MATH3720A - Lecture Notes

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6 Geometric approach for nonlinear equations

In all of our study, we have mainly focused on linear equations to compute explicit solutions. For nonlinear equations we have been restricted to techniques for separable equations, exact equations and Bernoulli equations. For nonlinear equations

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)}),$$

or nonlinear systems of equations

$$\vec{y}'(t) = \mathbb{P}(t, \vec{y}(t)),$$

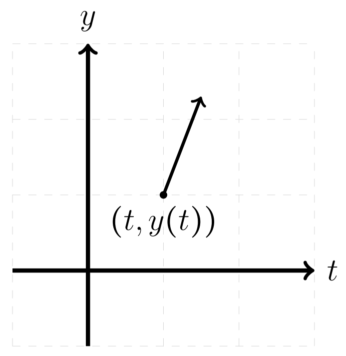
we can only say something about existence and uniqueness of solutions for small times if F or $\mathbb{P}$ are continuous with continuous derivatives. Unfortunately, explicit formulae for solutions are usually not available, but we can use geometric methods to deduce more information about the solutions. This will be the focus of this section.

6.1 First order equations

Given a first order nonlinear ODE

$$y'(t) = f(t, y(t)),$$

for continuous functions F and $\frac{\partial F}{\partial y}$, we know that there is exactly one solution to the IVP when initial conditions are given. Furthermore we can plot the graph of t vs y using the equation.



At each point $(t, y(t))$ we can draw a line segment with the slope $f(t, y(t))$. This gives a **direction field** for the ODE.

Example 6.1. *Consider the first order ODE*

$$y' = f(y) = 2 - y.$$

As f does not depend on t, the slopes of the line segments at a fixed y-coordinate are all the same. We plot the direction field in Fig. 1. Notice that the line segments

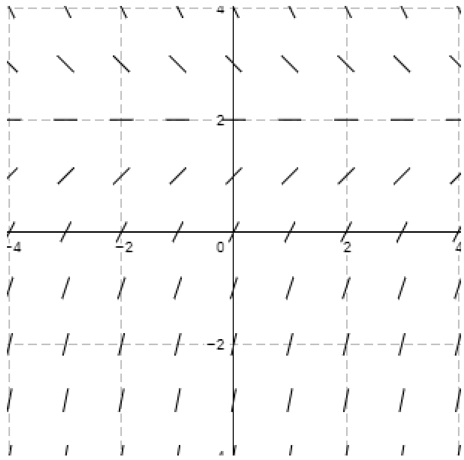


Figure 1: Direction field of $y' = 2 - y$.

have zero slope whenever the points lie on the line $\{y = 2\}$. Similarly, for the first order ODE

$$y' = y - 2,$$

we have the direction field Fig. 2.

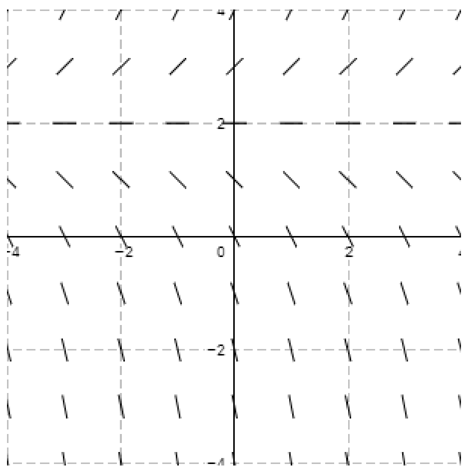


Figure 2: Direction field of $y' = y - 2$.

Now imagine affixing an arrow at the end of each line segment, this gives a vector field in the $t-y$ plane. Putting a particle inside this vector field at initial position (t_0, y_0) traces out a **trajectory** $\{(t, y(t)) : t \in I\}$ for the solution to the ODE.

In the first example $y' = 2-y$, all trajectories will go to the line $\{y = 2\}$ as $t \to \infty$, while for the second example $y' = y-2$, if trajectories start with initial position $y_0 = 2$, then the trajectories will stay on the line $\{y = 2\}$ as $t \to \infty$. However, if $y_0 > 2$, then the trajectories will move up and away from $\{y = 2\}$, and correspondingly in $y_0 < 2$, the trajectories will move down and away from $\{y = 2\}$.

In the above examples, the line $\{y = 2\}$ is what we will call **equilibrium/stationary** solutions to the ODE, as the values of y do not change as time progresses.

Definition 6.1 (Critical point and stationary solutions). *Given a continuous function $f(t, y)$, suppose $y_* \in \mathbb{R}$ is a point such that*

$$\boxed{f(t, y_*) = 0 \quad \forall t \in I},$$

then y_* is a ***critical point*** *of f. We call the constant function*

$$\boxed{\phi(t) = y_* \quad \forall t \in I}$$

a ***stationary solution*** *to the ODE $y' = f(t, y)$.*

Aside from the direction fields, another useful graphic for nonlinear **autonomous** ODEs

$$y' = f(y)$$

is the graph y vs $f(y)$.

Example 6.2. *Recall the Logistic equation: for positive constants r and K,*

$$y' = f(y) = ry\left(1 - \frac{y}{K}\right)$$

for population dynamics. It is easy to see that $y = 0$ and $y = K$ are stationary solutions, and if y is not equal to 0 or K, then

$$y(t) = \frac{y_0 K}{y_0 + (K - y_0)e^{-rt}} \qquad \text{for } y(0) = y_0.$$

Hence, we can deduce that for nonnegative initial values y_0,

$$y(t) \to K \text{ as } t \to \infty \text{ if } y_0 > 0, \quad y(t) = 0 \text{ for all } t > 0 \text{ if } y_0 = 0.$$

We can plot the direction field for the case $r = 1$ and $K = 4$, and observe in Fig. 3 below that the line segments with y-coordinate equal to 0 or 4 have zero slope. We now plot the graph y vs $f(y)$ in Fig. 4, which is a parabola that intersects the horizontal axis at two points $y = 0$ and $y = K$. The points that intersect the horizontal

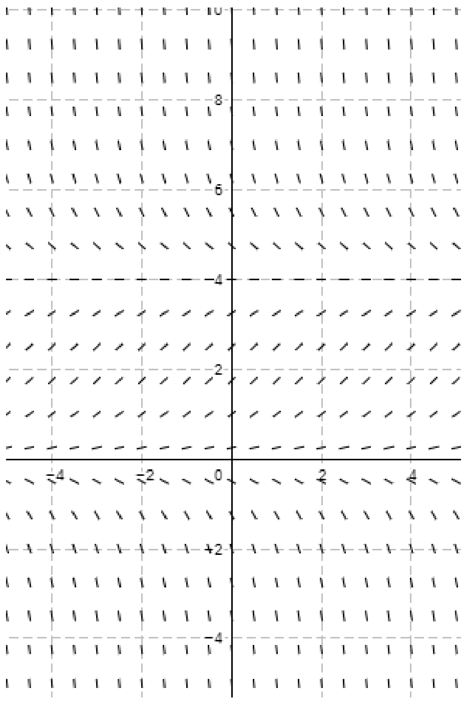


Figure 3: Direction field for the Logistic equation with $r = 1$ and $K = 4$.

axis are the stationary solutions. From this plot we can also deduce some behaviour of the solution to the ODE. Suppose we start with an initial condition x_0 in between 0 and K, then $f(x_0)$ is positive and so the solution y will increase in value, until it reaches $y = K$ where the derivative y' is zero. Similarly, if we start with an initial condition $x_1 > K$, then $f(x_1)$ is negative. Hence, the solution y will decrease in value, until it reaches $y = K$. Similarly, if we start with an initial value $x_2 < 0$, then $f(x_2)$ is negative and the solution y will decrease, moving away from the stationary solution $y = 0$. This can be summarized in Fig. 5, where we include arrows to demonstrate the behaviour of the solution.

Example 6.3. *We study a modification of the Logistic equation, called Logistic equation with threshold. Let $r > 0, 0 < T < K$ be positive constants, and consider the equation*

$$y' = f(y) = -r\left(1 - \frac{y}{T}\right)\left(1 - \frac{y}{K}\right)y.$$

First we identify the critical points, which are $y_1 = 0$, $y_2 = T$ and $y_3 = K$. Next, plotting the graph y vs $f(y)$ (see Fig. 6 for $r = 1$, $T = 4$ and $K = 8$) leads to a cubic graph with the following observations:

- *if initial condition $y(0) = y_0 \in (0, T)$, then $f(y_0)$ is negative and the solution y should decrease;*
- *if initial condition $y(0) = y_0 \in (T, K)$, then $f(y_0)$ is positive and the solution y should increase;*

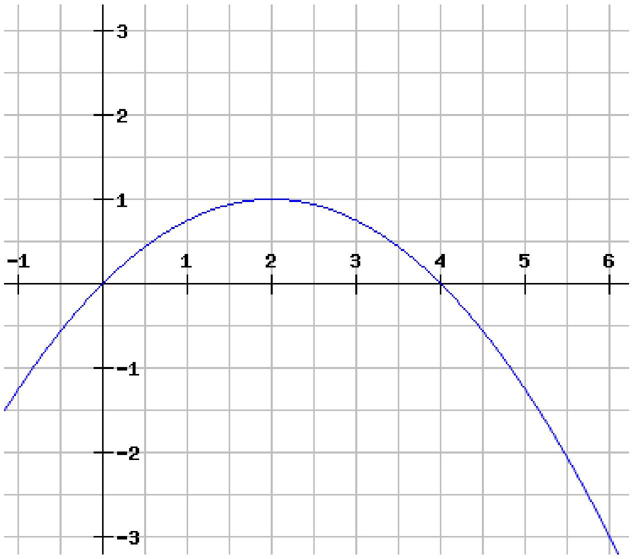


Figure 4: The plot y vs $f(y)$ for the Logistic equation.

- *if initial condition $y(0) = y_0 > K$, then $f(y_0)$ is negative and the solution y should decrease.*

From this we deduce that

$$y(t) \to 0 \text{ if } 0 < y_0 < T, \quad y(t) \to T \text{ if } y_0 = T, \quad y(t) \to K \text{ if } y_0 > T.$$

Furthermore, the direction plot Fig. 7 also supports our observations on the solution behaviour.

The idea of using direction fields and the plot y vs $f(y)$ to study the behaviour of the solution without actually solving the ODE is the heart of this section. In the above examples we saw that there are instances where if we start "close" to a stationary solution, we either converge to the stationary solution, or we move away to another stationary solution, or even possibly the solution $y(t)$ goes to $\pm\infty$ as $t \to \infty$. We characterize this type of phenomena with the following definition.

Definition 6.2 (Stability)**.** *Given an autonomous first order ODE $y' = f(y)$, and a stationary solution y_*. We say that y_* is **asymptotically stable** if there is an $\delta_0 > 0$ (depending only on y_*) such that for any solution $\phi(t)$ to the IVP*

$$y' = f(y) \text{ for } t \in I, \quad y(t_0) = y_0 \text{ with } t_0 \in I,$$

the following property is fulfilled:

$$\boxed{|y_0 - y_*| < \delta_0 \quad \Longrightarrow \quad \phi(t) \to y_* \text{ as } t \to \infty},$$

i.e., if we start close to y_*, we will move towards y_* as time progresses. The stationary solution y_* is called **stable** if for every $\varepsilon > 0$ there is a $\delta > 0$ (depending only on y_* and ε) such that

$$\boxed{|y_0 - y_*| < \delta \quad \Longrightarrow \quad |\phi(t) - y_*| < \varepsilon \quad \forall t \geq t_0},$$

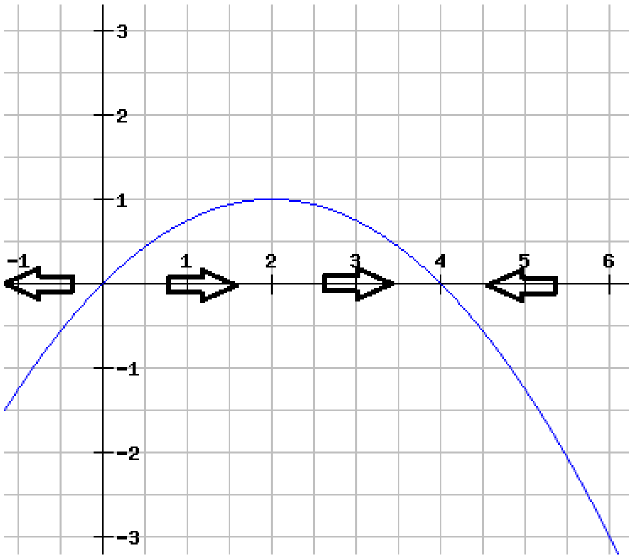


Figure 5: The plot y vs $f(y)$ for the Logistic equation. Arrows indicate the behaviour of the solution.

i.e., if we start close to y_*, then it is guaranteed that we do not move too far away from y_*. However, the trajectory $\{\phi(t)\}_{t\geq t_0}$ does not need to approach y_* as $t \to \infty$. If y_* is not stable, then we call it **unstable**.

