}{2} 4t\,dt + e^{-t}\int \frac{e^{t}}{2} 4t\,dt$$

$$= -4\int t\,dt + 2e^{t}\int t e^{-t}\,dt + 2e^{-t}\int t e^{t}\,dt$$

$$= -4 - 2t^2$$

General solution

$$y = Y + c_1 y_1 + c_2 y_2 + c_3 y_3$$

$$= -2t^2 + c_1 e^{t} + c_2 e^{-t} + c_3 \quad c_1, c_2, c_3 \in \mathbb{R}.$$