MATH3720A Ordinary Differential Equations
2017 - 18
Midterm Exam (24 October) - Solutions

1. Find all solutions to the following equations. Show your calculations.
   (a) [5 pts] $(te^{ty} - 6y)y' = 2t - ye^{ty}$.
   (b) [5 pts] $y^{(3)} - 7y'' + 12y' = 0$.
   (c) [5 pts] $y' + \sin(t)y^{10} = 0$.
   (d) [5 pts] $y''y' + (y')^2 = 0$.

Solutions.

(a) This ODE can be written in terms of

$$(ye^{ty} - 2t) + (te^{ty} - 6y)y' = 0 \quad \Rightarrow M(t,y) := ye^{ty} - 2t, \quad N(t,y) := te^{ty} - 6y.$$

Computing

$$M_y = e^{ty} + tye^{ty}, \quad N_t = e^{ty} + tye^{ty} \Rightarrow M_y = N_t,$$

and so the equation is exact. We can now compute for the function $\Psi(t,y)$ by integrating M with respect to t:

$$\Psi(t,y) = \int M(t,y)\,dt = e^{ty} - t^2 + h(y).$$

Differentiating then gives

$$N(t,y) = te^{ty} + h'(y) \Rightarrow h'(y) = -6y \Rightarrow h(y) = -3y^2,$$

and so the general solution to the ODE is

$$\Psi(t,y(t)) = e^{ty(t)} - t^2 - 3y(t)^2 = c, \quad c \in \mathbb{R}.$$

(b) Substituting $v = y'$ yields

$$v'' - 7v' + 12v = 0.$$

The characteristic equation for this second order ODE is

$$r^2 - 7r + 12 = (r-4)(r-3) = 0 \Rightarrow r_1 = 4, \quad r_2 = 3.$$

Hence the general solution to the second order ODE is

$$v(t) = c_1 e^{4t} + c_2 e^{3t},$$

and upon integrating for y gives

$$y(t) = \frac{c_1}{4} e^{4t} + \frac{c_2}{3} e^{3t} + c_3, \quad c_1, c_2, c_3 \in \mathbb{R}.$$

There is also the solution $v = y' = 0$ which implies that

$$y(t) = b, \quad b \in \mathbb{R},$$

but this is already included as part of the general solution.

(c) First note that $y = 0$ is a solution. If $y \neq 0$, then dividing by y^{10} leads to a separable equation:

$$\frac{y'}{y^{10}} = \frac{d}{dt}\left(-\frac{1}{9y^9}\right) = -\sin(t) \Rightarrow -\frac{1}{9y^9} = \cos(t) + c \Rightarrow y(t) = \frac{-1}{(9\cos(t) + c)^{1/9}}$$

for $c \in \mathbb{R}$.

(d) Note that we can factorise

$$y''y' + (y')^2 = y'(y'' + y') = 0 \Rightarrow y' = 0 \text{ or } y'' + y' = 0.$$

For the first case we have the solution

$$y = a,$$

for constant $a \in \mathbb{R}$, and for the second case we have

$$y'' + y' = 0 \Rightarrow v' + v = 0 \quad (v = y') \Rightarrow v = ce^{-t} \Rightarrow y = -ce^{-t} + d,$$

for constants $c, d \in \mathbb{R}$.

2. Give examples of the following. Show your reasoning.

   (a) [5 pts] An initial value problem

   $$y' = f(y), \quad y(t_0) = y_0$$

   where f and $f'(y)$ are continuous everywhere, but the interval of existence is not $\mathbb{R}$.

   (b) [5 pts] A non-exact first order ODE.

   (c) [5 pts] A second order homogeneous ODE for which $y_1 = e^{2t}$ and $y_2 = e^{-8t}$ form a fundamental set of solutions.

   (d) [5 pts] A pair of functions f and g that are linearly independent, but their Wronskian $W(f, g)[t]$ is zero for all t.

Solution.

(a) One example is

$$y' = y^2, \quad y(0) = 1,$$

which has $f(y) = y^2$ and $f'(y) = 2y$ that are continuous for all $y \in \mathbb{R}$. However, as the equation is separable one obtains

$$y(t) = \frac{1}{1-t} \to \infty \text{ as } t \to 1,$$

and so the interval of existence cannot be $\mathbb{R}$.

(b) Any example would do, e.g.

$$y' + y = 0 \Rightarrow M(t,y) = y, \quad N(t,y) = 1 \Rightarrow M_y = 1 \neq 0 = N_t.$$

(c) If $y_1 = e^{2t}$ and $y_2 = e^{-8t}$, then $r_1 = 2$ and $r_2 = -8$ are the roots of the characteristic equation, which implies that the characteristic equation is

$$(r-2)(r+8) = r^2 + 6r - 16$$

and so the ODE is

$$y'' + 6y' - 16y = 0.$$

(d) One example that was encountered in the Homework 2 is

$$f(t) = t^2 |t|, \quad g(t) = t^3.$$

Then, we have $f'(t) = 3t|t|$ and $g'(t) = 3t^2$ so that

$$W(f,g)[t] = g'(t)f(t) - f'(t)g(t) = 0 \quad \forall t \in \mathbb{R}.$$

However, if

$$\alpha_1 t^2 |t| + \alpha_2 t^3 = 0 \quad \forall t \in \mathbb{R}$$

and plugging in $t = 1$ and $t = -1$ we obtain

$$\alpha_1 + \alpha_2 = 0, \quad \alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2 = 0.$$

Hence f and g are linearly independent but the Wronskian is zero.