Note that for an unstable stationary solution y_*, aside from $y_0 = y_*$, which implies that $y(t) = y_*$ for all $t \geq t_0$, any other initial condition would lead to $|y(t) - y_*| \nrightarrow 0$, and so the solution $y(t)$ will never reach y_* for an unstable stationary solution.

For the Logistic equation, the stationary solution $y_* = 0$ is unstable, and $y_* = K$ is asymptotically stable for any initial condition $y_0 > 0$. For the Logistic equation with threshold, $y_* = 0$ and $y_* = K$ are asymptotically stable and $y_* = T$ is unstable.

6.2 First order linear systems

We now turn to first order linear systems of the form

$$\vec{y}'(t) = \mathbb{A}\vec{y}(t),$$

where $\mathbb{A} \in \mathbb{R}^{2\times 2}$ is a constant matrix with real coefficients. For the upcoming analysis, we will make the following assumptions:

$$\boxed{\mathbb{A} \text{ is non-singular } \Leftrightarrow \det\mathbb{A} \neq 0, \text{ and } 0 \text{ is not an eigenvalue of } \mathbb{A}}.$$

Then, the only possible solution to

$$\mathbb{A}\vec{x} = \vec{0}$$

is the zero vector $\vec{x} = \vec{0}$. This implies that $\vec{0}$ is the **unique critical point**.

Compare to first order equations, the solution $\vec{y}$ to $\vec{y}'(t) = \mathbb{A}\vec{y}(t)$ is a vector, and in this case $\vec{y} = (y_1, y_2)$. Without going to a three dimensional plot $(t, y_1(t), y_2(t))$, we can still obtain information on the behaviour of the solution $\vec{y}(t)$.

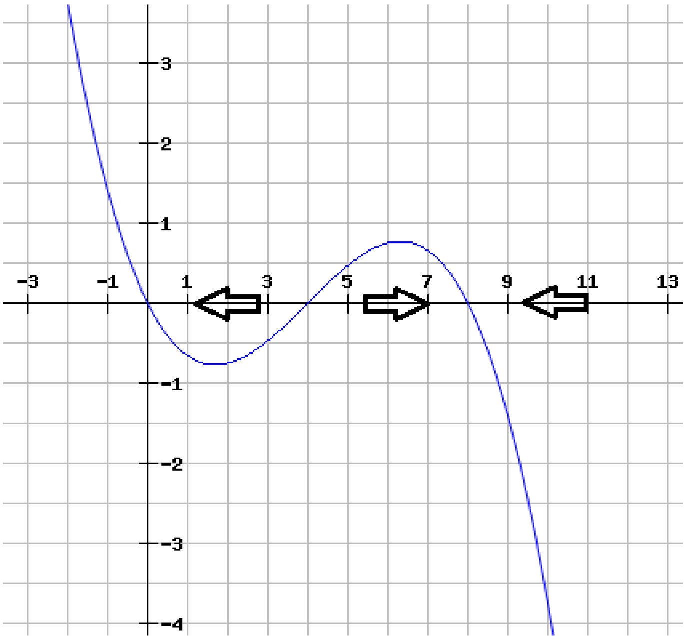


Figure 6: The plot y vs $f(y)$ for the Logistic equation with threshold. Arrows indicate the behaviour of the solution.

Definition 6.3. *We call the (y_1, y_2) plane as the* ***phase plane****. A solution $\vec{y}(t) = (y_1(t), y_2(t))$ for $t \in I$ traces out a curve in the phase plane, which we call a* ***trajectory****. As it is impossible to draw all trajectories, for a representative set of trajectories we call a* ***phase portrait****.*

The phase portrait will yield crucial information about the stability of the critical points - which are determined by the eigenvalues of the matrix $\mathbb{A}$. For a 2×2 matrix, we have the following three possibilities for eigenvalues:

(a) Real, distinct eigenvalues $r_1 \neq r_2$,

(b) Complex conjugate pairs of eigenvalues $r_1 = \lambda + i\mu$, $r_2 = \overline{r_1}$,

(c) Real, repeated eigenvalues $r_1 = r_2$.

6.2.1 Real distinct eigenvalues with the same sign

Recall that if $r_1 \neq r_2$, then the eigenvectors $\vec{\xi}_1$, $\vec{\xi}_2$ corresponding to r_1 and r_2 are linearly independent, and the general solution to $\vec{y}'(t) = \mathbb{A}\vec{y}(t)$ is

$$\vec{y}(t) = c_1\vec{\xi}_1 e^{r_1 t} + c_2\vec{\xi}_2 e^{r_2 t}.$$

If both r_1 and r_2 are negative, then as $t \to \infty$, we have that $\vec{y}(t) \to \vec{0}$. In particular all solutions tend to the critical point. We now illustrate how this happens in the phase portrait.

First, if the initial condition $\vec{y}(0) = \vec{x}$ with $\vec{x}$ is parallel to $\vec{\xi}_1$, then $c_2 = 0$ and so $\vec{y}(t) = c_1\vec{\xi}_1 e^{r_1 t}$ and thus the solution **always stays** on the line spanned by the

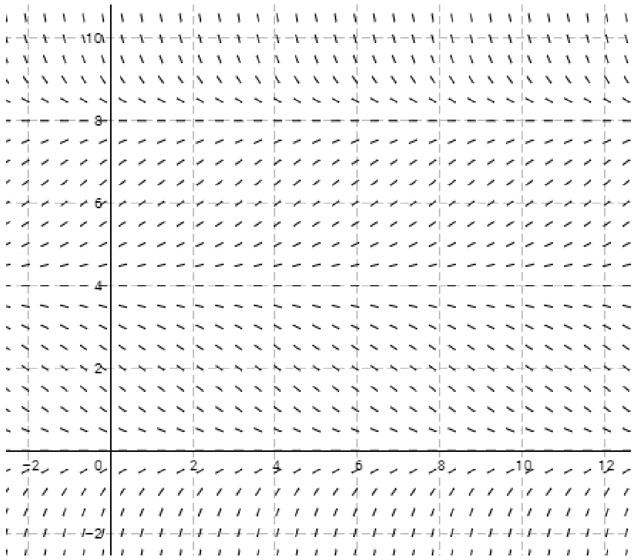


Figure 7: Direction field for the Logistic equation with threshold.

vector $\vec{\xi}_1$. Analogously if $\vec{x}$ is parallel to $\vec{\xi}_2$, then $c_1 = 0$ and $\vec{y}(t)$ always stays on the line spanned by $\vec{\xi}_2$. This is illustrated in Fig. 8 for the matrix $\mathbb{A} = \begin{pmatrix} -1 & 0 \\ -1 & -0.25 \end{pmatrix}$ with eigenvalues $r_1 = -1$, $r_2 = -0.25$ and eigenvectors $\vec{\xi}_1 = (3, 4)$ and $\vec{\xi}_2 = (0.1)$.

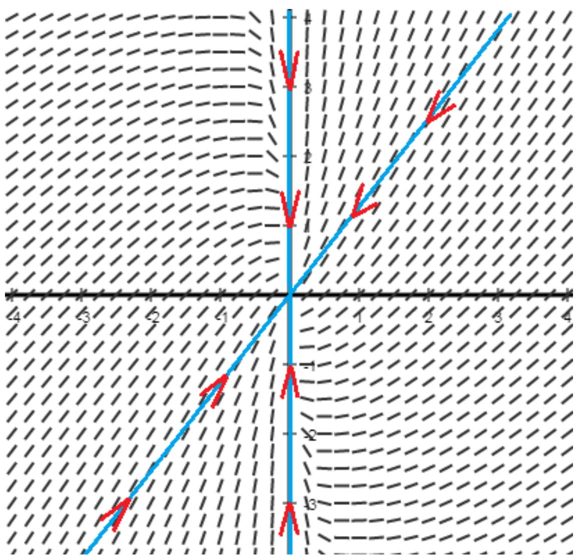


Figure 8: Behaviour of solution to $\vec{y}'(t) = \mathbb{A}\vec{y}(t)$ if eigenvalues are negative and distinct. The blue lines indicate the lines spanned by the eigenvectors $\vec{\xi}_1$ and $\vec{\xi}_2$, which need not be perpendicular.

So how does $\vec{y}(t)$ approaches the critical point $\vec{0}$ if the initial condition $\vec{x}$ does not lie on the lines spanned by $\vec{\xi}_1$ or $\vec{\xi}_2$? We rewrite the expression for the general solution into

$$\vec{y}(t) = e^{r_2 t}\left(c_1\vec{\xi}_1 e^{(r_1-r_2)t} + c_2\vec{\xi}_2\right) \text{ if } r_1 < r_2,$$
$$\vec{y}(t) = e^{r_1 t}\left(c_1\vec{\xi}_1 + c_2\vec{\xi}_2 e^{(r_2-r_1)t}\right) \text{ if } r_2 < r_1.$$

If $r_1 < r_2$, then as $t \to \infty$, the term $c_1\vec{\xi}_1 e^{(r_1-r_2)t}$ is negligible. Therefore, the trajectories **tend towards** the line spanned by $\vec{\xi}_2$. Note that as $t \to -\infty$ (running backwards in time), the term $c_1\vec{\xi}_1 e^{(r_1-r_2)t}$ is dominating the expression. Hence, the trajectories would have nearly the same slope as $\vec{\xi}_1$ as $t \to \infty$. This leads to the picture in Fig. 9.

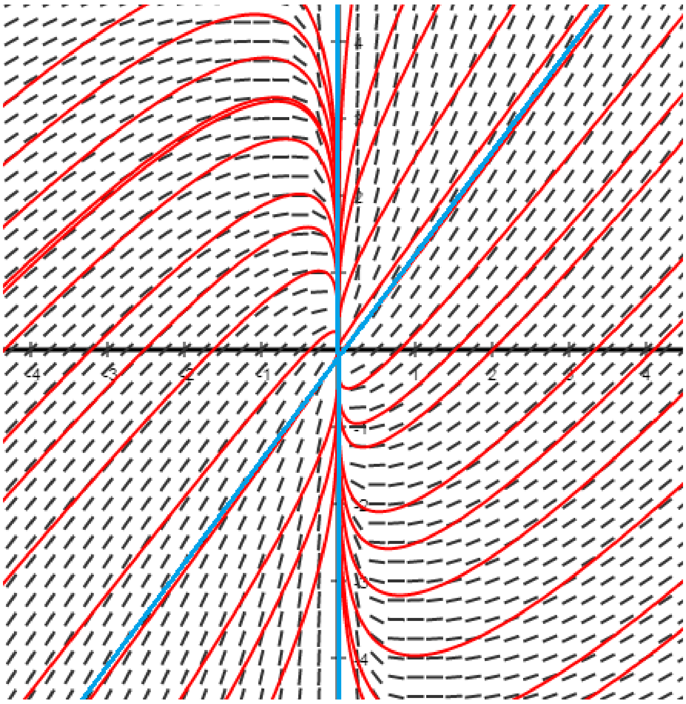


Figure 9: Phase portrait of $\vec{y}'(t) = \begin{pmatrix} -1 & 0 \\ -1 & -0.25 \end{pmatrix}$.

Analogously, if $r_2 < r_1$, then we have a similar situation, namely the trajectories coming from past time $(t \to \infty)$ would have the same slope as $\vec{\xi}_2$, and approaches the slope of $\vec{\xi}_1$ as $t \to \infty$.

For this case where the eigenvalues are negative, we call the critical point $\vec{0}$ a **nodal sink**, since all trajectories point towards $\vec{0}$. If $r_1, r_2 > 0$, then we get the same phase portrait, but the direction of motion is reversed. Hence, trajectories will move away from the critical point $\vec{0}$. In this case we call $\vec{0}$ a **nodal source**.

6.2.2 Real distinct eigenvalues with opposite sign

Without loss of generality, suppose $r_2 < 0 < r_1$. Then, from the expression for the general solution

$$\vec{y}(t) = c_1\vec{\xi}_1 e^{r_1 t} + c_2\vec{\xi}_2 e^{r_2 t},$$

we observe the following:

(a) rewriting

$$\vec{y}(t) = e^{r_1 t}\left(c_1\vec{\xi}_1 + c_2\vec{\xi}_2 e^{(r_2-r_1)t}\right)$$

for $t \to \infty$, with $r_2 - r_1 < 0$, the term $c_2\vec{\xi}_2 e^{(r_2-r_1)t}$ is negligible compared to the other term $c_1\vec{\xi}_1$. Hence, the trajectories approach the line spanned by $\vec{\xi}_1$ as $t \to \infty$.

(b) Writing

$$\vec{y}(t) = e^{r_2 t}\left(c_1\vec{\xi}_1 e^{(r_1-r_2)t} + c_2\vec{\xi}_2\right),$$

for $t \to -\infty$, with $r_1 - r_2 > 0$, the term $c_1\vec{\xi}_1 e^{(r_1-r_2)t}$ is negligible compared to the other term $c_2\vec{\xi}_2$. Hence, the trajectories originate from the line spanned by $\vec{\xi}_2$.

