MATH3720A Ordinary Differential Equations
2017 - 18 Term 1
Homework Assignment 3

Please hand in your assignment to the assignment boxes on 2/F LSB by 5pm on **Friday 24th November**.

1. [5 pts] Let $\vec{y}^{(1)}, \vec{y}^{(2)}$ and $\vec{y}^{(3)}$ be three solutions to the homogeneous system

$$\vec{Y}'(t) = \mathbb{P}(t)\vec{Y}(t), \quad t \in I,$$

for a 3×3 matrix $\mathbb{P}(t)$, and I is an open interval of $\mathbb{R}$. Show that

$$\frac{d}{dt}W[t] = (P_{11}(t) + P_{22}(t) + P_{33}(t))W[t],$$

where $W[t] = W(\vec{y}^{(1)}, \vec{y}^{(2)}, \vec{y}^{(3)})[t]$ is the Wronksian.

2. Find the general solution to the following equations

   (a) [3 pts] $y^{(4)} + 2y'' + y = 8 + \sin(2t)$;

   (b) [3 pts] $y''' + y'' + y' + y = 2e^{-t} + 5t^2$;

   (c) [3 pts] $y''' + y' = \tan(t) + 5\sec(t)$ for $t \in (-\frac{\pi}{2}, \frac{\pi}{2})$.

   (d) [3 pts] $t^3y''' + t^2y'' - 2ty' + 2y = 2t^4$ for $t > 0$ given that $t, t^2, \frac{1}{t}$ are solutions to the homogeneous equation.

3. [9 pts] Find the general solution to the system of equations $\frac{d}{dt}\vec{Y}(t) = \mathbb{A}\vec{Y}(t)$, where the matrix $\mathbb{A}$ is given as

$$\text{(a) } \mathbb{A} = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}, \quad \text{(b) } \mathbb{A} = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & -2 \\ 3 & 2 & 1 \end{pmatrix}, \quad \text{(c) } \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix}.$$

4. (a) [6 pts] Use the method of undetermined coefficients to find a particular solution to the system of equations $\frac{d}{dt}\vec{Y}(t) = \mathbb{A}\vec{Y}(t) + \vec{H}(t)$, where

$$\text{(i) } \mathbb{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix}, \vec{H} = \begin{pmatrix} e^{2t} \\ t^2 \end{pmatrix}, \text{ (ii) } \mathbb{A} = \begin{pmatrix} 1 & \sqrt{2} \\ \sqrt{3} & -1 \end{pmatrix}, \vec{H} = \begin{pmatrix} e^t \\ \sqrt{3}e^{-t} \end{pmatrix}.$$

   (b) (i) [3 pts] Find a fundamental set of solutions for the homogeneous system

$$\frac{d}{dt}\vec{Y}(t) = \mathbb{A}\vec{Y}(t), \quad \mathbb{A} = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix},$$

   and write down the fundamental matrix $\mathbb{F}(t)$.

   (ii) [2 pts] Find the fundamental matrix $\mathbb{G}(t)$ such that $\mathbb{G}(0) = \mathbb{I}$.

(iii) [5 pts] Use the method of variation of parameter to find a particular solution to

$$\frac{d}{dt}\vec{Y}(t) = \mathbb{A}\vec{Y}(t) + \vec{H}(t), \quad \vec{H}(t) = \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}.$$

(iv) [10 pts] Repeat (i)–(iii) for the system with

$$\mathbb{A} = \begin{pmatrix} -5/4 & 3/4 \\ 3/4 & -5/4 \end{pmatrix}, \quad \vec{H}(t) = \begin{pmatrix} 4t \\ e^{t} \end{pmatrix}.$$

