MATH3720A - Lecture Notes - Week 1

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September 7, 2017

1 Introduction

1.1 Motivation for ODEs

Many phenomena in the world around us:

- growth of plants and animals;
- movement of people, objects and goods (in an economy);
- value of stock prices;
- flow of fluids/gases, melting of ice;

are "**dynamic**" in nature. We can associate "**dynamic**" with "**change (in time)**", and to model these changes with mathematics, we can use equations of the form:

$$\text{rate of change of a quantity} = \text{source} - \text{depletion} .$$

As the **derivative** of a function provides the rate at which it is **changing** with respect to its variable, the equations we use to model change will involve derivatives, and we call these equations as **differential equations**.

We begin with a definition.

Definition 1.1. *A **differential equation** is an equation that involves the **derivatives** of an unknown function.*

Notation: Let $x = x(t)$ be a function of t. For any $n \in \mathbb{N}$, we write

$$x^{(n)} = \frac{d^n x}{dt^n}, \quad \text{and often} \quad x' = x^{(1)}, \quad x'' = x^{(2)}.$$

The main subject of study in this course is the class of differential equations called "**Ordinary differential equations**" (ODEs), and a problem involving ODEs has 5 components:

(1) the **independent variable** - usually time, denoted by t;

(2) the **dependent variable** (also the **quantity of interest**) - such as distance, price, number of people, denoted by y;

(3) the equation - specifying how the quantity of interest changes with respect to the independent variable;

(4) the **interval of definition** - denoted by $I \subset \mathbb{R}$, which is the **range** for which the solution to the ODE is defined;

(5) the **initial conditions** - specific conditions related to the particular problem we want to model.

Example 1.1 (Motion of a falling object)**.** *Consider an object falling in the atmosphere with mass $m > 0$. We are interested in the* ***velocity*** *v of the object as time progresses. So we think of v as a function of t and derive an equation for the rate of change $\frac{dv}{dt}$. A crucial idea is the* ***balances of forces*** *- which is* ***Newton's third law****. There are two (opposing) forces acting on the object as it falls:*

(1) Gravity exerts a (downwards) force $F_g = mg$, where g is the gravitational constant;

(2) Movement through the air generates ***air resistance/drag forces****, which we take as proportional to the velocity. This gives a (upwards) force $F_a = \gamma v$, where $\gamma > 0$ is the drag coefficient.*

The ***net*** *force pointing downwards is therefore*

$$F = F_g - F_a = mg - \gamma v.$$

Using ***Newton's second law*** *- which relates net force with the product of mass and acceleration, and recalling acceleration is the rate of change of velocity with respect to time, we are led to*

$$mg - \gamma v = F = ma = m\frac{dv}{dt}$$
$$\Rightarrow \boxed{m\frac{dv}{dt} = mg - \gamma v}.$$

In the above, we see that

1. *the independent variable is time t;*

2. *the dependence variable is the velocity v;*

3. *the equation is $mv' = mg - \gamma v$;*

4. *the interval of definition can be taken as $I = [0, \infty)$ - modelling the motion of the object from time $t = 0$ onwards;*

5. *as initial condition we can take $v(0) = 0$.*

Example 1.2 (Population dynamics)**.** *Let us model the growth of a* ***herd of cows****. We are interested in the number of cows after a certain period of time. Setting* t *as the independent variable and* $p(t)$ *- the number of cows - as the dependent variable, we now derive an ODE for* $p(t)$*. Note that for physical reasons, only the case* $p(t) \geq 0$ *makes sense.*

The modelling assumptions we will make are the following:

- *the rate at which population changes is* ***proportional*** *to the population at present time - leading to the equation*

$$\frac{dp}{dt} = h(p)p,$$

  for some function h*;*

- *when* $p(t)$ *is small,* $h(p(t))$ *is positive (less competition, more food for everyone);*

- *when* $p(t)$ *is large,* $h(p(t))$ *is negative (more competition, less food for everyone);*

