TUT 8: NON-HOMOGENEOUS LINEAR SYSTEM

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1. Use the method of undetermined coefficients to find a particular solution of the following non-homogeneous system

$$\vec{x}' = \mathbb{A}\vec{x} + \vec{H}(t)$$

with

(a)

$$\mathbb{A} = \begin{pmatrix} 1 & 0 \\ 3 & -1 \end{pmatrix}, \quad \vec{H}(t) := \vec{H}^{(1)}(t) + \vec{H}^{(2)}(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

(b)

$$\mathbb{A} = \begin{pmatrix} 4 & -3 \\ 3 & -2 \end{pmatrix}, \quad \vec{H}(t) := \vec{H}^{(1)}(t) + \vec{H}^{(2)}(t) = \begin{pmatrix} e^t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ t \end{pmatrix}.$$

(c)

$$\mathbb{A} = \begin{pmatrix} 2 & -5 \\ 1 & -2 \end{pmatrix}, \quad \vec{H}(t) := \vec{H}^{(1)}(t) + \vec{H}^{(2)}(t) = -\begin{pmatrix} \cos 2t \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ \sin t \end{pmatrix}.$$

Solution. (a) It is clear that the eigenvalues of the coefficient matrix $\mathbb{A}$ are $\lambda = 1$ and -1. Let $\vec{X}(t) = \vec{X}^{(1)}(t) + \vec{X}^{(2)}(t)$ be a particular solution with

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\vec{X}^{(1)}(t) = \mathbb{A}\vec{x}(t) + \vec{H}^{(1)}; \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\vec{X}^{(2)}(t) = \mathbb{A}\vec{x}(t) + \vec{H}^{(2)}. \end{cases}$$

Since $\lambda = 1$ is a simple eigenvalue, take $\vec{X}^{(1)}(t) = \left(\vec{a}t + \vec{b}\right) e^t$, then the first equation implies

$$\left(\vec{a}t + \vec{b}\right) + \vec{a} = \mathbb{A}\left(\vec{a}t + \vec{b}\right) + (1,0)^T.$$

This identity can be rewritten as

$$(\mathbb{A} - \mathbb{I})\vec{a}\,t + (\mathbb{A} - \mathbb{I})\vec{b} - \vec{a} + (1,0)^T = 0,$$

which holds true for any t if

$$(\mathbb{A} - \mathbb{I})\vec{a} = 0, \quad (\mathbb{A} - \mathbb{I})\vec{b} = \vec{a} - (1,0)^T \quad \text{with} \quad \mathbb{A} - \mathbb{I} = \begin{pmatrix} 0 & 0 \\ 3 & -2 \end{pmatrix}.$$

Furthermore these equations can be reduced to

$$3a_1 - 2a_2 = 0, \quad a_1 - 1 = 0, \quad 3b_1 - 2b_2 = a_2.$$

Take $b_1 = 1$, then $a_1 = 1, a_2 = 3/2, b_2 = 3/4$ and

$$\vec{X}^{(1)}(t) = \left[\begin{pmatrix} 1 \\ 3/2 \end{pmatrix} t + \begin{pmatrix} 1 \\ 3/4 \end{pmatrix}\right] e^t.$$

Note that 0 is not an eigenvalue of $\mathbb{A}$, we take $\vec{X}^{(2)}(t) = \vec{u}\,t + \vec{v}$, then

$$\vec{u} = \mathbb{A}(\vec{u}\,t + \vec{v}) + t\,(0,1)^T.$$

This identity holds for all t only if

$$(\mathbb{A} - \mathbb{I})\vec{u} = -(0,1)^T \quad \text{and} \quad \mathbb{A}\vec{v} = \vec{u}.$$

Taking $u_2 = -1$ leads to $u_1 = v_1 = 1, v_2 = 4$ and then

$$\vec{X}^{(2)}(t) = \begin{pmatrix} 1 \\ -1 \end{pmatrix} t + \begin{pmatrix} 1 \\ 4 \end{pmatrix}.$$

In conclusion, a particular solution can be $\cdots$.

(b) Let $\vec{X}(t) = \vec{X}^{(1)}(t) + \vec{X}^{(2)}(t)$ be a particular solution with

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\vec{X}^{(1)}(t) = \mathbb{A}\vec{x}(t) + \vec{H}^{(1)}; \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\vec{X}^{(2)}(t) = \mathbb{A}\vec{x}(t) + \vec{H}^{(2)}. \end{cases}$$

Since

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & -2-\lambda \end{vmatrix} = (\lambda - 1)^2,$$

$\lambda = 1$ is an eigenvalue of $\mathbb{A}$ with algebraic multiplicity 2. Take $\vec{X}^{(1)}(t) = \left(\vec{a}t^2 + \vec{b}t + \vec{c}\right)e^t$, then

$$\vec{a}t^2 + (2\vec{a} + \vec{b})t + \vec{b} + \vec{c} = \mathbb{A}\vec{a}t^2 + \mathbb{A}\vec{b}t + \mathbb{A}\vec{c} + e^t(1,0)^T,$$

which implies

$$\begin{cases} (\mathbb{A} - \mathbb{I})\vec{a} = 0, \\ (\mathbb{A} - \mathbb{I})\vec{b} = 2\vec{a}, \\ (\mathbb{A} - \mathbb{I})\vec{c} = \vec{b} - (1,0)^T. \end{cases}$$