3. (a) [4 pts] Find a fundamental set of solutions to the homogeneous ODE

$$y'' - 4y' + 4y = 0.$$

(b) [8 pts] Use the method of undetermined coefficients to find a particular solution to

$$y'' - 4y' + 4y = 2\sin(t) + e^{-2t} + t^3.$$

(c) [8 pts] Use the method of variation of parameter to find a particular solution to

$$y'' - 4y' + 4y = e^{2t}\ln(t).$$

You may use the fact that $\frac{d}{dt}\left(\frac{1}{4}t^2(2\ln(t) - 1)\right) = t\ln(t)$.

Solution.

(a) The characteristic equation for the ODE is

$$r^2 - 4r + 4 = (r-2)^2 = 0 \Rightarrow r_1 = r_2 = 2.$$

Hence, we consider the pair (e^{2t}, te^{2t}). By standard computation of the Wronskian

$$W(e^{2t}, te^{2t}) = e^{4t} \neq 0 \quad \forall t \in \mathbb{R},$$

and so (e^{2t}, te^{2t}) forms a fundamental set of solutions to the ODE.

(b) First we find a particular solution to

$$y'' - 4y' + 4y = 2\sin(t).$$

We try

$$Y_1(t) = A\cos(t) + B\sin(t),$$

for undetermined constants A and B. Differentiating the substituting into the ODE gives

$$Y_1'' - 4Y_1' + 4Y_1 = \cos(t)(3A - 4B) + \sin(t)(3B + 4A) = 2\sin(t),$$

and so

$$3A - 4B = 0, \quad 3B + 4A = 2 \Rightarrow A = \frac{8}{25}, \quad B = \frac{6}{25}.$$

Thus,

$$Y_1(t) = \frac{8}{25}\cos(t) + \frac{6}{25}\sin(t).$$

For a particular solution to

$$y'' - 4y' + 4y = e^{-2t},$$

since $r_1, r_2 \neq -2$, we can try

$$Y_2(t) = Ae^{-2t}.$$

Differentiating and substituting leads to

$$Y_2'' - 4Y_2' + 4Y_2 = 16Ae^{-2t} = e^{-2t} \Rightarrow A = \frac{1}{16},$$

and so

$$Y_2(t) = \frac{1}{16}e^{-2t}.$$

For a particular solution to

$$y'' - 4y' + 4y = t^3,$$

we try

$$Y_3(t) = At^3 + Bt^2 + Ct + D,$$

so that

$$Y_3'' - 4Y_3' + 4Y_3 = 4At^3 + (4B - 12A)t^2 + (6A - 8B + 4C)t + (4D - 4C + 2B) = t^3$$

and so

$$A = \frac{1}{4}, \quad B = \frac{3}{4}, \quad C = \frac{9}{8}, \quad D = \frac{3}{4}$$
$$\Rightarrow Y_3(t) = \frac{1}{4}t^3 + \frac{3}{4}t^2 + \frac{9}{8}t + \frac{3}{4}.$$

Therefore, a particular solution to the non-homogeneous ODE is

$$Y(t) = \frac{8}{25}\cos(t) + \frac{6}{25}\sin(t) + \frac{1}{16}e^{-2t} + \frac{1}{4}t^3 + \frac{3}{4}t^2 + \frac{9}{8}t + \frac{3}{4}.$$

(c) From (a) the Wronskian is $W[t] = e^{4t}$. Hence, for $y_1 = e^{2t}$ and $y_2 = te^{2t}$ a particular solution to the ODE using the variation of parameter formula is

$$Y(t) = -y_1 \int \frac{y_2 e^{2t}\ln(t)}{W[t]}\,dt + y_2 \int \frac{y_1 e^{2t}\ln(t)}{W[t]}\,dt.$$

We compute (neglecting constants of integration)

$$\int \frac{y_1 e^{2t}\ln(t)}{W[t]}\,dt = \int \ln(t)\,dt = t(\ln(t) - 1),$$
$$\int \frac{y_2 e^{2t}\ln(t)}{W[t]}\,dt = \int t\ln(t)\,dt = \frac{1}{4}t^2(2\ln(t) - 1).$$

Hence, the particular solution is

$$Y(t) = -\frac{1}{4}e^{2t}(t^2(2\ln(t) - 1)) + t^2e^{2t}(\ln(t) - 1).$$

4. (a) [10 pts] Let $p(t)$ and $q(t)$ be functions that are continuous for all $t \in \mathbb{R}$. Can $y(t) = t^2e^t$ be a solution to the equation

$$y'' + p(t)y' + q(t)y = 0$$

satisfied for all $t \in \mathbb{R}$? If yes, construct such functions $p(t)$ and $q(t)$. If no, explain why.