- *the function* $h(p)$ *should decrease as* p *grows larger.*

To obtain the ODE we simply need to find a suitable function for h*. One simple choice is*

$$h(p) = r - ap,$$

where $r, a > 0$ *are the* ***reproduction*** *and* ***elimination*** *rates, respectively. Setting*

$$K := \frac{r}{a},$$

which is also known as the ***carrying capacity****, the ODE for* p *now reads as*

$$\boxed{\frac{dp}{dt} = (r - ap)p = rp\left(1 - \frac{p}{K}\right)}.$$

The above equation is also called the ***Logistic equation****. Once again, we are interested in the values of* p *from* $t = 0$ *onwards, so we set* $I = [0, \infty)$*. As for initial condition, we set* $p(0) = p_0$*, where* p_0 *is the initial number of cows the farmer has.*

Example 1.3 (Motion of a pendulum)**.** *An object of mass* $m > 0$ *is attached to a rigid rod of length* L *and affixed to the ceiling. The object is allowed to swing in one direction and its motion traces out an* ***arc****. We are interested at how the angle* θ *between the rod and the centreline changes in time as the pendulum swings. Again we think of* θ *as a function of* t *and derive an equation for* $\frac{d\theta}{dt}$*. The key idea is also the* ***balance of forces****. Note that in the weight of the object* $W = mg$ *exerts two forces that are perpendicular to each other. A* ***tangential force*** $F_s = mg\sin\theta$

that drives the motion of the pendulum, and a ***normal force*** $F_c = mg\cos\theta$ *that is perpendicular and does not contribute to the motion.*

By Newton's second law, taking into account that the tangential force F_s *points in the opposite direction to the motion of the pendulum, leads to*

$$mg\sin\theta = F_s = -ma \Rightarrow a = -g\sin\theta.$$

Meanwhile, the velocity v *of the object is given as* $v = L\frac{d\theta}{dt}$*, and the acceleration* $a = \frac{dv}{dt}$ *can be computed as* $a = L\frac{d^2v}{dt^2}$*. Altogether we now have the* ***pendulum equation***

$$\boxed{L\frac{d^2\theta}{dt^2} = -g\sin\theta}.$$

For the interval of definition, we again consider $I = [0,\infty)$*, and for the initial conditions we have to prescribe an initial angle (position of the pendulum)* $\theta(0) = \theta_0$*, and an initial velocity* $\frac{d\theta}{dt}(0) = \frac{v_0}{L}$ *(how hard you initially swing the pendulum).*

In all of the examples above, we have to set an initial condition(s) for the ODE. What is the reason for this? If we consider the simple ODE example:

$$\text{Solve} \quad \frac{dy}{dt} = t.$$

Integration gives

$$\boxed{y(t) = \frac{1}{2}t^2 + c},$$

where $c \in \mathbb{R}$ is a **constant of integration**. We call the above formula the **general solution** to the ODE.

Since $c \in \mathbb{R}$ is an arbitrary constant, we have in fact obtained an **infinite number** of solutions to the ODE. The question becomes:

Which solution should I take as the "correct one"?

To remove this ambiguity, we prescribe an initial condition to **fix** the constant c: Suppose we set $y(t_0) = y_0$ for some **given** constants $t_0 \in I$ and $y_0 \in \mathbb{R}$, then it turns out that

$$y_0 = \frac{1}{2}t_0^2 + c \Rightarrow c = y_0 - \frac{1}{2}t_0^2 \Rightarrow \boxed{y(t) = \frac{1}{2}(t^2 - t_0^2) + y_0}.$$

We call the above formula a **particular solution** to the ODE.

Rule of thumb - The **number** of initial conditions needed is the same as the **highest** number of derivatives appearing in the ODE.

Definition 1.2. *An ordinary differential equation is an equation involving* ***ONE*** *independent variable $t \in I$ and* ***ONE*** *dependent variable y of the form*

$$F(t, y, y', y'', \dots, y^{(n)}) = 0.$$

Given constants $t_0, t_1, \dots, t_{n-1} \in I$ and $y_0, y_1, \dots, y_{n-1} \in \mathbb{R}$, we call

$$\begin{cases} F(t, y, y', y'', \dots, y^{(n)}) = 0, \\ y(t_0) = y_0,\ \frac{dy}{dt}(t_1) = y_1, \dots,\ \frac{d^{(n-1)}y}{dt^{n-1}}(t_{n-1}) = y_{n-1}, \end{cases}$$

an ***initial value problem*** *(IVP).*

Definition 1.3.