(c) If $c_1 = 0$, and $c_2 \neq 0$, then $\vec{y}(t) = c_2\vec{\xi}_2 e^{r_2 t} \to \vec{0}$ as $t \to \infty$.

(d) If $c_2 = 0$, and $c_1 \neq 0$, then $\vec{y}(t) = c_1\vec{\xi}_1 e^{r_1 t} \to \vec{0}$ as $t \to -\infty$.

These four observations yields the phase portrait in Fig. 10 for the matrix $\mathbb{A} = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix}$ with eigenvalues $r_1 = 2$, $r_2 = -1$ and eigenvectors $\vec{\xi}_1 = (-2, 1)$ (lower blue line), $\vec{\xi}_2 = (-1, 2)$ (higher blue line).

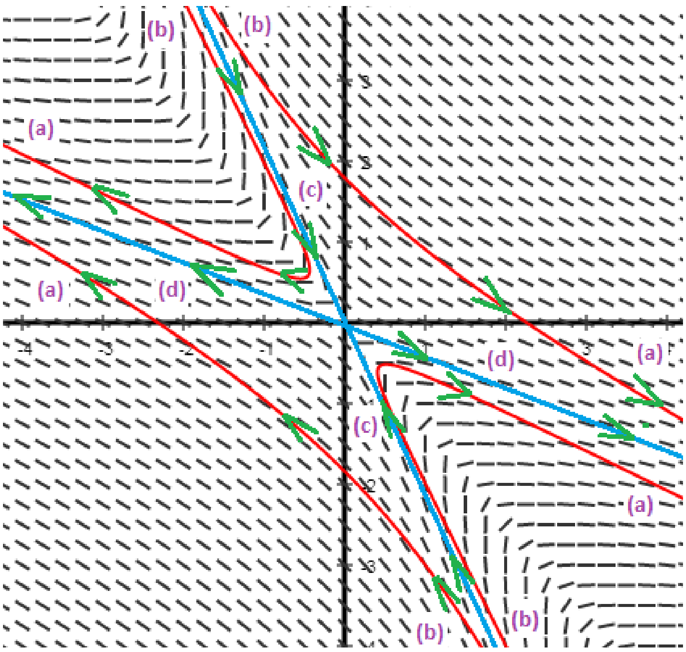


Figure 10: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} 3 & 2 \\ -2 & -2 \end{pmatrix}\vec{y}(t)$.

For the case $r_1 < 0 < r_2$ we obtain the same portrait, but the arrows are reversed.

6.2.3 Equal eigenvalues, two L.I. eigenvector

Assume we have a repeated eigenvalue $r_1 = r_2 = r$, with two linearly independent eigenvectors $vec\xi_1$ and $\vec{\xi}_2$. Then the expression for the general solution is

$$\vec{y}(t) = \left(c_1\vec{\xi}_1 + c_2\vec{\xi}_2\right) e^{rt}.$$

If $r < 0$ then $\vec{y}(t) \to \vec{0}$ as $t \to \infty$ independently of the sign of c_1 and c_2. This means that every trajectory is a <u>**straight line**</u> through the critical point $\vec{0}$, see Fig. 11. Similarly, if $r > 0$, then $\vec{y}(t) \to \vec{0}$ as $t \to -\infty$ independently of the sign of c_1 and c_2. The trajectories are also straight lines through the critical point. In such a case, we call the critical point a <u>**proper node**</u> or a <u>**star point**</u>.

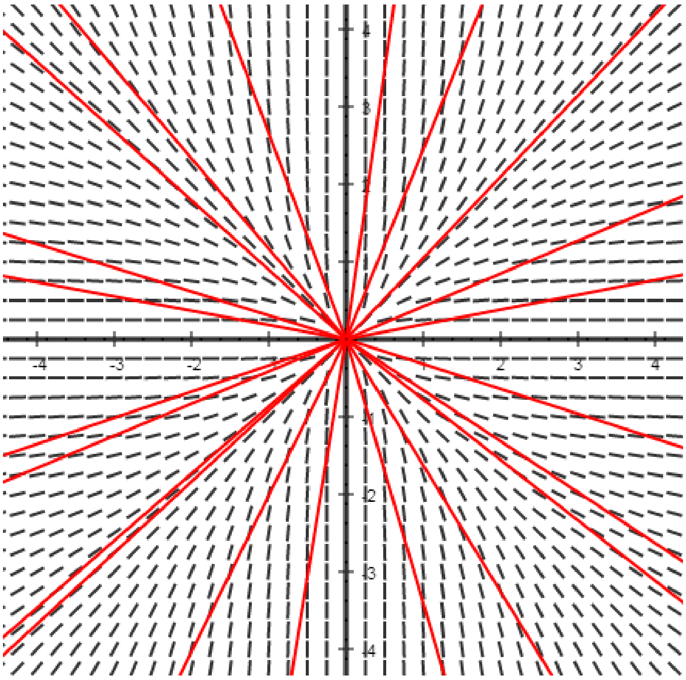


Figure 11: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}\vec{y}(t)$.

6.2.4 Equal eigenvalues, 1 L.I. eigenvector

Let $r_1 = r_2 = r$ be the repeated eigenvalue, and $\vec{\xi}$ the associated eigenvector. Then, the general solution is

$$\begin{aligned}\vec{y}(t) &= c_1\vec{\xi}e^{rt} + c_2\vec{\xi}te^{rt} + c_2\vec{\eta}e^{rt}\\ &= e^{rt}\left((c_1\vec{\xi} + c_2\vec{\eta}) + tc_2\vec{\xi}\right) =: e^{rt}\vec{z}(t),\end{aligned}$$

where we recall that $\vec{\eta}$ is a <u>**generalized eigenvector**</u> to the eigenvalue r, i.e.,

$$(\mathbb{A} - r\mathbb{I})\vec{\xi} = \vec{0}, \quad (\mathbb{A} - r\mathbb{I})\vec{\eta} = \vec{\xi}.$$

To sketch the trajectories, note for fixed c_1 and c_2, the vector function $\vec{z}(t) = (c_1\vec{\xi} + c_2\vec{\eta}) + tc_2\vec{\xi}$ is a straight line through the point $c_1\vec{\xi} + c_2\vec{\eta}$ in the direction $\vec{\xi}$. Writing the solution $\vec{y}(t)$ as $\vec{y}(t) = e^{rt}\vec{z}(t)$ allows us to interpret that $\vec{z}(t)$ determines the **direction** of the trajectory and e^{rt} is the magnitude.

The simplest case is $c_2 = 0$ and $c_1 \neq 0$, then we have

$$\vec{y}(t) = c_1\vec{\xi}e^{rt},$$

which for $r < 0$, we have $\vec{y}(t) \to \vec{0}$ as $t \to \infty$. This can be see in Fig. 12 for the matrix $\mathbb{A} = \begin{pmatrix} 1 & 4 \\ -4 & 7 \end{pmatrix}$ with a repeated eigenvalue $r = -3$ and eigenvector $\vec{\xi} = (-1, 1)$, where along the blue line the trajectories move towards the critical point $\vec{0}$ if $r < 0$.

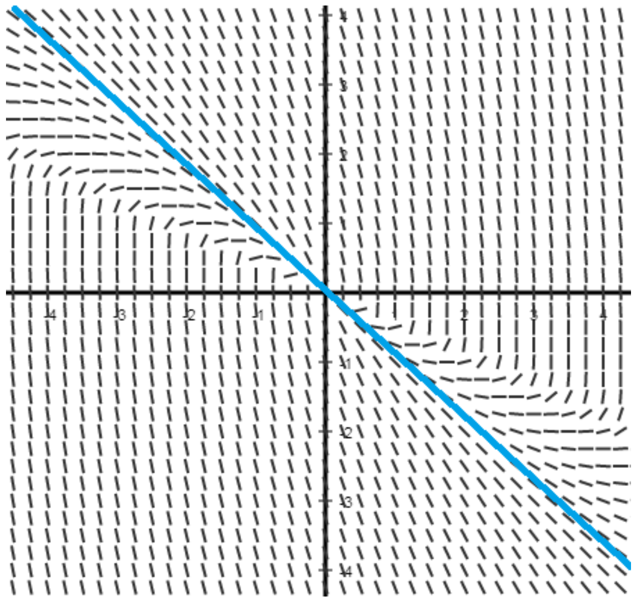


Figure 12: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix}\vec{y}(t)$.

To fill in the rest of the phase portrait, we need to draw the trajectories for $c_1, c_2 \neq 0$. The first thing to draw is the line given by $c_1\vec{\xi} + c_2\vec{\eta} + tc_2\vec{\xi}$, and **take note** which is the direction of **increasing** t. Note that the direction of increasing t is different for $c_2 > 0$ and for $c_2 < 0$. This is given in Fig. 13.

From the expression

$$\vec{y}(t) = (c_1\vec{\xi} + c_2\vec{\eta} + c_2t\vec{\xi})e^{rt},$$

the dominating term is $c_2t\vec{\xi}e^{rt}$ for large (positive/negative) values of t, and so we expect the trajectories to be parallel to the line spanned by $\vec{\xi}$.

As t increases, the direction of the trajectories follow the direction of increasing t, but due to $r < 0$, the magnitude is shrinking exponentially. So we expect that the trajectory to travel along the direction of increasing t, but do a **sharp turn** to go back to the origin. This is reflected in Fig. 14.

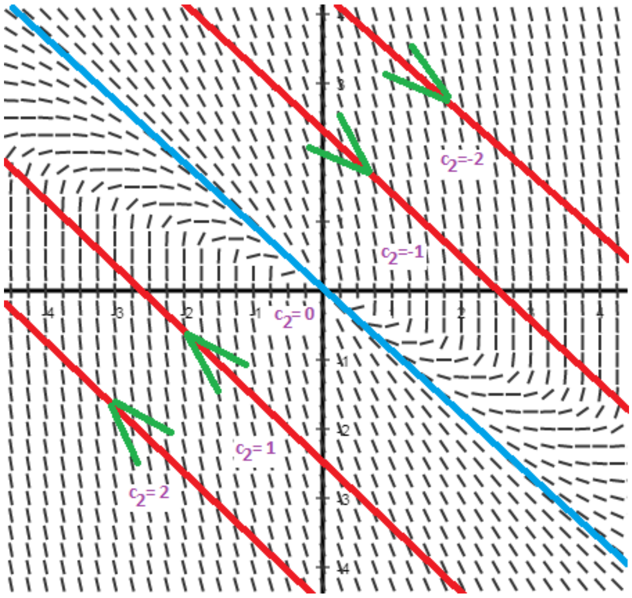


Figure 13: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix} \vec{y}(t)$. The green arrows indicate the direction of increasing t.

It is worth pointing out that not all trajectories **make a turn**. If the trajectory does not **overshoot** the origin, it will go straight towards the critical point, as shown in the blue trajectories in Fig. 14.

For the case $r > 0$, we have the same phase portrait, but the direction of the trajectories are reversed, i.e., the origin is unstable and every trajectories is leaving the origin. See for example Fig. 15 for the matrix $\mathbb{A} = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix}$ with repeated eigenvalue $r = 1$ and eigenvector $\vec{\xi} = (-1, 2)$. This gives an unstable critical point at the origin.

In these cases, where the geo. mult. of the repeated eigenvalue is equal to one, we call the critical point an **improper node** or a **degenerate node**.

6.2.5 Complex eigenvalues, non-zero real part

Suppose $r_1 = \lambda + i\mu$, for $\lambda, \mu \in \mathbb{R}$ with $\lambda \neq 0$, and so $r_2 = \lambda - i\mu$. Denoting the corresponding eigenvectors to be $\vec{\xi}_1 = \vec{u} + i\vec{v}$ with $\vec{\xi}_2 = \vec{u} - i\vec{v}$, and set

$$\vec{y}_1(t) = e^{\lambda t}(\vec{u}\cos(\mu t) - \vec{v}\sin(\mu t)) = e^{\lambda t}\vec{z}_1(t),$$
$$\vec{y}_2(t) = e^{\lambda t}(\vec{u}\sin(\mu t) + \vec{v}\cos(\mu t)) = e^{\lambda t}\vec{z}_2(t),$$

we have the general solution

$$\vec{y}(t) = e^{\lambda t}(c_1\vec{z}_1(t) + c_2\vec{z}_2(t)),$$

for $c_1, c_2 \in \mathbb{R}$ arbitrary. Note that $\vec{z}_1$ and $\vec{z}_2$ are functions of cosine and sine, and therefore are periodic functions in t. We expect that

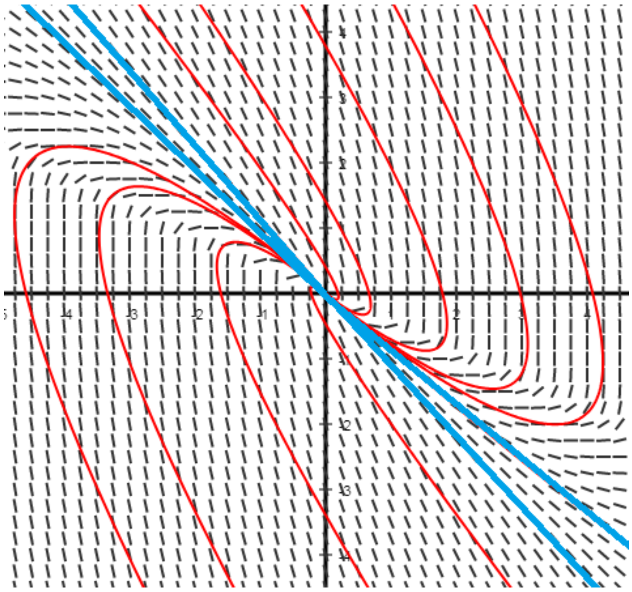


Figure 14: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} 1 & 4 \\ -4 & -7 \end{pmatrix} \vec{y}(t)$. Blue trajectories do not turn around to reach the origin.