Note that

$$\mathbb{A} - \mathbb{I} = \begin{pmatrix} 3 & -3 \\ 3 & -3 \end{pmatrix}.$$

Therefore

$$a_1 - a_2 = 0, \quad 3(b_1 - b_2) = 2a_1 = 2a_2, \quad 3(c_1 - c_2) = b_1 - 1 = b_2.$$

Then $b_1 - b_2 = 1$ and $a_1 = a_2 = 3/2$. Taking $b_1 = 1, c_1 = 1$, then $b_2 = 0, c_2 = 1$ and

$$\vec{X}^{(1)}(t) = \left[\begin{pmatrix} 3/2 \\ 3/2 \end{pmatrix} t^2 + \begin{pmatrix} 1 \\ 0 \end{pmatrix} t + \begin{pmatrix} 1 \\ 1 \end{pmatrix}\right] e^t.$$

Now take $\vec{X}^{(2)}(t) = \vec{u}t + \vec{v}$, then

$$\vec{u} = \mathbb{A}(\vec{u}t + \vec{v}) + t(0,1)^T.$$

This yields

$$\mathbb{A}\vec{u} = -(0,1)^T, \quad \mathbb{A}\vec{v} = \vec{u}.$$

Solving directly gets

$$\vec{u} = \begin{pmatrix} -3 \\ -4 \end{pmatrix}, \quad \vec{v} = \begin{pmatrix} -6 \\ -7 \end{pmatrix} \quad \text{and then} \quad \vec{X}^{(2)}(t) = \begin{pmatrix} -3 \\ -4 \end{pmatrix} t + \begin{pmatrix} -6 \\ -7 \end{pmatrix}.$$

(c) The eigenvalues are given by

$$\begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = \lambda^2 + 1 = 0, \quad \lambda = \pm i.$$

Let $\vec{X}(t) = \vec{X}^{(1)}(t) + \vec{X}^{(2)}(t)$ be a particular solution with

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\vec{X}^{(1)}(t) = \mathbb{A}\vec{x}(t) + \vec{H}^{(1)}, \quad \vec{H}^{(1)}(t) = -(\cos 2t, 0)^T; \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\vec{X}^{(2)}(t) = \mathbb{A}\vec{x}(t) + \vec{H}^{(2)}, \quad \vec{H}^{(2)}(t) = (0, \sin t)^T. \end{cases}$$

Firstly, take $\vec{X}^{(1)}(t) = \vec{a}\cos 2t + \vec{b}\sin 2t$, then $\frac{\mathrm{d}}{\mathrm{d}t}\vec{X}^{(1)}(t) = -2\vec{a}\sin 2t + 2\vec{b}\cos 2t$ and

$$-2\vec{a}\sin 2t + 2\vec{b}\cos 2t = \mathbb{A}(\vec{a}\cos 2t + \vec{b}\sin 2t) - \cos 2t\,(1,0)^T.$$

Rewriting the above identity leads to

$$\Big[\mathbb{A}\vec{a} - 2\vec{b} - (1,0)^T\Big]\cos 2t + \Big[\mathbb{A}\vec{b} + 2\vec{a}\Big]\sin 2t = 0,$$

which holds true for any t if

$$\mathbb{A}\vec{a} = 2\vec{b} + (1,0)^T, \quad \text{and} \quad \mathbb{A}\vec{b} = -2\vec{a}.$$

Observe that

$$\mathbb{A}^2\vec{b} = -2\mathbb{A}\vec{a} = -4\vec{b} - (2,0)^T, \quad \mathbb{A}^2 = -\mathbb{I}.$$

This gives $\vec{b} = -(2/3, 0)^T$, and then $\vec{a} = -2^{-1}\mathbb{A}\vec{b} = 2/3(1,1)^T$. Combining this with the special form chosen for $\vec{X}^{(1)}(t)$, one has

$$\vec{X}^{(1)}(t) = \frac{2}{3}\begin{pmatrix} 1 \\ 1 \end{pmatrix}\cos 2t - \frac{2}{3}\begin{pmatrix} 1 \\ 0 \end{pmatrix}\sin 2t.$$

