MATH3720A Ordinary Differential Equations
2017 - 18 Term 1
Homework Assignment 2

Please hand in your assignment to the assignment boxes on 2/F LSB by 3pm on **Friday 13th October**.

1. Given an ODE of the form $M(t,y) + N(t,y)y' = 0$, where $M_y \neq N_t$ (so that the ODE is not exact), consider a function μ depending on the variable $z = f(t)g(y)$, i.e., $\mu(z) = \mu[f(t)g(y)]$.

   (a) (2 point) What is the condition to be satisfied if

   $$\mu(f(t)g(y))M(t,y) + \mu(f(t)g(y))N(t,y)y' = 0$$

   is to be an exact equation? Write down the equation satisfied by μ.

   (b) (3 points) For the ODE

   $$\left(\frac{\sin y}{y} - 3e^{-t}\sin t\right) + \left(\frac{\cos y + 3e^{-t}\cos t}{y}\right)y' = 0,$$

   show that if $z = ye^t$, i.e., $f(t) = e^t$ and $g(y) = y$, then the integrating factor $\mu(z) = \mu(ye^t)$ satisfies the ODE

   $$\mu' = \frac{\mu}{z}.$$

2. Consider the differential *inequality*

   $$\frac{dz}{dt} \le p(t)z,$$

   where $p(t)$ is a continuous function on $I \subset \mathbb{R}$. Let $P(t)$ be the anti-derivative of $p(t)$, i.e., $P'(t) = p(t)$.

   (a) (3 points) Following the method of integrating factors, find a non-negative function $\mu(t)$ so that after multiplying the inequality with $\mu(t)$ one gets

   $$\frac{d}{dt}F(t,z(t)) \le 0 \qquad (1)$$

   for some function F. Identify the function F. Then, integrating both sides of (1) from t_0 to s, show the following inequality can be derived:

   $$z(s) \le z(t_0)e^{P(s)-P(t_0)}.$$

   (b) Given fixed constants $\alpha, \beta, \gamma, \delta$ with rectangle $R := (\alpha,\beta) \times (\gamma,\delta)$. Let $y_1(t)$ and $y_2(t)$ be two solutions to the IVP

   $$y' = f(t,y), \quad y(0) = y_0,$$

   where $0 \in (\alpha,\beta)$, $y_0 \in (\gamma,\delta)$, and f is continuous in R.

(i) (2 points) Write down the IVP satisfied by the difference $z(t) = y_1(t) - y_2(t)$, and hence show that

$$\frac{d}{dt}|z|^2 = 2z(f(t, y_1(t)) - f(t, y_2(t))).$$

(ii) (1 point) Under the assumption that the function f satisfies

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2|$$

for some positive constant L, derive the differential inequality

$$\frac{d}{dt}|z|^2 \leq 2L|z|^2.$$

(iii) (3 points) Solve the differential inequality and deduce that $z(t) = 0$ for all $t \geq 0$. What is the consequence for the solutions y_1 and y_2?

3. (a) (3 points) Let f and g be two continuous functions defined on $\mathbb{R}$. Suppose the Wronskian $W(f, g)[t]$ is non-zero for some $t \in I$. Show that f and g are linearly independent.

(b) It turns out that the converse (Linear independence $\Rightarrow$ Wronskian non-zero) is not true for general functions that are not the solution to some ODE.

(i) (4 points) Compute the derivative of the function $f(t) = t^2|t|$ for all $t \in \mathbb{R}$.

(ii) (3 points) Show that the Wronskian of $f(t) = t^2|t|$ and $g(t) = t^3$ is zero.

(iii) (3 points) Show that $f(t) = t^2|t|$ and $g(t) = t^3$ are linearly independent.

(c) (4 points) Given a second order linear homogeneous ODE

$$y'' + p(t)y' + q(t)y = 0 \tag{2}$$

with continuous coefficients p and q. Let $y_1(t)$ be a non-zero solution to the ODE. Use Abel's theorem to show that if y_2 is a function with $W(y_1, y_2)[t] \neq 0$ for all $t \in I$ and also satisfies the ODE

$$y_1(t)\frac{dy_2}{dt} - y_1'(t)y_2 = ce^{-\int p(t)\,dt},$$

where c is the constant from Abel's theorem, then y_2 is a solution to the ODE (2).