(a) The ***order*** *of an ODE is the* ***highest order*** *of derivative that appears.*

(b) An ODE $F(t, y, y', \dots, y^{(n)}) = 0$ is ***linear*** *if F is a* ***linear function*** *of $y, \frac{dy}{dt}, \dots, \frac{d^n y}{dt^n}$. Otherwise, it is a* ***non-linear*** *ODE. The general* ***linear*** *ODE of order n is*

$$a_0(t)y^{(n)}(t) + a_1(t)y^{(n-1)}(t) + \dots + a_n(t)y(t) = f(t),$$

for some given functions $a_0, a_1, \dots, a_n$ and f.

(c) An ODE is called ***autonomous*** *if the independent variable does not appear explicitly (only in the derivatives). Otherwise it is a* ***non-autonomous*** *ODE.*

Looking at the ODEs we have studied so far:

ODE	Order	Linear ?	Autonomous ?
$mv' = mg - \gamma v$	1	✓	✓
$p' = rp(1 - p/K)$	1	✗	✓
$L\theta'' = -g \sin\theta$	2	✗	✓
$y' = t$	1	✓	✗

While one expects the solution y to an ODE is a function depending on t, there is also a **special class** of solutions worth looking at:

Definition 1.4. *For a first order ODE $y' = F(t, y)$, we say that a function y_* is an* ***equilibrium solution*** *(or a* ***stationary solution****) to the ODE if*

$$F(t, y_*) = 0 \quad \forall t \in I.$$

Note that the equilibrium solution y_* does not depend on t, i.e., $\frac{dy_*}{dt} = 0$, and so it automatically satisfies the ODE.

Example 1.4.

1. *For the motion of the falling object, the ODE is* $mv' = mg - \gamma v = F(t,v)$*. So if* $F(t, v_*) = 0$ *for some function* v_**, we compute to see that*

$$0 = F(t, v_*) = mg - \gamma v_* \Rightarrow v_* = \frac{mg}{\gamma}$$ *(related to the* ***terminal velocity****).*

2. *For the population dynamics, the ODE is* $p' = rp(1 - p/K) = F(t,p)$*. So if* $F(t,p_*) = 0$ *for some function* p_**, we have*

$$0 = F(t,p_*) = rp_*(1 - p_*/K) \Rightarrow p_* = 0 \text{ or } p_* = K.$$

Interpretation: if $p_* = 0$*, then we have no cows left (extinction), and if* $p_* = K > 0$*, then the number of cows is equal to the* ***carrying capacity*** *- the maximum number that is supported by the environment.*

3. *For the pendulum (although it is a second order ODE), suppose* θ_* *is a function satisfying* $F(t,\theta_*) = -g\sin\theta_* = 0$*. Then*

$$0 = -g\sin\theta_* \Rightarrow \theta_* = 0,$$

and the interpretation is that the pendulum lies on the centreline.

1.2 Goal of this course

Given an ODE

$$\begin{cases} F(t, y, y', \ldots, y^{(n)}) = 0, \\ y(t_0) = y_0, \frac{dy}{dy}(t_1) = y_1, \ldots, \frac{d^{(n-1)}y}{dt^{n-1}}(t_{n-1}) = y_{n-1}, \end{cases} \tag{$\star$}$$

we attempt to answer the following questions:

1. can we find an **explicit formula** for the solution $y(t)$?
2. if not, can we **prove** that there **exists** a solution $y(t)$? If a solution exists, is it a unique solution?
3. if a solution $y(t)$ exists, what is its behaviour as t varies?
4. are there stationary solutions to $(\star)$?

2 First order equations

2.1 Two linear ODE example

We begin our study with two examples of first order linear ODEs.

2.1.1 Example 1

For given real constants a, b, t_0, y_0, solve

$$\begin{cases} \frac{dy}{dt} = ay + b, \\ y(t_0) = y_0. \end{cases}$$

This is a linear and autonomous ODE. Let us consider the case $a = 0$. Then the ODE becomes

$$y' = b, \quad y(t_0) = y_0.$$

Integrating yields the general solution

$$y(t) = bt + c, \quad c \in \mathbb{R},$$

and the initial condition gives the particular solution

$$\boxed{y(t) = y_0 + b(t - t_0)}.$$

For the case $a \neq 0$, we rearrange the ODE into another form:

$$y' = ay + b = a(y + b/a) \Rightarrow \frac{1}{y + \frac{b}{a}} \frac{dy}{dt} = a.$$