Secondly, since i is an eigenvalue to $\mathbb{A}$, take $\vec{X}^{(2)}(t) = (\vec{u}^{(1)}t + \vec{v}^{(1)})\cos t + (\vec{u}^{(2)}t + \vec{v}^{(2)})\sin t$. Then

$$\begin{aligned} &\vec{u}^{(1)}\cos t - (\vec{u}^{(1)}t + \vec{v}^{(1)})\sin t + \vec{u}^{(2)}\sin t + (\vec{u}^{(2)}t + \vec{v}^{(2)})\cos t \\ &= \mathbb{A}\Big[(\vec{u}^{(1)}t + \vec{v}^{(1)})\cos t + (\vec{u}^{(2)}t + \vec{v}^{(2)})\sin t\Big] + (0,1)^T\sin t, \end{aligned}$$

which can be rewritten as

$$\begin{aligned} \Big[\mathbb{A}\vec{u}^{(1)} - \vec{u}^{(2)}\Big]t\cos t + \Big[\mathbb{A}\vec{u}^{(2)} + \vec{u}^{(1)}\Big]t\sin t + \Big[\mathbb{A}\vec{v}^{(1)} - \vec{u}^{(1)} - \vec{v}^{(2)}\Big]\cos t \\ + \Big[\mathbb{A}\vec{v}^{(2)} - \vec{u}^{(2)} + \vec{v}^{(1)} + (0,1)^T\Big]\sin t = 0. \end{aligned}$$

This identity holds for any t when

$$\begin{cases} \mathbb{A}\vec{u}^{(1)} = \vec{u}^{(2)}, \quad \mathbb{A}\vec{u}^{(2)} = -\vec{u}^{(1)}, \\ \mathbb{A}\vec{v}^{(1)} = \vec{u}^{(1)} + \vec{v}^{(2)}, \\ \mathbb{A}\vec{v}^{(2)} = \vec{u}^{(2)} - \vec{v}^{(1)} - (0,1)^T. \end{cases}$$

Note that $\mathbb{A}^2 = -\mathbb{I}$. Let $\mathbb{A}$ act on the third equation, using the last identity and rewriting a little bit imply

$$0 = (\mathbb{A}^2 + \mathbb{I})\vec{v}^{(1)} = \mathbb{A}\vec{u}^{(1)} + \vec{u}^{(2)} - (0,1)^T.$$

This combined with the first identity yields

$$\mathbb{A}\vec{u}^{(1)} = (0, 1/2), \quad \text{and then} \quad \vec{u}^{(1)} = (5/2, 1)^T, \quad \vec{u}^{(2)} = (0, 1/2)^T.$$

Take $\vec{v}^{(1)} = 0$, then $\vec{v}^{(2)} = -\vec{u}^{(1)} = -(5/2, 1)^T$ and

$$\vec{X}^{(2)}(t) = \begin{pmatrix} 5/2 \\ 1 \end{pmatrix} t\cos t + \left[\begin{pmatrix} 0 \\ 1/2 \end{pmatrix} t - \begin{pmatrix} 5/2 \\ 1 \end{pmatrix}\right] \sin t.$$

In conclusion, a particular solution is given by $\vec{X}(t) = \vec{X}^{(1)}(t) + \vec{X}^{(2)}(t)$ with $\vec{X}^{(1)}(t)$ and $\vec{X}^{(2)}(t)$ given above.

□

2. Use the method of variation of parameters to find a particular solution to these linear systems in the previous question.

Proof. Let's try 1(a) only. It is easy to see for matrix $\mathbb{A}$, eigenvalue $\lambda_1 = 1$ is associated with eigenvector $\vec{\xi}^{(1)} = (2,3)^T$, and $\lambda_2 = -1$ associated with eigenvector $\vec{\xi}^{(2)} = (0,1)^T$. Hence

$$\vec{x}^{(1)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix} e^t, \quad \vec{x}^{(2)} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t}$$

forms a fundamental set of solutions to the homogeneous system. Denote the corresponding fundamental matrix by

$$\mathbb{F}(t) = \begin{pmatrix} 2e^t & 0 \\ 3e^t & e^{-t} \end{pmatrix} \quad \text{and then} \quad \mathbb{F}' = \mathbb{A}\mathbb{F}.$$

Now we are going to find a particular solution in the form

$$\vec{X}(t) = \mathbb{F}(t)\vec{u}(t).$$

Then

$$\mathbb{F}'(t)\vec{u}(t) + \mathbb{F}(t)\vec{u}\,'(t) = \mathbb{A}\mathbb{F}(t)\vec{u}(t) + (e^t, t)^T.$$

Using the property of $\mathbb{F}$ and integrating yield

$$\vec{u}(t) = \int \mathbb{F}^{-1}(t) \begin{pmatrix} e^t \\ t \end{pmatrix} \mathrm{d}t + C.$$

Note that

$$\mathbb{F}^{-1}(t) = \frac{1}{2}\begin{pmatrix} e^{-t} & 0 \\ -3e^t & 2e^t \end{pmatrix}, \quad \text{and then} \quad \mathbb{F}^{-1}(t)\begin{pmatrix} e^t \\ t \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 \\ -3e^{2t} + 2te^t \end{pmatrix}.$$

Take $C = 0$, then

$$\vec{u}(t) = \int \frac{1}{2}\begin{pmatrix} 1 \\ -3e^{2t} + 2te^t \end{pmatrix} \mathrm{d}t = \begin{pmatrix} t/2 \\ -3e^{2t}/4 + te^t - e^t \end{pmatrix}.$$

In conclusion, a particular solution is

$$\vec{X}(t) = \mathbb{F}(t)\vec{u}(t) = \begin{pmatrix} te^t \\ 3te^t/2 - 3e^t/4 + t - 1 \end{pmatrix}.$$

□