4. Consider two first order linear ODEs that are *coupled*:

$$u' = v, \quad v' = -p(t)v - q(t)u$$

for continuous functions $p(t)$ and $q(t)$ that are bounded, i.e., $|p(t)| \leq P$ and $|q(t)| \leq Q$ for all $t \in \mathbb{R}$, and initial conditions

$$u(0) = u_0, \quad v(0) = v_0.$$

(a) (2 points) Show that

$$\frac{d}{dt}(u^2 + v^2) = 2(1 - q(t))uv - 2p(t)v^2.$$

(b) (2 points) Using the (Young's) inequality $|uv| \leq \frac{1}{2}(u^2 + v^2)$, show that

$$\frac{d}{dt}(u^2 + v^2) \leq (1 + Q + 2P)(u^2 + v^2).$$

(c) (6 points) Using the results of Question 2 (b), show that for $t \geq 0$,

$$u^2(t) + v^2(t) \leq (u_0^2 + v_0^2)e^{(1+Q+2P)t},$$

and argue that there cannot exists a $t_* \in (0, \infty)$ such that $u^2(t) \to \infty$ as $t \to t_*$. Use this result to show that any solution y to the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(0) = y_0, \quad y'(0) = y_1,$$

with bounded continuous coefficients $p(t)$ and $q(t)$ cannot blow up in finite time.

5. (a) (3 points) The motion of a pendulum can be described by a second order ODE for the angle θ that the pendulum makes with the downward vertical. For small swing motions, the ODE is given as

$$\theta'' = -w^2\theta,$$

for some constant $w > 0$. Show that the general solution to the above ODE can be written as

$$\theta(t) = M\cos(wt - \phi) \tag{3}$$

for some constants M and ϕ. [Hint use the double angle formula]

(b) (3 points) We now add *damping* (i.e., friction/resistance) to the pendulum, leading to the new ODE

$$\theta'' + \lambda\theta' + w^2\theta = 0, \tag{4}$$

for some constant $\lambda > 0$. Obtain the general solutions and comment on the behaviour as $t \to \infty$ for each of the following cases:

(i) Over-damping $\lambda^2 > 4w^2$.

(ii) Critical damping $\lambda^2 = 4w^2$.

(iii) Under-damping $\lambda^2 < 4w^2$.

From now on we will always denote by the value $\lambda_c := 2w$ as the *critical value of damping* associated to the ODE (4).

(c) Oscillations on a foot bridge can also be modelled with the same type of ODE. As people walk on the bridge their footsteps create small wobbling. Denote by x the deviation of a point on the bridge from its normal position, and assume x satisfies the ODE (when there are no pedestrians)

$$Mx'' + kx' + hx = 0.$$

(i) (1 point) For the values $M = 4 \times 10^5$ kg, $k = 5 \times 10^4$ kg/s and $h = 10^7$ kg/s^2, show that the level of damping here is about 1% of the critical level.

(ii) (2 points) If there are N pedestrians on the bridge, they exert a forcing and leads to the new ODE

$$Mx'' + kx' + hx = 300Nx'. \tag{5}$$

Find the critical number of pedestrians N_0 (in terms of k, h, M) such that if there are more than N_0 pedestrians, then the bridge is no longer damped.

(iii) (2 points) Show that with $N = 200$ pedestrians, and the values $M = 4 \times 10^5$ kg, $k = 5 \times 10^4$ kg/s and $h = 10^7$ kg/s^2, the general solution to the ODE (5) contains oscillations with frequency approximately 0.8 hertz (oscillations per second) and amplitude growing like $e^{t/80}$. [For a function of the form

$$f(t) = A\cos(\mu t - \phi),$$

the amplitude is the value A and the frequency is $\frac{\mu}{2\pi}$.]

(d) Given a system consisting of a spring attached to an object mass, the displacement u of the spring (difference between the lengths before and after the object is hung onto the spring) satisfies the IVP

$$mu'' + ku = 0, \quad u(0) = a, \quad u'(0) = b.$$

(i) (1 point) Define the *kinetic energy* $K(t) = \frac{1}{2}m(u'(t))^2$ and the *potential energy* $P(t) = \frac{1}{2}k(u(t))^2$, show that the total energy (kinetic + potential) is conserved throughout time, i.e.,

$$K(t) + P(t) = K(0) + P(0) \quad \forall t \geq 0.$$

[Hint: Conservation of energy is equivalent to $\frac{d}{dt}(K(t) + P(t)) = 0$.]

(ii) (3 point) Suppose we add damping to the system, leading to

$$mu'' + \lambda u' + ku = 0, \quad \lambda > 0.$$

Show that one obtains the inequality

$$\frac{d}{dt}(K(t) + P(t)) \leq 0.$$

[Hint: compute $\frac{d}{dt}(K(t) + P(t))$] What conclusions can you derive about the change in the total energy if u is not constant in time?