(1) if $\lambda < 0$, then $\vec{y}(t) \to \vec{0}$ as $t \to \infty$;

(2) if $\lambda > 0$, then $\vec{y}(t) \to \vec{0}$ as $t \to -\infty$.

So, for $\lambda < 0$, we expect the trajectories to encircle the critical point while tending towards $\vec{0}$ like a spiral, see Fig. 16 for $\mathbb{A} = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix}$ with $r_1 = -1 + \sqrt{2}i$. For $\lambda > 0$, we have the same phase portrait, but we spiral outwards.

One way to determine whether the trajectories spiral "clockwise" or "anticlockwise" is to look at the transformation of a point by the matrix $\mathbb{A}$. For example

$$\vec{y}'(t) = \begin{pmatrix} -0.5 & 1 \\ -1 & -0.5 \end{pmatrix} \vec{y}(t)$$

has a matrix $\mathbb{A}$ with eigenvalues $-0.5 \pm i$. Applying the matrix $\mathbb{A}$ to the point $\vec{x} = (0, 1)$ yields

$$\vec{x}' = \mathbb{A}\vec{x} = \begin{pmatrix} 1 \\ -0.5 \end{pmatrix}.$$

The vector $(1, -0.5)$ provides a direction which the trajectories will be traveling. Therefore, if a trajectory starts at $(0, 1)$, it will move in a direction $(1, -0.5)$ and so the trajectories spiral in a clockwise direction.

We call the critical point $\vec{0}$ a **spiral point** in the case where the eigenvalues of the matrix $\mathbb{A}$ are complex conjugate pairs with non-zero real part. If $\lambda < 0$ we have a **spiral sink** and if $\lambda > 0$ we have a **spiral source**.

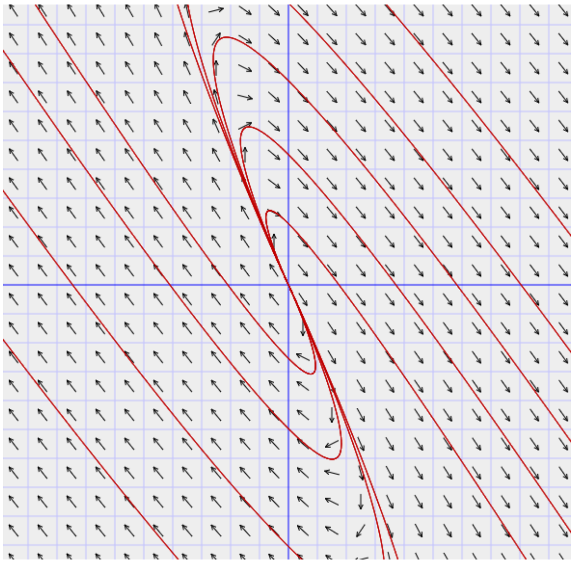


Figure 15: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} 3 & 1 \\ -4 & -1 \end{pmatrix} \vec{y}(t)$.

6.2.6 Purely imaginary eigenvalues

We now consider the case where the eigenvalues of the matrix $\mathbb{A}$ are purely imaginary, i.e., $r_1 = i\mu$, $r_2 = -i\mu$ for $\mu \in \mathbb{R}$. In this case the general solution is

$$\begin{aligned}\vec{y}(t) &= c_1\vec{z}_1(t) + c_2\vec{z}_2(t)\\ &= c_1(\vec{u}\cos(\mu t) - \vec{v}\sin(\mu t)) + c_2(\vec{u}\sin(\mu t) + \vec{v}\cos(\mu t)),\end{aligned}$$

where $\vec{\xi}_1 = \vec{u} + i\vec{v}$ is the eigenvector corresponding to r_1. Due to the periodic nature of $\vec{z}_1$ and $\vec{z}_2$ we expect the trajectories to encircle the critical point, but neither approach nor move away as $t \to \infty$. This can also be seen from rewriting the general solution:

$$\begin{aligned}\vec{y}(t) &= \sqrt{c_1^2+c_2^2}\left(\vec{u}\cos(\mu t)\frac{c_1}{\sqrt{c_1^2+c_2^2}} + \vec{u}\sin(\mu t)\frac{c_2}{\sqrt{c_1^2+c_2^2}}\right)\\ &\quad + \sqrt{c_1^2+c_2^2}\left(\vec{v}\cos(\mu t)\frac{c_2}{\sqrt{c_1^2+c_2^2}} - \vec{v}\sin(\mu t)\frac{c_1}{\sqrt{c_1^2+c_2^2}}\right)\\ &= \sqrt{c_1^2+c_2^2}\left(\vec{u}\sin(\theta+\mu t) + \vec{v}\cos(\theta+\mu t)\right),\end{aligned}$$

where $\theta \in [0, 2\pi]$ is a constant such that $\sin(\theta) = \frac{c_1}{\sqrt{c_1^2+c_2^2}}$ and $\cos(\theta) = \frac{c_2}{\sqrt{c_1^2+c_2^2}}$. The last line shows that the trajectory $\{\vec{y}(t)\}_{t\in I}$ can be seen as a ellipse centered at the origin with a fixed distance that is not changing in time. See for example Fig. 17 for the matrix $\mathbb{A} = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix}$ with eigenvalues $r_1 = i$, $r_2 = -i$.

Again, the direction of the trajectories "clockwise" or "anticlockwise" can be determined by testing one point with the matrix $\mathbb{A}$. For example starting from the

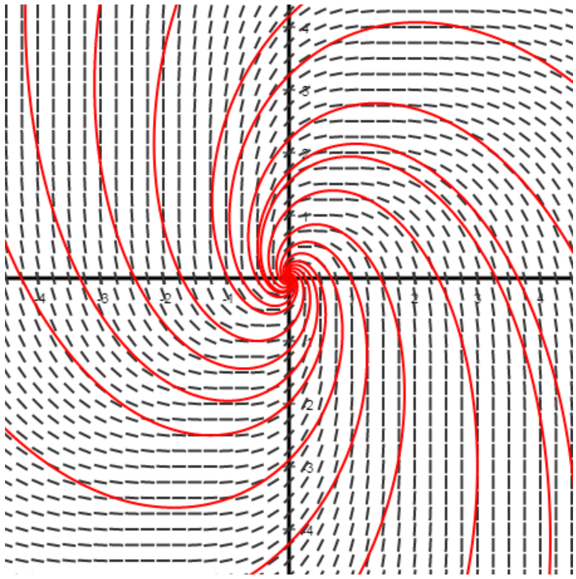


Figure 16: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} -1 & -1 \\ 2 & -1 \end{pmatrix} \vec{y}(t)$.

point $\vec{x} = (0,1)$ we have

$$\vec{x}' = \mathbb{A}\vec{x} = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

From this we expect the trajectories to move clockwise. For the case where the eigenvalues of $\mathbb{A}$ are purely imaginary, we call the critical point a **center**.

Summary. The behaviour of trajectories for the system $\vec{y}'(t) = \mathbb{A}\vec{y}(t)$ where the origin $\vec{0}$ is a critical point depends heavily on the non-zero eigenvalues r_1, r_2. One of the following three situations can occur:

- All trajectories approach $\vec{0}$ as $t \to \infty$, then $\vec{0}$ is either a **nodal sink** or a **spiral sink**.
- All trajectories remains bounded (contained in a bounded set in the phase space) but do not approach $\vec{0}$ as $t \to \infty$. Then $\vec{0}$ is a **centre**.
- Some trajectories (possibly all) except the trajectory $\vec{y}_*(t) = \vec{0}$ for all t, becomes unbounded as $t \to \infty$. Then $\vec{0}$ is either a **nodal source**, a **spiral source** or a **saddle point**.

Note that due to the **uniqueness**, through each point (y_1, y_2) of the phase plane, there is **only** one trajectory passing through that point. This implies that trajectories **do not cross each other**.

Similar to the case of scalar equations, we now give a notion of stability for systems of equations.

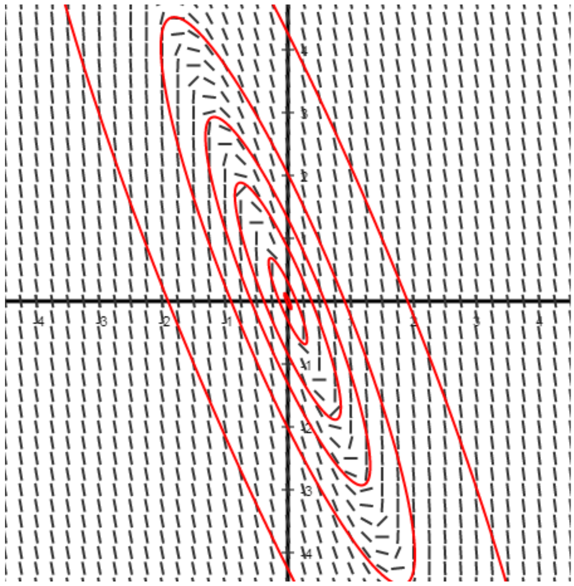


Figure 17: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} 2 & 1 \\ -5 & -2 \end{pmatrix} \vec{y}(t)$.

Definition 6.4 (Stability). *Let $\vec{y}_*$ be a critical point of the autonomous system*

$$\vec{y}'(t) = \vec{f}(\vec{y}(t)) \qquad \text{for } t \geq 0,$$

i.e., $\vec{f}(\vec{y}_*) = \vec{0}$. We say that

*(1) $\vec{y}_*$ is **stable** if for any $\varepsilon > 0$, there exists a $\delta > 0$ (depending on y_* and ε) such that any solution $\vec{y} = \vec{\phi}(t)$ to $\vec{y}'(t) = \vec{f}(\vec{y}(t))$ satisfies*

$$|\phi(0) - \vec{y}_*| < \delta \qquad \Longrightarrow \qquad \left|\vec{\phi}(t) - \vec{y}_*\right| < \varepsilon \quad \forall t \geq 0.$$

1. *$\vec{y}_*$ is **unstable** if it is not stable.*

2. *$\vec{y}_*$ is **asymptotically stable** if it is **stable** and there exists $\delta_0 > 0$ (depending only on y_*) such that*

$$\left|\vec{\phi}(0) - \vec{y}_*\right| < \delta_0 \qquad \Longrightarrow \qquad \vec{\phi}(t) \to \vec{y}_* \text{ as } t \to \infty.$$

Note that asymptotic stability is a stronger property than stability. Furthermore, the stability property means that the trajectories do not have to tend towards $\vec{y}_*$, they just have to remain close by.

We can now classify for $\mathbb{A} \in \mathbb{R}^{2\times 2}$ the stability of the critical point $\vec{0}$:

6.2.7 Zero as one of the eigenvalues

What if 0 is an eigenvalue of $\mathbb{A}$? That is $r_1 = 0$ and $r_2 \neq 0$. Then note that the corresponding eigenvector $\vec{\xi}$ to the eigenvalue 0 satisfies

$$\mathbb{A}\vec{\xi} = \vec{0},$$

Eigenvalues	Type	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 > 0 > r_2$	Saddle	Unstable
$r_1 < r_2 < 0$	Node	Asym. stable
$r_1 = r_2 < 0$	Proper / improper node	Asym. stable
$r_1 = r_2 > 0$	Proper / improper node	Unstable
$r_1 = \lambda + i\mu$	Spiral	Unstable ($\lambda > 0$), Asym. stable ($\lambda < 0$)
$r_1 = i\mu$	Center	Stable

Table 1: Type and stability of the critical point based on the eigenvalues

and so every point on the straight line $\{t\vec{\xi} : t \in \mathbb{R}\}$ is a critical point. For example, consider the matrix

$$\mathbb{A} = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \quad \text{with } r_1 = 0, \quad r_2 = -1, \quad \vec{\xi}_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Thus, all points on the vertical axis $\{t(0,1) : t \in \mathbb{R}\}$ is a critical point. The phase portrait is plotted in Fig. 18. Note that once a trajectory hits the vertical axis, it stops and does not appear on the other side of the axis. Another example is with the matrix $\mathbb{A} = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}$ whose phase portrait is given in Fig. 19.

