MATH 3270B - Ordinary Differential Equations - 2017/18

Quiz One (May 25, 2018)

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Answer all the questions. Show your detail steps.

1. Solve all the following initial value problems:

   (a) $ty' + y = te^{-t} \qquad y(1) = 1$

   (b) $ty' + 2y = \sin t \qquad y(\pi) = \dfrac{1}{\pi}$

   Solutions:

   (a) $ty' + y = \dfrac{d(ty)}{dt} = te^{-t}$, integrate on both side we get:

   $ty = -te^{-t} - e^{-t} + C$, where C is a constant. Hence,

   $y = -e^{-t} - \dfrac{1}{t}e^{-t} + \dfrac{C}{t}$. From $y(1) = 1$ we know:

   $C = 1 + \dfrac{2}{e}$. Hence $y = -e^{-t} - \dfrac{1}{t}e^{-t} + \dfrac{1 + \frac{2}{e}}{t}$

   (b) Multiply the equation by t on both sides, we have:

   $t^2y' + 2ty = \frac{d(t^2y)}{dt} = t\sin t$. Integrate the equation:

   $t^2y = -t\cos t + \sin t + C$, where C is a constant. Hence,

   $y = -\dfrac{1}{t}\cos t + \dfrac{1}{t^2}\sin t + \dfrac{C}{t^2}$. From $y(\pi) = \dfrac{1}{\pi}$ we know:

   $C = 0$. Hence $y = -\dfrac{1}{t}\cos t + \dfrac{1}{t^2}\sin t$.

2. Find the solutions to the following initial value problem with $y_0 = \dfrac{1}{2}$ and $y_0 = \dfrac{e}{e-1}$, respectively, and study the limitations of the solutions as $t \to +\infty$:

$$y' = (1 - y)y, \qquad y(0) = y_0.$$

Solutions: Let $y = y(t)$, $y' = \dfrac{dy}{dt}$. Hence

$\dfrac{1}{(1-y)y} dy = 1dt$. Integrate on both sides we have:

$-\ln|1-y| + \ln|y| = t + C,\ i.e.\ \ln|\dfrac{y}{1-y}| = t + C$, where C is a constant.

Hence, $y = \dfrac{e^{t+C}}{1+e^{t+C}} = \dfrac{e^t}{C+e^t}$, where C is a constant.

(a) If $y_0 = \dfrac{1}{2}$, $C = 1$, and $y = \dfrac{e^t}{1+e^t}$;

(b) if $y_0 = \dfrac{e}{e-1}$, $C = -\dfrac{1}{e}$, and $y = \dfrac{e^{t+1}}{e^{t+1}-1}$.

As $t \to \infty$, $y \to 1$.

3. Find the general solutions to the ODEs:

(a) $y' = y + \frac{t}{y^2}$

(b) $t^2 y' = y^2 + 3ty + t^2$

Solutions:

(a) We first multiply $3y^2$ to both sides: $3y^2 \dfrac{dy}{dt} = 3y^3 + 3t$.

Then add 1 on both sides: $3y^2 \dfrac{dy}{dt} + 1 = 3(y^3 + t) + 1$.

Let $\nu(t) = y^3 + t$, then we see that

$$\nu' = 3\nu + 1$$

. This equation is separable: $\dfrac{1}{3\nu + 1} d\nu = 1dt$.

Hence $\nu = \dfrac{Ce^t - 1}{3} = y^3 + 1$, and $y = \sqrt[3]{\dfrac{Ce^t - 1}{3}} - t$, where C is a constant.

(b) Divide on both sides by t^2, we see that

$$y' = (\frac{y}{t})^2 + 3(\frac{y}{t}) + 1$$

This is a homogeneous equation. Take $\nu = \dfrac{y}{t}$, then $\dfrac{dy}{dt} = \nu + t\nu'$.

Hence $\nu + t\nu' = \nu^2 + 3\nu + 1$. This is a separable equation:

$$\frac{1}{(\nu+1)^2} d\nu = \frac{1}{t} dt, \quad (\nu \neq -1)$$

. Integrate on both sides, we have $\nu = \dfrac{-1}{\ln|t| + C} - 1$, and $y = t\nu = \dfrac{-t}{\ln|t| + C} - t$, where C is a constant.

When $\nu = -1$, it is easy to check that $y = -t$ is also a solution.

4. (a) Determine whether the following equation is exact. If it is exact, then find the general solution.

$$\left(x - \frac{y}{x^2 + y^2}\right) dx + \left(y + \frac{x}{x^2 + y^2}\right) dy = 0.$$

(b) Show that the following equation is NOT exact and find an integrating factor (nonzero) μ such that the equation is exact when multiplied by μ.

$$ydx + (2x + ye^y + 2e^y)dy = 0.$$

Solutions:

(a) $M = x - \dfrac{y}{x^2 + y^2}, \quad M_y = \dfrac{y^2 - x^2}{(x^2 + y^2)^2},$
$N = y + \dfrac{x}{x^2 + y^2}, \quad N_x = \dfrac{y^2 - x^2}{(x^2 + y^2)^2}.$
Hence we see that $M_y = N_x$, this equation is exact.
$\Phi_x = M \implies \Phi = \dfrac{1}{2}x^2 - \arctan\dfrac{x}{y} + g(y);$
$\Phi_y = N \implies \dfrac{x}{x^2 + y^2} + g'(y) = y + \dfrac{x}{x^2 + y^2},$
$\implies g(y) = \dfrac{1}{2}y^2.$
Hence the solution is $\Phi(x, y) = \dfrac{1}{2}x^2 + \dfrac{1}{2}y^2 - \arctan\dfrac{x}{y} = C$, where $C \in \mathbb{R}$ is a constant.