If there exists a function $H(y)$ such that $H'(y) = (y + b/a)^{-1}$, then the ODE becomes (via the Chain rule)

$$H'(y)\frac{dy}{dt} = \frac{d}{dt}H(y(t)) = a.$$

It turns out that

$$H(y) = \ln(y + b/a),$$

and so we have

$$\ln(y(t) + b/a) = at + c, \quad c \in \mathbb{R}.$$

Taking exponential then leads to the general solution

$$\boxed{y(t) = \kappa \exp(at) - \frac{b}{a}}, \quad \kappa := \exp(c).$$

Using the initial condition $y(t_0) = y_0$ we obtain the particular solution

$$\boxed{y(t) = (y_0 + b/a)\exp(a(t - t_0)) - \frac{b}{a}}.$$

In summary we find that

$$y(t) = \begin{cases} y_0 + b(t - t_0) & \text{for } a = 0, \\ (y_0 + b/a)\exp(a(t - t_0)) - \frac{b}{a} & \text{for } a \neq 0. \end{cases}$$

This example shows that the explicit formula for the solution can **depend** on the values of the given coefficients. **Always** keep this in mind before starting to solve the ODE.

2.1.2 Example 2

For a given function $p(t)$, find the general solution to

$$\frac{dy}{dt} = p(t)y.$$

Note that $y(t) \equiv 0$ is one solution! Suppose that $y(t_*) \neq 0$ for some $t_* \in I$, then we can rearrange the ODE into the form

$$\frac{1}{y}\frac{dy}{dt} = p(t) \Rightarrow \frac{d}{dt}\ln(y(t)) = p(t).$$

Integrating yields

$$\ln(y(t)) = \int p(t)\, dt + c, \quad c \in \mathbb{R},$$

and taking exponential gives the general solution

$$\boxed{y(t) = \kappa \exp\left(\int p(t)\, dt\right)}, \quad \kappa := \exp(c).$$

Remark 2.1. *In the above examples, it should have been*

$$\ln|y(t)| = \int p(t)\, dt + c.$$

Once we take the exponential we find that

$$|y(t)| = \exp\left(\int p(t)\, dt\right)\exp(c).$$

Setting

$$\kappa = \begin{cases} \exp(c) & \text{if } y(t) > 0, \\ -\exp(c) & \text{if } y(t) < 0, \end{cases} \Rightarrow y(t) = \kappa \exp\left(\int p(t)\, dt\right).$$

The sign of the constant κ does not matter once we use the initial condition to determine its value.

2.2 Linear first order equations - method of integrating factors

In the above Example 2, we obtain that the general solution to the ODE

$$\frac{dy}{dt} = p(t)y$$

is

$$y(t) = \kappa \exp\left(\int p(t)\, dt\right),$$

for some arbitrary non-negative constant κ. We now study the general linear first order ODE:

$$\begin{cases} \frac{dy}{dt} = p(t)y + q(t), \\ y(t_0) = y_0 \end{cases} \tag{2.1}$$

for some given functions $p(t)$, $q(t)$ and constants t_0 and y_0. One example is the equation for the motion of the falling object: $mv' = mg - \gamma v$, where we set $y = v$, $p = -\gamma/m$ and $q = g$. The method we use is called the **method of integrating factors**.

Idea: Multiply the ODE (2.1) by a function $\mu(t)$, leading to

$$\mu(t)\frac{dy}{dt} - \mu(t)p(t)y(t) = \mu(t)q(t). \tag{2.2}$$

Suppose

$$\boxed{\mu(t)\frac{dy}{dt} - \mu(t)p(t)y(t) = \frac{d}{dt}\left(\mu(t)y(t)\right)}, \tag{2.3}$$

then, the multiplied ODE (2.2) becomes

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)q(t) \Rightarrow \boxed{\mu(t)y(t) = \int \mu(t)q(t)\, dt + c}, \quad c \in \mathbb{R}. \tag{2.4}$$

If in addition, $\mu(t)$ is **non-zero**, we can divide by $\mu(t)$ and end up with the general solution

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)\, dt + c\right]}. \tag{2.5}$$

Definition 2.1. *If such a function $\mu(t)$ exists satisfying (2.3), then we call $\mu(t)$ the **integrating factor**.*