(b) [10 pts] Given that $y_1(t) = t$ is a solution to the homogeneous ODE

$$t^2y'' - t(t + 2)y' + (t + 2)y = 0, \quad t > 0,$$

find a solution y_2 that is linearly independent to y_1, and show that y_1 and y_2 form a fundamental set of solutions.

Solution.

(a) The answer is NO. Consider the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(0) = 0, \quad y'(0) = 0.$$

Then,$y_* = 0$ is a solution and as p, q are continuous we see that $y_* = 0$ is the only solution. However, the function $y(t) = t^2e^t$ satisfies

$$y(0) = 0, \quad y'(0) = 0$$

but $t^2e^t \neq 0$ for all $t \in \mathbb{R}$. Hence we have a contradiction. Therefore, $y(t) = t^2e^t$ cannot be a solution to the ODE for all $t \in \mathbb{R}$.

(b) Since $t > 0$ we can divide by t to obtain the standard form

$$y'' - \frac{(t+2)}{t}y' + \frac{t+2}{t^2}y = 0.$$

Suppose another solution z exists, then by Abel's theorem, we know the Wronskian is given by

$$W(y_1, z)[t] = ce^{\int 1+\frac{2}{t}\,dt} = ce^tt^2.$$

Meanwhile, by the definition of the Wronskian

$$W(y_1, z)[t] = z'y_1 - y_1'z = tz' - y_2 = e^tt^2.$$

Solving for the linear first order ODE

$$z' - \frac{1}{t}z = te^t$$

with the method of integrating factors, where the integrating factor $\mu(t)$ is computed as $\mu(t) = \frac{1}{t}$, we see that

$$\frac{d}{dt}\frac{z(t)}{t} = e^t \Rightarrow z(t) = te^t + ct, \quad c \in \mathbb{R}.$$

Since $y_1 = t$ is a solution to the ODE we find that the other solution is

$$y_2(t) = te^t.$$

Computing the Wronksian now gives

$$W(y_1, y_2)[t] = t^2e^t \neq 0 \text{ for } t > 0,$$

and so (t, te^t) forms a fundamental set of solutions to the ODE.

5. (a) [4 pts] State Abel's theorem for a n-th order linear ODE.

   (b) [8 pts] Show that $W(5, \sin^2(t), \cos(2t)) = 0$ for all $t \in \mathbb{R}$ without evaluating the Wronskian.

(c) [8 pts] Let $p(t), q(t), r(t)$ be continuous functions on $\mathbb{R}$, suppose the functions $y_1(t) = t$, $y_2(t) = t^2$ and $y_3(t) = t^3$ are solutions to the linear ODE

$$y''' + p(t)y'' + q(r)y' + r(t)y = 0.$$

Compute the Wronskian $W(t, t^2, t^3)$, and use your answer to part (a) to derive the interval $I \subset \mathbb{R}$ for which $\{t, t^2, t^3\}$ can be a fundamental set of solutions to the above ODE. You may use the following formula

$$\det\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & k \end{pmatrix} = a(ek - hf) - d(bk - hc) + g(bf - ec).$$

Solution.

(a) Let I be an open interval with continuous functions $P_{n-1}(t), \ldots, P_0(t)$. Let $(y_1, \ldots, y_n)$ be solutions to the homogeneous equation

$$y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = 0 \quad \forall t \in I.$$

Then, the Wronskian is given as

$$W(y_1, \ldots, y_n)[t] = ce^{-\int P_{n-1}(t)\,dt}$$

for some constant c not depending on $t \in I$.

(b) Using the fact that if y_1, y_2, y_3 are linearly dependent, then the Wronskian $W(y_1, y_2, y_3)[t]$ is zero, we see that for $y_1 = 5$, $y_2 = \sin^2(t)$, $y_3 = \cos(2t) = \cos^2(t) - \sin^2(t)$ that

$$y_1 = 10y_2 + 5y_3,$$

and so

$$y_1(t) - 10y_2(t) - 5y_3(t) = 0 \quad \forall t \in \mathbb{R}.$$

Therefore, y_1, y_2, y_3 are linearly dependent.

(c) The Wronskian $W(t, t^2, t^3)$ is given as

$$W(t, t^2, t^3) = \begin{vmatrix} t & t^2 & t^3 \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = 2t^3,$$

which is non-zero for $I = (0, \infty)$ or $I = (-\infty, 0)$. Hence (t, t^2, t^3) can be a fundamental set of solutions of the ODE for $t \in (-\infty, 0)$ or for $t \in (0, \infty)$.

— End of question paper —