(b) $M_y = 1, \quad N_x = 2$, $M_y \neq N_x$, this equation is not exact.
To find an integrating factor, note that $\dfrac{N_x - M_y}{M} = \dfrac{1}{y}$, only depends on y.
Then we can find $\mu(y)$ such that $\mu(y)' = \dfrac{1}{y}\mu(y)$. Take $\mu(y) = y$ is an integrating factor.

5. Find all the solutions to the following initial value problems:

   (a) $y' = ye^{\sin y} \cos t, \qquad y(0) = 0.$

   (b) $y' = y^3 t, \qquad y(0) = 0.$

   Solutions:

   (a) We first note that $y \equiv 0$ is a solution and satisfies initial condition.

   Let $f(y,t) = ye^{\sin y} \cos t$, we have $f(y,t)$ and $\frac{\partial f}{\partial y} = e^{\sin y} \cos t + y \cos(y) e^{\sin y} \cos t$ are continuous on $\mathbb{R}^2$.

   By uniqueness asserted in Thm 2.4.2, $y \equiv 0$ is only solution to the IVP.

   (b) We first note that $y \equiv 0$ is a solution and satisfies initial condition.

   The equation is separable: $\frac{1}{y^3} dy = t dt$,

   integrate on both sides we have: $-\frac{1}{2} y^{-2} = \frac{1}{2} t^2 + C$.

   Hence $y = \pm \frac{1}{\sqrt{-t^2 + C}}$, where C is a constant.

   However no $C \in \mathbb{R}$ satisfies the initial condition, we conclude that $y \equiv 0$ is only solution to the IVP.

6. Is the solution to the following initial value problem unique?

$$y' = y^{\frac{2}{3}}, \qquad y(0) = 0.$$

   If yes, then give the proof; otherwise, find more than one solutions.

   Solutions: Not unique.

   First observe that $y \equiv 0$ is a solution to the IVP.

   The equation is separable, $y^{-2/3} dy = 1 dt$,

   integrate on both sides: $3y^{1/3} = t + C \implies y = (\frac{t+C}{3})^3$, C is a constant.

   $y(0) = 0$ gives $C = 0$, hence $y = \frac{t^3}{27}$ is also a solution to the IVP.

7. Find the general solutions to the following ODEs:

(a) $y'' + 3y' + 2y = 0$

(b) $y'' + 2y' + 3y = 0$

(c) $y'' + 2\sqrt{3}y' + 3 = 0$

Solutions:

(a) The characteristic equation is $r^2 + 3r + 2 = 0$,

$\implies (r+1)(r+2) = 0,\ r_1 = -1,\ r_2 = -2.$

$\implies y_1 = e^{-t},\ y - 2 = e^{-2t},$

$\implies$ general solution is: $y = C_1 e^{-t} + C_2 e^{-2t},\quad C_1, C_2 \in \mathbb{R}.$

(b) The characteristic equation is $r^2 + 2r + 3 = 0$,

$\implies (r+1)^2 = -2,\quad r_1 = -1 + \sqrt{2}i,\ r_2 = -1 - \sqrt{2}i.$

$\implies y_1 = e^{-t}\cos(\sqrt{2}t),\ y_2 = e^{-t}\sin(\sqrt{2}t).$

Hence general solution is $y = e^{-t}(C_1\cos(\sqrt{2}t) + C_2\sin(\sqrt{2}t)),\quad C_1, C_2 \in \mathbb{R}.$

(c) (Both give full marks)

It you solve $y'' + 2\sqrt{3}y' + 3y = 0$:

The characteristic equation is $r^2 + 2\sqrt{3}r + 3 = 0$,

$\implies (r + \sqrt{3})^2 = 0,\quad r_1 = r_2 = -\sqrt{3}.$

Hence $y_1 = e^{-\sqrt{3}t},\ y_2 = te^{-\sqrt{3}t},$

the general solution is $y = e^{-\sqrt{3}t}(C_1 + C_2 t),\quad C_1, C_2 \in \mathbb{R}.$

If you solve $y'' + 2\sqrt{3}y' + 3 = 0$:

first we see that $Y(t) = -\frac{\sqrt{3}}{2}t$ is a particular solution.

The general solution is $c_1 e^{-2\sqrt{3}t} + c_2 - \frac{\sqrt{3}}{2}t$, where $c_1,\ c_2 \in \mathbb{R}$.