MATH3720A Ordinary Differential Equations
2017 - 18 Term 1
Homework Assignment 1

Please hand in your assignment to the assignment boxes on 2/F LSB by 3pm on **Friday 22nd September**.

1. (5 points) For each of the following ODEs, determine its order, whether it is linear or non-linear, and whether it is autonomous or non-autonomous:

   (a) $t^2y'' + ty' + 3y = \cos(t)$

   (b) $y^{(4)} + \tan(t)y^2 = e^t$

   (c) $y'' + \ln(t)\cosh^2(y^2) = \ln(t)\sinh^2(y^2)$

   (d) $y' + \cos(y'') - \ln(y) = t\exp(t - y)$

   (e) $y' + \tanh^3(y) - \tanh(y) = -\text{sech}^2(y)\tanh(y)$

2. The velocity v of the falling object with mass $m > 0$ satisfies the IVP

$$mv' = mg - \gamma v, \quad v(0) = 0,$$

   where $g > 0$ and $\gamma > 0$ are given constants.

   (a) (2 points) Obtain an explicit formula for the velocity $v(t)$. What can you say above the behaviour of the solution $v(t)$ as $t \to \infty$?

   (b) (2 point) If the object weights $m = 10$kg, and the drag coefficient $\gamma = 2$kg/s. Using that $g = 9.8\text{m/s}^2$ and your answer to (a) to deduce the time that must elapse for the object to reach 90 % of its limiting velocity. Compute also how far the object has fallen in that time period, using the formula

$$x'(t) = v(t), \quad x(0) = 0.$$

   (c) (2 points) Assume now the drag force is proportional to the square of the velocity, changing the IVP to

$$mv' = mg - \gamma v^2, \quad v(0) = 0.$$

   Find an expression for the solution $v(t)$, and comment on its behaviour as $t \to \infty$.

3. (a) (3 point) Solve the IVP and determine the interval of definition

$$\left(\frac{2ty}{t^2+1} - 2t\right) - (2 - \ln(t^2+1))y' = 0, \quad y(5) = 0.$$

(b) (3 points) Find the values of b for which the following equation is exact

$$y\exp(2ty) + bt\exp(2ty)y' = 0.$$

For other values of b, compute the integrating factor μ for which after multiplying gives an exact equation.

(c) (4 points) Find an integrating factor and solve the equations

$$\text{i.)}\ y + (2ty - y\exp(-2y))y' = 0,$$
$$\text{ii.)}\ 1 + (t/y - \cos(y))y' = 0.$$

4. Consider a container that contains 10 gallons of fresh water. We introduce water contaminated with a chemical into the container at a constant rate r gallons per minute and assume the mixture (fresh + contaminated) flows out of the container at the same rate (r gallons per minute). The concentration (measured in grams per gallon) of chemical in the incoming supply is modelled by the function $\gamma(t)$. The goal is to determine the amount of chemical in the container at any time.

   (a) (2 points) Let $Q(t)$ denote the amount of chemical (measured in grams) present in the container. Formulate an ODE for $Q(t)$ based on the description above. Pay attention that each term of the equation has to have units of "grams per minute".

   (b) (1 point) Find the general solution to the ODE.

   (c) (3 point) Obtain a particular solution to the ODE with the following information: $Q(0) = 0$ and $\gamma(t) = 2 + \sin(2t)$.

5. For positive constants r and K, the logistic equation for population dynamics is

$$p' = pr(1 - p/K).$$

   (a) (1 point) Given the initial condition $p(0) = p_0$, solve the IVP and find the particular solution.

   (b) (1 point) Determine for any value of $p_0 \geq 0$ the behaviour of the solution $p(t)$ as $t \to \infty$.

   (c) (3 points) Another model for population growth is the Gompertz equation

$$p' = rp\ln(K/p).$$

   First determine the equilibrium solutions of the Gompertz equation, and then solve the IVP with the initial condition $p(0) = p_0$ (Hint: use the transformation $u = \ln(y/K)$ and derive an ODE for u). Then comment on the behaviour of the solution $p(t)$ as $t \to \infty$ for any value of $p_0 \geq 0$.