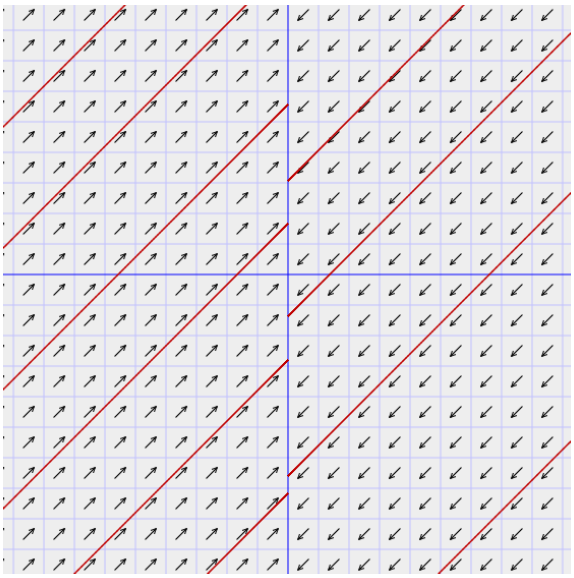


Figure 18: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} -1 & 0 \\ -1 & 0 \end{pmatrix} \vec{y}(t)$.

The question is whether the critical point $\vec{0}$ is stable or unstable or asym. stable. If $r_2 < 0$, then $\vec{0}$ is stable but not asym. stable, and if $r_2 > 0$, then $\vec{0}$ is unstable.

Example 6.4. *For*

$$\vec{y}'(t) = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} \vec{y}(t) =: \mathbb{A}\vec{y}(t)$$

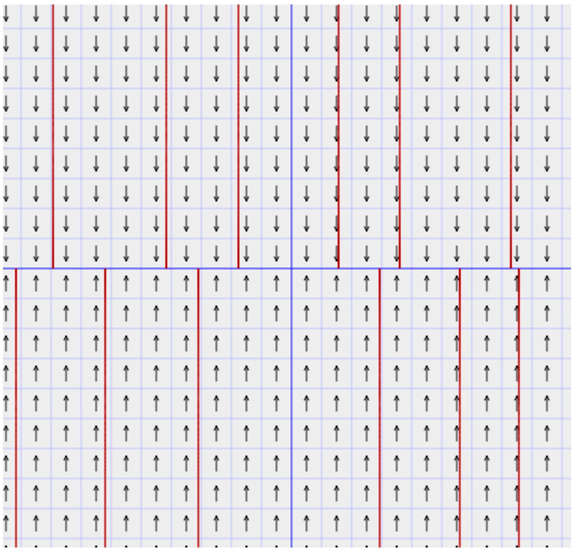


Figure 19: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \vec{y}(t)$.

determine the critical points and their stability. Draw the phase portrait.

(1) *$\vec{x}$ is a critical point if $\mathbb{A}\vec{x} = \vec{0}$. Since* $\det\mathbb{A} = 6 \neq 0$, *the matrix $\mathbb{A}$ is invertible and so $\vec{0}$ is the only critical point.*

(2) *Computing the eigenvalues, we find that*

$$\det(\mathbb{A} - r\mathbb{I}) = (r+6)(r+1) = 0$$

and so $r_1 = -6$ and $r_2 = -1$. These are real distinct eigenvalues and thus $\vec{0}$ is an asym. stable node.

(3) *Computing the eigenvectors yields*

$$\vec{\xi}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} 1/4 \\ 1 \end{pmatrix},$$

then the general solution is

$$\vec{y}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-6t} + c_2 \begin{pmatrix} 1/4 \\ 1 \end{pmatrix} e^{-t} = e^{-t} \left(c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t} + c_2 \begin{pmatrix} 1/4 \\ 1 \end{pmatrix} \right).$$

For large $t > 0$, $\vec{y}(t) \to \vec{0}$ with trajectories parallel to $\begin{pmatrix} 1/4 \\ 1 \end{pmatrix}$. For large $t < 0$, the dominating term is $c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-5t}$ and so the trajectories are parallel to $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$. This yields the phase portrait Fig. 20.

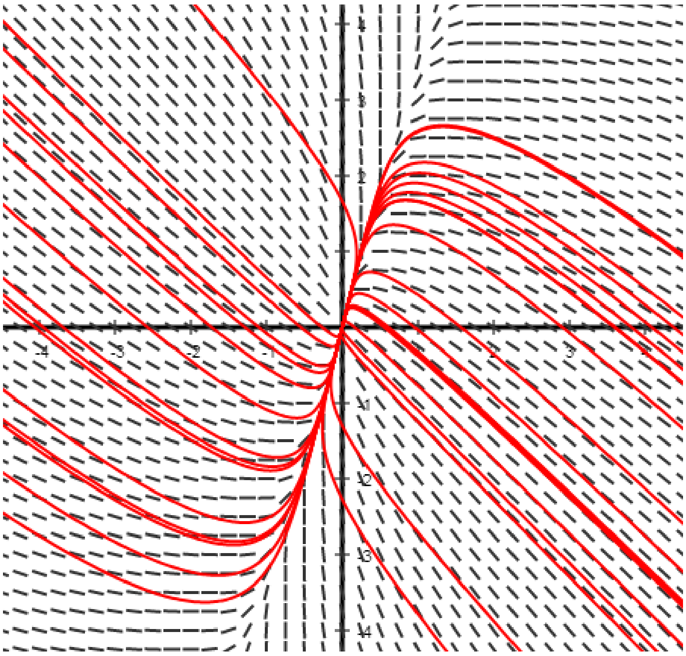


Figure 20: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} -5 & 1 \\ 4 & -2 \end{pmatrix} \vec{y}(t)$.

Example 6.5. *For*

$$\vec{y}'(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \vec{y}(t) - \begin{pmatrix} 4 \\ 0 \end{pmatrix} =: \mathbb{A}\vec{y}(t) - \begin{pmatrix} 4 \\ 0 \end{pmatrix},$$

find all critical points and examine their stability. Draw the phase portrait.

Note that this is a non-homogeneous system of equations. Nevertheless we can still find the critical points. We see that $\vec{x}$ is a critical point if

$$\mathbb{A}\vec{x} - \begin{pmatrix} 4 \\ 0 \end{pmatrix} = \vec{0} \quad \Longrightarrow \quad \vec{x} = \mathbb{A}^{-1} \begin{pmatrix} 4 \\ 0 \end{pmatrix}.$$

Since $\det\mathbb{A} = -2$, *the matrix is invertible and we find the critical point $\vec{y}_*$ to be*

$$\vec{y}_* = \begin{pmatrix} 2 \\ 2 \end{pmatrix}.$$

Next, to examine the stability of $\vec{y}_*$, we transform the system. Set

$$\vec{z}(t) = \vec{y}(t) - \vec{y}_* \quad \Longrightarrow \quad \vec{z}'(t) = \mathbb{A}\vec{z}(t).$$

Note that the critical point is now $\vec{z}_* = \vec{0}$. Therefore stability of $\vec{z}_*$ is equivalent to stability of $\vec{y}_*$.

Computing the eigenvalues and eigenvectors, we find that

$$\det(\mathbb{A} - r\mathbb{I}) = r^2 - 2 = 0 \quad \Longrightarrow \quad r_1 = \sqrt{2}, \quad r_2 = -\sqrt{2},$$

and so we have distinct real eigenvalues with different sign. Therefore, $\vec{z}_*$ (and so $\vec{y}_*$) is a saddle point which is unstable. We also find that

$$\vec{\xi}_1 = \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix},$$

with the general solution

$$\begin{aligned} \vec{z}(t) &= c_1 \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} e^{\sqrt{2}t} + c_2 \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{-\sqrt{2}t} \\ \Leftrightarrow \vec{y}(t) &= c_1 \begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix} e^{\sqrt{2}t} + c_2 \begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix} e^{-\sqrt{2}t} + \begin{pmatrix} 2 \\ 2 \end{pmatrix}. \end{aligned}$$

For $t \to \infty$, the trajectories are parallel to $\begin{pmatrix} 1 \\ \sqrt{2}-1 \end{pmatrix}$ and for $t \to -\infty$, the trajectories are parallel to $\begin{pmatrix} 1 \\ -1-\sqrt{2} \end{pmatrix}$. This leads to the following phase portrait for $\vec{z}$ in Fig. 21 and for $\vec{y}$ in Fig. 22.

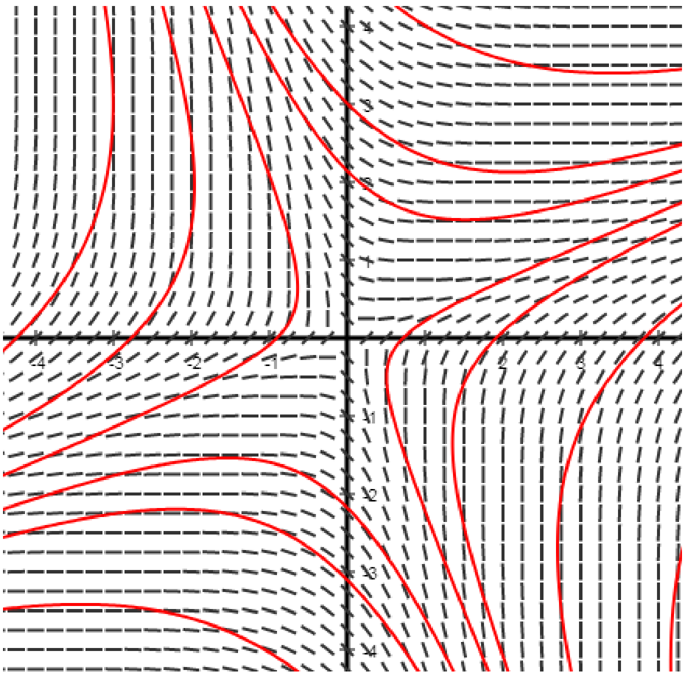


Figure 21: Phase portrait for $\vec{z}'(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \vec{z}(t)$.

6.3 Locally linear systems

For linear homogeneous systems with constant coefficients:

$$\vec{y}'(t) = \mathbb{A}\vec{y}(t), \quad \mathbb{A} \in \mathbb{R}^{2\times 2},$$

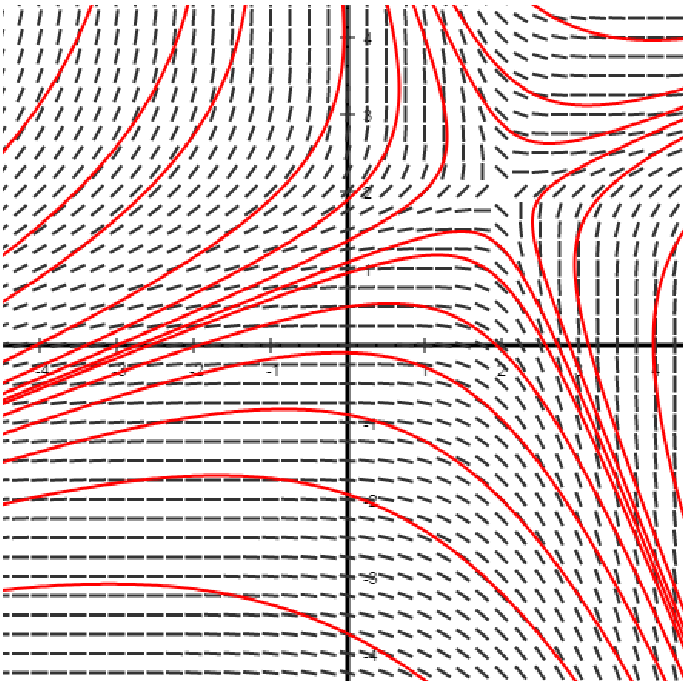


Figure 22: Phase portrait for $\vec{y}'(t) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} \vec{y}(t) - \begin{pmatrix} 4 \\ 0 \end{pmatrix}$.

the behaviour of trajectories in the phase plane can be more or less determined by the eigenvalues of $\mathbb{A}$. Hence, the stability of critical points can also be deduced. However, for nonlinear autonomous systems, this is **not true** due to the following reasons:

- several or many critical points competing for influence of the trajectories;
- nonlinearity far away can affect stability of critical points.

To investigate nonlinear systems, one idea to to approximate them with linear systems. However, when approximating we have to introduce the influence of small perturbations, which can have significant effects on stability of critical points.

6.3.1 Perturbations for linear systems

For a matrix $\mathbb{A} \in \mathbb{R}^{2\times 2}$ with eigenvalues r_1 and r_2, even small changes in the entries of $\mathbb{A}$ will lead to changes in the eigenvalues. For example, the eigenvalues r_1 may change sign from positive to negative and vice versa, or even become complex-valued. We give two examples on how this changes the stability of the critical point $\vec{0}$.