5. [3 pts each] Let $\vec{0}$ be the only critical point of the system $\frac{d}{dt}\vec{Y}(t) = \mathbb{A}\vec{Y}(t)$. Identify the type of the critical point $\vec{0}$ (node, spiral, saddle point or center) and sketch the phase portrait if the eigenvalues r_1, r_2, eigenvectors $\vec{\xi}_1, \vec{\xi}_2$, or generalized eigenvector $\vec{\eta}$ of $\mathbb{A}$ are given as
   (i) $r_1 = 1$, $\vec{\xi}_1 = (1,1)$, $r_2 = -2$, $\vec{\xi}_2 = (1,-1)$;
   (ii) $r_1 = -1$, $\vec{\xi}_1 = (0,1)$, $r_2 = -2$, $\vec{\xi}_2 = (2,3)$;
   (iii) $r_1 = r_2 = 4$, $\vec{\xi}_1 = (1,3)$, $\vec{\xi}_2 = (2,0)$ (geo. mult. = 2);
   (iv) $r_1 = r_2 = -1$, $\vec{\xi} = (0,1)$, $\vec{\eta} = (1,0)$ (geo. mult. = 1);
   (v) $r_1 = 1 + 3i, r_2 = 1 - 3i$, $\vec{\xi} = (1, 2+i)$, $\vec{\xi}_2 = (1, 2-i)$.
   (vi) $r_1 = 3i$, $r_2 = -3i$, $\vec{\xi}_1 = (1+2i, 1-3i)$, $\vec{\xi}_2 = (1-2i, 1+3i)$.

   For the last two, assume that the trajectories move in a clockwise direction. In all phase portrait label the axes, eigenvectors, and direction of the trajectories.

6. (a) [6 pts] Compute the Jacobian matrix of the following functions $\vec{f}(y_1, y_2)$:

$$\text{(i)} \begin{pmatrix} y_1(8 - 4y_2 - y_2) \\ y_2(3 - 3y_1 - y_2) \end{pmatrix}, \quad \text{(ii)} \begin{pmatrix} (2 + y_1)\sin(y_2) \\ 9\cos(y_1)e^{y_1} \end{pmatrix},$$

$$\text{(iii)} \begin{pmatrix} 6y_2^2 + 9y_1 \\ \ln(y_1)y_2 - \cos(y_2) \\ \cosh(y_1 + y_2) \end{pmatrix}$$

   (b) [5 pts] Show that $\vec{0} = (0,0)$ is a critical point to the system

$$x' = x + 2y^2, \quad y' = x + y.$$

   Compute the Jacobian matrix at $\vec{0}$ and show that the system is locally linear near the critical point $\vec{0}$. Discuss the type and stability of the critical point by examining the corresponding linear system.

   (c) For the system

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \vec{F}((x,y)) = \begin{pmatrix} x(1 - 2x - y) \\ y(-2 + 6x) \end{pmatrix}. \tag{1}$$

   (i) [3 pts] Identify the three critical points $\vec{X}_{1,*}, \vec{X}_{2,*}, \vec{X}_{3,*}$.

(ii) [9 pts] Compute the Jacobian matrix of $\vec{F}$ and determine if the system (1) is locally linear at each of the three critical points. [You may find using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ useful].

(iii) [9 pts] Discuss the type and stability of each critical points by studying the corresponding linear system (if they are locally linear), and sketch the phase portraits in the neighbourhood of the individual critical points.

7. (a) [3 pts] Identify if the following functions are positive or negative definite:

$$V_1(x,y) = x^8 + y^4,$$
$$V_2(x,y) = 2x^2 + y^2 + xy,$$
$$V_3(x,y) = -x^2 - 7y^2 + 2xy.$$

(b) [6 pts] Show that for any positive constant $a > 0$, the function $V(x,y) = a(x^2 + y^2)$ is a Liapunov function for the systems

$$\text{(i) } x' = -x^3 + 2xy^2, \quad y' = -2x^2y - y^3,$$
$$\text{(ii) } x' = x^5 - x^3y, \quad y' = y^3 + x^4.$$

[Recall that $V(x,y)$ is a Liapunov function for a system of equations

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \vec{F}((x,y)),$$

if V is positive definite and there is a positive/negative definite function W such that

$$\frac{d}{dt}V(x(t),y(t)) = W(x(t),y(t)).]$$

(c) [4 pts] For the system of equations in (b)(i) and (b)(ii), identify the stability of the critical point $\vec{0}$ using Liapunov's theorems for stability and instability.

(d) [8 pts] Find a Liapunov function of the form $V(x,y) = ax^2 + cy^2$ for the system

$$x' = -y - x(x^2 + y^2), \quad y' = x - y(x^2 + y^2),$$

and deduce that the critical point $\vec{0}$ is asymptotically stable.