But does such a function $\mu(t)$ exists? If it doesn't then this is a useless method. What is the equation satisfied by μ? From (2.3) we see that

$$\begin{aligned} \mu(t)y'(t) - \mu(t)p(t)y(t) &= \frac{d}{dt}(\mu y) = \mu'(t)y(t) + \mu(t)y'(t) \\ \Rightarrow y(t)\left(\frac{d\mu}{dt} + p(t)\mu(t)\right) &= 0. \end{aligned}$$

The above equation is satisfied if $y(t) = 0$ or $\mu'(t) + p(t)\mu(t) = 0$. The first case $y(t) = 0$ is not desirable, since if the initial condition y_0 is non-zero, we have a contradiction. Therefore, we consider the second case and obtain the equation

$$\boxed{\frac{d\mu}{dt} = -p(t)\mu} \tag{2.6}$$

as the ODE for μ. But this type of equation has been encountered before. From Example 2, we see that the general solution is

$$\mu(t) = \kappa \exp\left(-\int p(t)\,dt\right), \quad \kappa \in \mathbb{R}_{\geq 0}. \tag{2.7}$$

Take note of the minus sign! The question is what should we take the value of κ to be? Let us first substitute the formula (2.7) into the multiplied ODE (2.2):

$$\begin{aligned}
&\kappa \exp\left(-\int p(t)\,dt\right)\frac{dy}{dt} - \kappa \exp\left(-\int p(t)\,dt\right)p(t)y(t) = \kappa \exp\left(-\int p(t)\,dt\right)q(t) \\
\Rightarrow\ &\kappa\frac{d}{dt}\left(e^{-\int p(t)\,dt}y(t)\right) = \kappa e^{-\int p(t)\,dt}q(t).
\end{aligned}$$

It turns out that κ appears on both sides of the equation, and thus we can cancel out κ. In effect, we can choose $\kappa = 1$, which we will do so from now on. This implies that we take the integrating factor $\mu(t)$ to be

$$\boxed{\mu(t) = \exp\left(-\int p(t)\,dt\right)}, \tag{2.8}$$

and the general solution $y(t)$ to the ODE $y' = p(t)y + q(t)$ is given as

$$\boxed{y(t) = e^{\int p(t)\,dt}\left[\int e^{-\int p(t)\,dt}q(t)\,dt + c\right]}. \tag{2.9}$$

The particular solution and the constant c can be computed with the initial condition $y(t_0) = y_0$, which we will not do here.

Example 2.1. *Derive the general solution to the ODE*

$$t\frac{dy}{dt} + 2y = 4t^2.$$

Step 1. *Write the ODE in the form $y' = p(t)y + q(t)$ and identify p and q:*

$$t\frac{dy}{dt} + 2y = 4t^2 \Rightarrow \frac{dy}{dt} = -\frac{2}{t}y + 4t \Rightarrow p(t) = -\frac{2}{t}, \quad q(t) = 4t.$$

Step 2. *Compute the integrating factor $\mu(t)$:*

$$\mu(t) = \exp\left(-\int p(t)\,dt\right) = \exp\left(\int \frac{2}{t}\,dt\right) = t^2.$$

Step 3. *Plug into the formula* (2.9)

$$y(t) = \frac{1}{t^2}\left[\int t^2 \times 4t\,dt + c\right] = t^2 + \frac{c}{t^2}.$$

Remark 2.2. *The general solution* $y(t) = t^2 + \frac{c}{t^2}$, *for* $c \neq 0$, *is not defined at the point* $t = 0$. *So far in the course, we have not really discussed the interval of definition* $I \subset \mathbb{R}$. *In this case, the general solution is defined only for* $t \in (-\infty, 0) \cup (0, \infty) = \mathbb{R} \setminus \{0\}$. *If the graph of* $y(t)$ *is sketched we see that the graph has two parts, one to the left of the y-axis and one to the right of the y-axis. Which part we take depends on the initial condition.*

If we consider an initial condition $y(t_0) = y_0$, *where* $t_0 > 0$, *then we choose the right part - since we can determine the arbitrary constant c in the general solution only in the interval* $(0, \infty)$. *In this case the interval of definition is* $I = (0, \infty)$. *Similarly, if* $t_0 < 0$, *then we choose the left part as the solution, with* $I = (-\infty, 0)$. *This example serves as a reminder that the solution* $y(t)$ *to ODEs may not be defined for all values of* $t \in \mathbb{R}$, *and the initial condition plays a role in determining the interval of definition.*