Example 6.6. *Let $\mathbb{A}$ be the original matrix with eigenvalues r_1 and r_2, while $\mathbb{A}^*$ be the perturbed matrix with eigenvalues r_1^* and r_2^*. Suppose $r_1 = i\mu$ and $r_2 = -i\mu$ are purely imaginary. Then, the critical point $\vec{0}$ is a stable center. However, under small perturbations, the new eigenvalues r_1^* and r_2^* may still be complex conjugate pairs but most likely they will have **non-zero** real parts, as shown in Fig. 23. This changes the trajectories from ellipses orbiting the critical point $\vec{0}$ into spirals.*

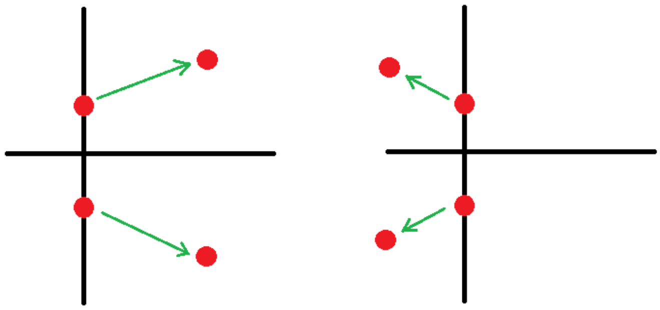


Figure 23: The horizontal axis is λ and the vertical axis is μ. Perturbation of purely imaginary eigenvalues may give rise to complex conjugate pairs with non-zero real part, changing the critical point $\vec{0}$ from a stable center to a (asym. stable or unstable) spiral point.

Example 6.7. *Let $\mathbb{A}$ be the original matrix with eigenvalues r_1 and r_2, while $\mathbb{A}^*$ be the perturbed matrix with eigenvalues r_1^* and r_2^*. Suppose $r_1 = r_2 < 0$, so that we have a repeated eigenvalue and the critical point $\vec{0}$ is a proper node that is asym. stable. Then the following can happen (see Fig. 24):*

(1) the new eigenvalues r_1^* and r_2^* are real but distinct, while the type of the critical point $\vec{0}$ can remain a node (if $r_1^* < r_2^* < 0$), or it might change into an unstable saddle point (if $r_1 = r_2$ is close to zero and $r_1^* < 0 < r_2^*$).

(2) the new eigenvalues are complex conjugate with non-zero imaginary parts. This changes the critical point $\vec{0}$ from a node to a spiral.

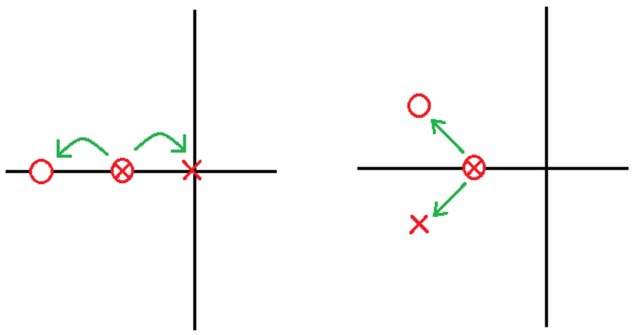


Figure 24: The horizontal axis is λ and the vertical axis is μ. Perturbation of repeated eigenvalues may give rise to (1) distinct real eigenvalues or (2) complex conjugate pairs with non-zero real part. This then changes the critical point $\vec{0}$ from a node to a spiral (if the new eigenvalues are complex).

6.3.2 Linear approximations to nonlinear systems

We now consider a nonlinear autonomous **two-dimensional** system of equations

$$\vec{y}\,'(t) = \vec{f}(\vec{y}(t)), \quad \vec{y} \in \mathbb{R}^2,$$

with a critical point at $\vec{x}_*$, i.e., $\vec{f}(\vec{x}_*) = \vec{0}$. Without loss of generality we can take $\vec{x}_*$ as the origin. Since, if $\vec{x}_* \neq \vec{0}$, then we can use the variable $\vec{z} = \vec{y} - \vec{x}_*$ and observe that

$$\vec{z}'(t) = \vec{y}'(t) = \vec{f}(\vec{y}(t)) = \vec{f}(\vec{z}(t) + \vec{x}_*) =: \vec{h}(\vec{z}(t)),$$

with

$$\vec{h}(\vec{0}) = \vec{f}(\vec{x}_*) = \vec{0}.$$

That is, $\vec{0}$ is a critical point of $\vec{z}'(t) = \vec{h}(\vec{z}(t))$.

Definition 6.5. *A critical point $\vec{x}_*$ is* ***isolated*** *if there is a circle about $\vec{x}_*$ where no other critical points are in the circle.*

The main idea of this section is to investigate the stability of the critical point $\vec{x}_*$ to the nonlinear system by studying an associated linear system. Using Taylor's theorem we can expand

$$\boxed{\vec{f}(\vec{y}) = \vec{f}(\vec{0}) + \mathrm{D}\vec{f}(\vec{0})\vec{y} + \vec{g}(\vec{y}) = \mathrm{D}\vec{f}(\vec{0})\vec{y} + \vec{g}(\vec{y})},$$

where $\mathrm{D}\vec{f}$ is the **Jacobian matrix** of $\vec{f}$ defined as

$$\mathrm{D}\vec{f}(\vec{x}) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{pmatrix},$$

and $\vec{g}(\vec{y})$ is a vector containing all higher order derivatives. Using this gives

$$\boxed{\vec{y}'(t) = \vec{f}(\vec{y}(t)) = \mathrm{D}\vec{f}(\vec{0})\vec{y}(t) + \vec{g}(\vec{y}(t))}.$$

The idea is that if $\vec{g}(\vec{y}(t))$ is "small" for trajectories $\{\vec{y}(t) : t \in I\}$ close to the critical point $\vec{0}$, then the nonlinear system $\vec{y}'(t) = \vec{f}(\vec{y}(t)) = \mathrm{D}\vec{f}(\vec{0})\vec{y}(t) + \vec{g}(\vec{y}(t))$ should be well approximated by the linear system

$$\boxed{\vec{y}'(t) = \mathrm{D}\vec{f}(\vec{0})\vec{y}(t)},$$

close to the critical point $\vec{0}$. This motivates the following definition.

Definition 6.6 (Locally linear systems)**.** *We say that the nonlinear system*

$$\vec{y}'(t) = \vec{f}(\vec{y}(t)), \quad \vec{y} \in \mathbb{R}^2,$$

with an isolated critical point $\vec{0}$ is ***locally linear*** *near $\vec{0}$ if there is a 2×2 matrix $\mathbb{A} \in \mathbb{R}^{2\times 2}$ and a vector function $\vec{g}(\vec{y}(t))$ such that*

$$\boxed{\vec{y}'(t) = \mathbb{A}\vec{y}(t) + \vec{g}(\vec{y}(t)), \quad \lim_{\vec{y}\to\vec{0}} \frac{\|\vec{g}(\vec{y})\|}{\|\vec{y}\|} = 0},$$

where $\|\vec{y}\| = \sqrt{y_1^2 + y_2^2}$ and $\|\vec{g}(\vec{y})\| = \sqrt{(g_1(y_1, y_2))^2 + (g_2(y_1, y_2))^2}$.

The condition

$$\lim_{\vec{y}\to\vec{0}} \frac{\|\vec{g}(\vec{y})\|}{\|\vec{y}\|} = 0$$

is how we quantify "smallness" of $\vec{g}(\vec{y})$, which means that the vector $\vec{g}(\vec{y})$ has smaller influence on the trajectories that the linear part $\mathbb{A}\vec{y}$. Furthermore, it is clear that the matrix $\mathbb{A}$ should be the Jacobian matrix $\mathrm{D}\vec{f}(\vec{0})$.

Example 6.8. *Consider the system*

$$\vec{y}'(t) = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix} \vec{y}(t) + \begin{pmatrix} -y_1^2 - y_1 y_2 \\ -0.75 y_1 y_2 - 0.25 y_2^2 \end{pmatrix}.$$

Then, setting

$$\mathbb{A} = \begin{pmatrix} 1 & 0 \\ 0 & 0.5 \end{pmatrix}, \quad \vec{g}(\vec{y}) = \begin{pmatrix} -y_1^2 - y_1 y_2 \\ -0.75 y_1 y_2 - 0.25 y_2^2 \end{pmatrix},$$

we proceed to check that

(1) $\vec{0}$ *is indeed a critical point;*

(2) The other critical points are $(0,2)$, $(1,0)$ *and* $(1/2,1/2)$. *Thus,* $\vec{0}$ *is an isolated critical point;*

1. *To verify the condition*

$$\lim_{\vec{y}\to\vec{0}} \frac{\|\vec{g}(\vec{y})\|}{\|\vec{y}\|} = 0,$$

it is useful to use polar coordinates: $y_1 = r\cos\theta$ *and* $y_2 = r\sin\theta$. *Then* $\|\vec{y}\| = r$ *and*

$$\begin{aligned} \|\vec{g}(\vec{y})\|^2 &= (r^2\cos^2\theta + r^2\cos\theta\sin\theta)^2 + (0.75 r^2\cos\theta\sin\theta + 0.25 r^2\sin^2\theta)^2 \\ &= r^4\left((\cos^2\theta + \cos\theta\sin\theta)^2 + (0.75\cos\theta\sin\theta + 0.25\sin^2\theta)^2\right). \end{aligned}$$

So that

$$\lim_{\vec{y}\to\vec{0}} \frac{\|\vec{g}(\vec{y})\|}{\|\vec{y}\|} = \lim_{r\to 0} r\left((\cos^2\theta + \cos\theta\sin\theta)^2 + (0.75\cos\theta\sin\theta + 0.25\sin^2\theta)^2\right)^{1/2} = 0.$$

Hence, the system is locally linear near $\vec{0}$.

Remark 6.1. *(a) Note that if the critical point is* $\vec{x}_*$ *instead of* $\vec{0}$, *then we want to derive a linear system close to* $\vec{x}_*$. *It is convenient to set* $\vec{z} = \vec{y} - \vec{x}_*$, *so that* $\vec{0}$ *is the critical point for the system in the variable* $\vec{z}$. *Then, as before by Taylor expansion*

$$\begin{aligned} \vec{y}'(t) = \vec{z}'(t) = \vec{f}(\vec{z} + \vec{x}_*) &\approx \vec{f}(\vec{x}_*) + \mathrm{D}\vec{f}(\vec{x}_*)\vec{z} + \vec{H}(\vec{z}) \\ &= \mathrm{D}\vec{f}(\vec{x}_*)(\vec{y} - \vec{x}_*) + \vec{H}(\vec{y} - \vec{x}_*), \end{aligned}$$

where $\vec{H}$ *contains terms of higher derivatives. The small condition now is given as*

$$\lim_{\vec{y}-\vec{x}_*\to\vec{0}} \frac{\|\vec{H}(\vec{y} - \vec{x}_*)\|}{\|\vec{y} - \vec{x}_*\|} = \lim_{\vec{z}\to\vec{0}} \frac{\|\vec{H}(\vec{z})\|}{\|\vec{z}\|} = 0.$$

(b) *If $\vec{f}$ is a twice continuously differentiable vector function, then $\vec{y}'(t) = \vec{f}(\vec{y}(t))$ is automatically locally linear near the critical point $\vec{0}$. That is, there is no need to check the condition $\lim_{\vec{y}\to\vec{0}} \frac{\|\vec{g}(\vec{y})\|}{\|\vec{y}\|}$.*

(c) *The matrix $\mathrm{D}\vec{f}(\vec{0})$ is a 2×2 matrix with constant coefficients.*

In showing that $\vec{g}(\vec{y})$ is "small" compared to the linear part $\mathrm{D}\vec{f}(\vec{0})\vec{y}$, we hope that the trajectories near the critical point $\vec{0}$ for the locally linear system $\vec{y}'(t) = \mathrm{D}\vec{f}(\vec{0})\vec{y} + \vec{g}(\vec{y})$ can be well approximated by studying the linear system $\vec{y}'(t) = \mathrm{D}\vec{f}(\vec{0})\vec{y}(t)$, which we have studied previously. This turns out to be true in most cases, but not all.

Theorem 6.1 (Stability for locally linear system). *Let r_1 and r_2 be the eigenvalues of $\mathrm{D}\vec{f}(\vec{0})$. Then, **aside** from the cases (a) $r_1 = i\mu$ for $\mu \in \mathbb{R}$ (and so $r_2 = -i\mu$) and (b) $r_1 = r_2 \in \mathbb{R}$, the type and stability of the critical point $\vec{0}$ for the locally linear system $\vec{y}'(t) = \mathrm{D}\vec{f}(\vec{0})\vec{y} + \vec{g}(\vec{y})$ and the linear system $\vec{y}'(t) = \mathrm{D}\vec{f}(\vec{0})\vec{y}(t)$ are the **same**.*

In particular, we have the following table:

r_1, r_2	Type	Stability
$r_1 > r_2 > 0$	Node	Unstable
$r_1 < r_2 < 0$	Node	Asym. stable
$r_1 < 0 < r_2$	Saddle	Unstable
$r_1, r_2 = \lambda \pm i\mu$	Spiral	Unstable ($\lambda > 0$), Asym. stable ($\lambda < 0$)

For the other cases, we have

r_1, r_2	Type	Stability
$r_1 = r_2 > 0$	Node or Spiral	Unstable
$r_1 = r_2 < 0$	Node or Spiral	Asym. stable
$r_1, r_2 = \pm i\mu$	Center or Spiral	Undetermined

The proof of the theorem is beyond the scope of this course. We make the following observations:

(1) Compare to Table 1, the type and stability of the critical point for the locally linear system are rather similar. Except for the case of equal eigenvalues and purely imaginary eigenvalues.

(2) The reason for this is that even though the nonlinear term $\vec{g}(\vec{y})$ is small compare to the linear term $\mathrm{D}\vec{f}(\vec{0})$, its influence on the eigenvalues can be large if we have purely imaginary eigenvalues or repeated eigenvalues, as discussed in Sec. 6.3.1.

(3) Small nonlinear terms may change the stable center (if we have purely imaginary eigenvalues) into a spiral point - which can be unstable or asym. stable. Therefore, when we computed that $\mathrm{D}\vec{f}(\vec{0})$ has purely imaginary eigenvalues, the theorem cannot determine the type and stability of the critical point.

(4) A similar thing happens for the case of repeated eigenvalues, where the nonlinear terms may change the node into a spiral. However, things are a bit better regarding the stability as the property of asym. stable or unstable remain unchanged.

We will present a method later to deduce the stability of the critical point when we encounter the case of purely imaginary eigenvalues. But first, we apply this to a physical application - the damped pendulum.

6.3.3 Damped pendulum

Recall from Chapter 1, the equation for the motion of a damped pendulum is

$$\theta'' + \gamma\theta' + w^2 \sin\theta = 0,$$

where θ is the angle the pendulum makes with the vertical line, and the parameter $\gamma > 0$ is a damping factor taking into account friction forces. As this is a second order nonlinear equation, we can express this into a first order system: Introducing the notation

$$y_1 = \theta, \quad y_2 = \theta',$$

then

$$\vec{y}'(t) = \begin{pmatrix} y_2 \\ -w^2 \sin y_1 - \gamma y_2 \end{pmatrix} =: \vec{f}(\vec{y}). \tag{6.1}$$

Step 1. The critical points of the above systems satisfy

$$y_2 = 0, \quad \sin y_1 = 0,$$

and so the critical points are $(\pm n\pi, 0)$ for $n \in \mathbb{Z}$.

Step 2. Check to see if (6.1) is locally linear near the critical points. First for $(0,0)$ we write (6.1) as

$$\vec{y}'(t) = \begin{pmatrix} 0 & 1 \\ -w^2 & \gamma \end{pmatrix} \vec{y} - w^2 \begin{pmatrix} 0 \\ \sin y_1 - y_1 \end{pmatrix} =: \mathbb{A}\vec{y}(t) + \vec{g}(\vec{y}(t)).$$

Then, for $\vec{y}$ close to $(0,0)$ we check

$$\lim_{\vec{y} \to \vec{0}} \frac{||\vec{g}(\vec{y})||}{||\vec{y}||} = 0.$$

For small y_1, by Taylor's expansion we have

$$\sin y_1 = y_1 - \frac{y_1^3}{3!} + \ldots,$$

and by polar coordinates $y_1 = r\cos\phi$, $y_2 = r\sin\phi$, we find that

$$||\vec{g}(\vec{y})|| = w^2|\sin y_1 - y_1| = w^2\left|r^3\frac{\cos^3\phi}{3!} - r^5\frac{\cos^5\phi}{5!} + \dots\right|,$$

and so

$$\lim_{\vec{y}\to 0}\frac{||\vec{g}(\vec{y})||}{||\vec{y}||} = \lim_{r\to 0} w^2 r^2\left|\frac{\cos^3\phi}{3!} - r^2\frac{\cos^5\phi}{5!} + \dots\right| = 0.$$

That is, (6.1) is locally linear near $(0,0)$. What about near $(\pi,0)$? For this we employ a transformation

$$\vec{z} = \vec{y} - \begin{pmatrix} \pi \\ 0 \end{pmatrix},$$

and so if $\vec{z}$ is small, then $\vec{y}$ is close to the critical point $(\pi,0)$. Then, it is clear that

$$\vec{z}'(t) = \vec{y}'(t) = \vec{f}(\vec{y}(t)) = \vec{f}(\vec{z}(t) + (\pi,0)) = \begin{pmatrix} z_2 \\ -w^2\sin(z_1+\pi) - \gamma z_2 \end{pmatrix} = \begin{pmatrix} z_2 \\ w^2\sin z_1 - \gamma z_2 \end{pmatrix},$$

upon using the addition formula for $\sin(\cdot)$:

$$\sin(z_1+\pi) = \sin z_1\cos\pi + \cos z_1\sin\pi = -\sin z_1.$$

Thus,

$$\vec{z}'(t) = \begin{pmatrix} 0 & 1 \\ w^2 & -\gamma \end{pmatrix}\vec{z}(t) + \begin{pmatrix} 0 \\ w^2(\sin z_1 - z_1) \end{pmatrix} =: \mathbb{B}\vec{z}(t) + \vec{h}(\vec{z}(t)). \tag{6.2}$$

Similar arguments as before show that

$$\lim_{\vec{z}\to\vec{0}}\frac{||\vec{h}(\vec{z})||}{||\vec{z}||} = 0$$

if we use polar coordinates and Taylor expansion for $\sin(\cdot)$ for small values of z_1. Hence, (6.1) is also locally linear near $(\pi,0)$.

The same arguments can be used to show that (6.1) is locally linear near all the critical points $(\pm n\pi,0)$ for $n\in\mathbb{Z}$. We now investigate the stability of the associated linear system and infer results for the locally linear system.

Step 3. Near the critical point $(0,0)$, (6.1) can be expressed as the locally linear system

$$\vec{y}'(t) = \begin{pmatrix} 0 & 1 \\ -w^2 & \gamma \end{pmatrix}\vec{y} - w^2\begin{pmatrix} 0 \\ \sin y_1 - y_1 \end{pmatrix} =: \mathbb{A}\vec{y}(t) + \vec{g}(\vec{y}(t)).$$

Note that one can also obtain the matrix $\mathbb{A}$ by computing the Jacobian matrix for $\vec{f}$, which we will do below:

$$\mathrm{D}\vec{f}(\vec{y}) = \begin{pmatrix} 0 & 1 \\ -w^2\cos y_1 & -\gamma \end{pmatrix}.$$

At the critical point $(0,0)$ we see that the Jacobian matrix $\mathrm{D}\vec{f}(\vec{0})$ coincides with $\mathbb{A}$. Now by computing the eigenvalues of $\mathbb{A}$, we first determine the type and stability of the critical point $\vec{0}$ to the linear system $\vec{y}'(t) = \mathbb{A}\vec{y}(t)$. We have

$$\det(\mathbb{A} - r\mathbb{I}) = r^2 + \gamma r + w^2 = 0,$$

and so

$$r_1 = -\frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 - 4w^2}, \quad r_2 = -\frac{\gamma}{2} - \frac{1}{2}\sqrt{\gamma^2 - 4w^2}.$$

The classification for the type and stability of the critical point $(0,0)$ is as follows:

(1) if $\gamma^2 > 4w^2$, then r_1, r_2 are distinct negative eigenvalues and $\vec{0}$ is an asym. stable node.

(2) if $\gamma^2 = 4w^2$, then r_1, r_2 are equal but negative eigenvalues and $\vec{0}$ is an asym. stable node.

(3) if $\gamma^2 < 4w^2$, then r_1, r_2 are complex conjugate pairs of eigenvalues with negative real part, and $\vec{0}$ is an asym. stable spiral.

In fact the same classification holds for all critical points of the form $(\pm 2m\pi, 0)$ for $m \in \mathbb{Z}$, since $\mathrm{D}\vec{f}((\pm 2m\pi, 0)) = \mathbb{A}$. Then, by Thm. 6.1, the critical points $(\pm 2m\pi, 0)$ for $m \in \mathbb{Z}$ to the system (6.1) has the same type and stability as stated above.

Now we look at the critical point $(\pi, 0)$, which we transform to $(0,0)$ when studying the locally linear system (6.2). Observe that

$$\mathrm{D}\vec{f}((\pi, 0)) = \begin{pmatrix} 0 & 1 \\ w^2 & -\gamma \end{pmatrix} = \mathbb{B},$$

and the eigenvalues for $\mathbb{B}$ are

$$r_1 = -\frac{\gamma}{2} + \frac{1}{2}\sqrt{\gamma^2 + 4w^2}, \quad r_2 = -\frac{\gamma}{2} - \frac{1}{2}\sqrt{\gamma^2 + 4w^2}.$$

Note that

$$\sqrt{\gamma^2 + 4w^2} > \sqrt{\gamma^2} = \gamma,$$

and so r_1 is positive and r_2 is negative. This implies that $(0,0)$ as a critical point to the transformed system (6.2) is an unstable saddle point. This means that after transforming back and also using Thm. 6.1, $(\pi, 0)$ is an unstable saddle point for the pendulum system (6.1). A similar analysis then shows that $(\pm(2m+1)\pi, 0)$ for $m \in \mathbb{Z}$ are all unstable saddle points.

Step 4. We summarize the above with a phase portrait. Thanks to (6.1), we now know that the "odd" critical points $(\pm(2m+1)\pi, 0)$ for $m \in \mathbb{Z}$ are all unstable saddle points, and depending on the values of γ and w^2, the "even" critical points $(\pm 2m\pi, 0)$ can be nodes or spirals, but they are always asym. stable if $\gamma > 0$. In Fig. 25, we show the phase portrait highlighting the critical points $(0,0)$, $(\pi, 0)$, $(2\pi, 0)$ and $(3\pi, 0)$ for the parameters $\gamma = 1$, $w = 1$ (so that we have spirals). In Fig. 26 we show the phase portrait for $\gamma = 5$, $w = 1$ (so that we have nodes).

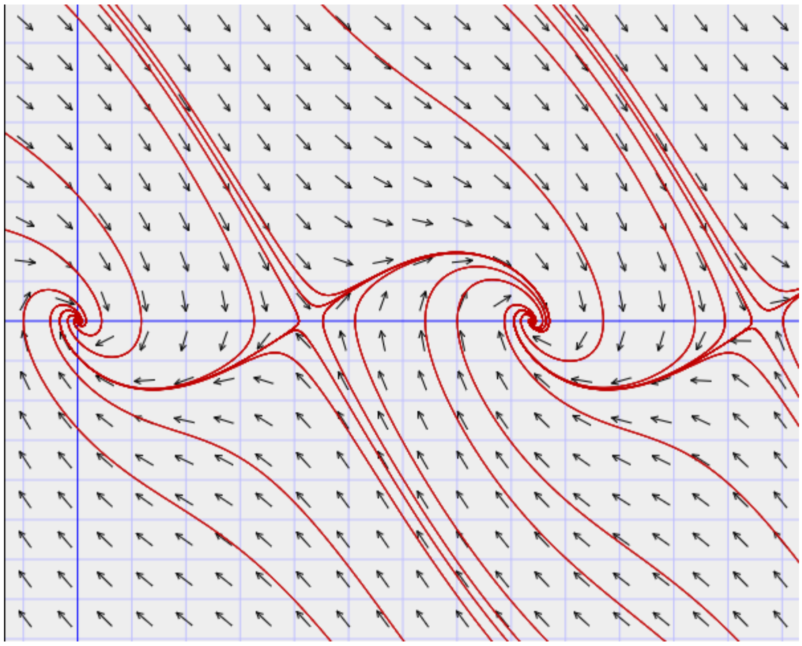


Figure 25: Phase portrait for the damped pendulum with $\gamma = w = 1$.

6.3.4 Undamped pendulum

Suppose we have no damping in the pendulum, which is equivalent to setting γ to zero in (6.1). Then, still we have critical points $(\pm n\pi, 0)$ for $n \in \mathbb{Z}$. Computing the Jacobian matrix near the "odd" critical points $(\pm(2m+1)\pi, 0)$ for $m \in \mathbb{Z}$ yields

$$\mathrm{D}\vec{f}((\pm(2m+1)\pi, 0)) = \begin{pmatrix} 0 & 1 \\ w^2 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = w$, $r_2 = -w$. This gives that $(\pm(2m+1)\pi, 0)$ are all unstable saddle points.

For the "even" critical points $(\pm 2m\pi, 0)$ for $m \in \mathbb{Z}$, we have

$$\mathrm{D}\vec{f}((\pm 2m\pi, 0)) = \begin{pmatrix} 0 & 1 \\ -w^2 & 0 \end{pmatrix},$$

with eigenvalues $r_1 = iw$, $r_2 = -iw$. Note that we have purely imaginary eigenvalues and thus Thm. 6.1 cannot be used to deduce the stability of the critical points $(\pm 2m\pi, 0)$.

6.4 Liapunov's second method.

We now present a method to infer stability information about the "even" critical points of the undamped pendulum. The approach we discuss now is called **Liapunov's second method**, sometimes known as the **direct method**, since this approach needs no knowledge of the solution to the system of equations, and conclusions about stability/instability of a critical point can be obtained.

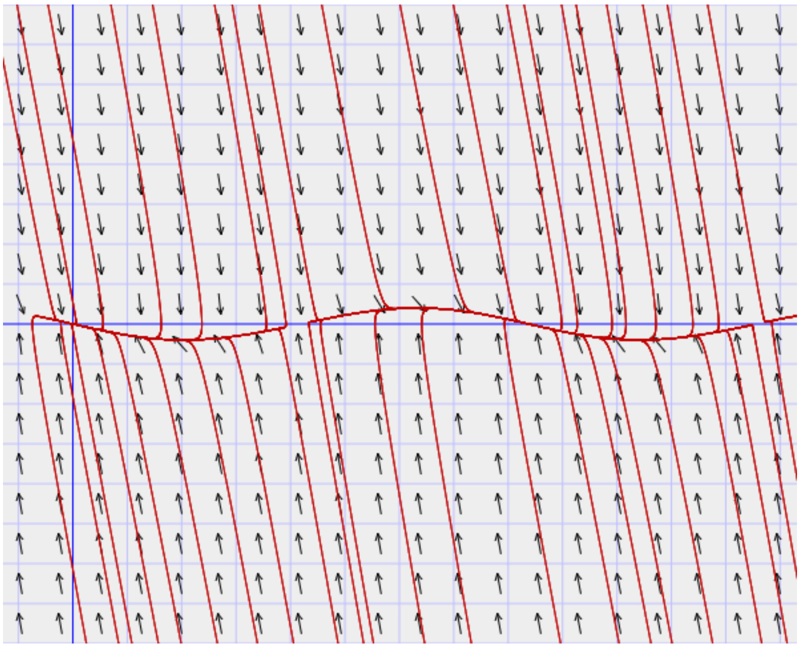


Figure 26: Phase portrait for the damped pendulum with $\gamma = 5, w = 1$.

Note that Liapunov's first method is about representing solutions in a series and then studying the convergence - we will not mention this in the course.

6.4.1 Application to the undamped pendulum

For the undamped pendulum, the original equation is

$$\theta'' + \frac{g}{L}\sin\theta = 0,$$

where we set $w = \sqrt{g/L}$ for convenience. Introducing the variables $y_1 = \theta$, $y_2 = \theta'$ we obtain the first order system

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -\frac{g}{L}\sin y_1 \end{pmatrix}. \tag{6.3}$$

From physics there are two energies associated to the pendulum:

(a) Potential energy given by $mgL(1-\cos y_1) = mgL(1-\cos\theta)$;

(b) Kinetic energy given by $\frac{1}{2}mL^2y_2^2 = \frac{1}{2}mL^2(\theta')^2$.

Let us make some observations

(i) The critical points to (6.3) are $(\pm n\pi, 0)$ for $n \in \mathbb{Z}$. We have previously studied the stability and type of the "odd" critical points which are unstable saddle points.

(ii) The potential energy is minimal (equal to zero) when $y_1 = \pm 2m\pi$ for $m \in \mathbb{Z}$, while the maximum potential energy (equal to $2mgL$) is achieved at $y_1 = \pm(2m+1)\pi$ for $m \in \mathbb{Z}$.

(iii) The total energy (the sum of the potential and kinetic energies) is

$$V(y_1, y_2) = mgL(1 - \cos y_1) + \frac{1}{2}mL^2 y_2^2$$

is conserved, i.e.,

$$\frac{d}{dt}V(y_1(t), y_2(t)) = 0.$$

And so, on trajectories $(y_1(t), y_2(t))_{t\in I}$ for an open interval $I \subset \mathbb{R}$, the total energy $V(y_1, y_2)$ remains unchanged.

The last point is the crucial part of Liapunov's second method. Note that at $y_1 = \pm 2m\pi$, $y_2 = 0$, both the potential and kinetic energies are zero, and so the total energy is zero at the "even" critical points. Hence, if we start with a trajectory $(y_1(t), y_2(t))_{t\in I}$ with initial condition (z_1, z_2), i.e., $y_1(t_0) = z_1$, $y_2(t_0) = z_2$, that is "close" to the "even" critical points, then by conservation of total energy we can infer that

$$V(y_1(t), y_2(t)) = V(z_1, z_2) \quad \forall t \in I,$$

and so the total energy for $t > t_0$ will remain small.

For example, pick (z_1, z_2) close to $(0, 0)$, and for small values of y_1, we can Taylor expand $\cos(\cdot)$ to obtain

$$\begin{aligned} V(y_1(t), y_2(t)) &= mgL(1 - \cos(y_1(t))) + \frac{1}{2}mL^2(y_2(t))^2 \\ &\approx \frac{1}{2}mgL(y_1(t))^2 + \frac{1}{2}mL^2(y_2(t))^2, \end{aligned}$$

and conservation of total energy gives

$$V(z_1, z_2) = V(y_1(t), y_2(t)) \approx \frac{1}{2}mgL(y_1(t))^2 + \frac{1}{2}mL^2(y_2(t))^2.$$

Roughly speaking, the trajectories $(y_1(t), y_2(t))_{t\in I}$ can be approximated by the equation

$$\boxed{\frac{y_1^2}{2\frac{V(z_1,z_2)}{mgL}} + \frac{y_2^2}{2\frac{V(z_1,z_2)}{mL^2}} = 1}.$$

This is the equation for an ellipse enclosing the critical point $(0, 0)$ where the major and minor axes are determined by the initial energy $V(z_1, z_2)$. In particular, the smaller the initial energy $V(z_1, z_2)$, the smaller the ellipse. Nevertheless this shows that $(0, 0)$ is a stable critical point (not asym. stable like in the damped pendulum).

What about the critical point $(2\pi, 0)$? Using a transformation

$$\vec{w} = \vec{y} - \begin{pmatrix} 2\pi \\ 0 \end{pmatrix},$$

we have

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} w_2 \\ -\frac{g}{L}\sin(w_1 + 2\pi) \end{pmatrix} = \begin{pmatrix} w_2 \\ -\frac{g}{L}\sin w_1 \end{pmatrix},$$

and so the critical point $(2\pi, 0)$ to the original undamped pendulum (6.3) is now the critical point $(0,0)$ to the transformed system for (w_1, w_2).

In the initial conditions (x_1, x_2) is close to $(2\pi, 0)$, and the initial total energy $V(x_1, x_2)$ is small, the trajectories $(w_1(t), w_2(t))_{t\in I}$ can be approximated by the equation

$$\boxed{\frac{y_1^2}{2\frac{V(x_1-2\pi,x_2)}{mgL}} + \frac{y_2^2}{2\frac{V(x_1-2\pi,x_2)}{mL^2}} = 1}.$$

In particular, we still get an ellipse, but the center is now at $(2\pi, 0)$ and so the critical point $(2\pi, 0)$ is a stable center. The same arguments can be used to show that the "even" critical points of the undamped pendulum are all stable centers. Fig. 27 shows the phase portrait for $w = 1$.

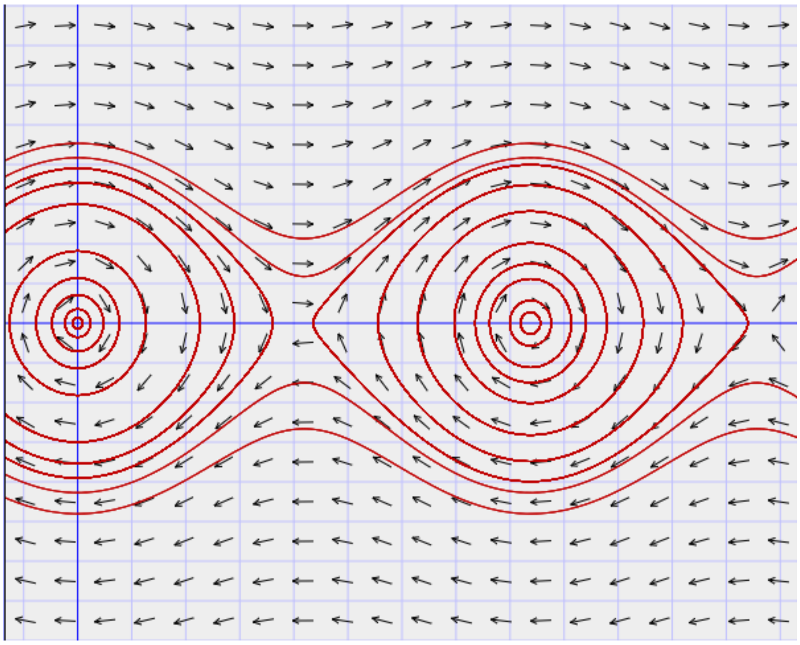


Figure 27: Phase portrait for the undamped pendulum with $w = 1$.

6.4.2 General theory

In the undamped pendulum example, the function V plays a significant role in helping us determine the stability of some critical points. Let us now consider a nonlinear autonomous system

$$y_1' = F_1(y_1, y_2), \quad y_2' = F_2(y_1, y_2) \text{ for } t \in I,$$

with a critical point $(0,0)$, i.e., $F_1(0,0) = F_2(0,0) = 0$. Denote by $D \subset \mathbb{R}^2$ a region containing $(0,0)$, and a trajectory by $(y_1(t), y_2(t))_{t\in I}$.

Definition 6.7 (Positive/negative definite). *Let $V : \mathbb{R}^2 \to \mathbb{R}$ be a function such that $V(z_1, z_2) < \infty$ for all $(z_1, z_2) \in D$. We say*

(a) V is ***positive definite*** *on D if $V(0,0) = 0$ and $V(z_1, z_2) > 0$ for all $(z_1, z_2) \in D \setminus \{(0,0)\}$;*

(b) V is ***negative definite*** *on D if $V(0,0) = 0$ and $V(z_1, z_2) < 0$ for all $(z_1, z_2) \in D \setminus \{(0,0)\}$;*

(c) V is ***positive semidefinite*** *on D if $V(0,0) = 0$ and $V(z_1, z_2) \geq 0$ for all $(z_1, z_2) \in D$;*

(d) V is ***negative semidefinite*** *on D if $V(0,0) = 0$ and $V(z_1, z_2) \leq 0$ for all $(z_1, z_2) \in D$.*

Note that in all of the above definitions, we always have the condition $V(0,0) = 0$.

Example 6.9. *The function*

$$V(x, y) = \sin(x^2 + y^2),$$

on the region

$$D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < \pi/2\},$$

which is a circle centre at the origin with radius strictly less than $\pi/2$. Then, it is easy to check that $V(0,0) = 0$ and $V(x, y) > 0$ for $(x, y) \in D \setminus \{(0,0)\}$. Hence, V is positive definite.

Example 6.10. *The function*

$$V(x, y) = (x + y)^2$$

on the region $D = \mathbb{R}^2$ satisfies $V(0,0) = 0$. But $V(-y, y) = 0$ and so V is zero also on the line $y = x$. This V is only positive semidefinite.

Returning to the nonlinear system

$$y_1' = F_1(y_1, y_2), \quad y_2' = F_2(y_1, y_2) \text{ for } t \in I,$$

and let V be a function of (y_1, y_2). Then,

$$\boxed{\frac{d}{dt}V(y_1(t), y_2(t)) = \frac{\partial V}{\partial y_1}y_1' + \frac{\partial V}{\partial y_2}y_2' = \left(\frac{\partial V}{\partial y_1}F_1 + \frac{\partial V}{\partial y_2}F_2\right)(y_1, y_2) =: W(y_1, y_2)}.$$

We now state two theorems - the first is about stability and the second is about instability.

Theorem 6.2 (Liapunov's stability theorem). *Consider the autonomous system*

$$y_1' = F_1(y_1, y_2), \quad y_2' = F_2(y_1, y_2) \text{ for } t \in I,$$

with an isolated critical point $(0,0)$. Suppose there is a function V that is continuous with continuous derivatives and is ***positive definite*** *on a region D. If* *(a) the function* $W(y_1, y_2) = \left(\frac{\partial V}{\partial y_1} F_1 + \frac{\partial V}{\partial y_2} F_2\right)(y_1, y_2)$ *is* ***negative definite*** *on* D, *then* $(0,0)$ *is* ***asym. stable****.*

(b) the function $W(y_1, y_2)$ *is* ***negative semidefinite*** *on* D*, then* $(0,0)$ *is* ***stable****.*

Let's apply this to the undamped pendulum: Recall we have the total energy

$$V(y_1, y_2) = mgL(1 - \cos y_1) + \frac{1}{2} mL^2 y_2^2.$$

Consider the region D given as

$$D := (-\pi/2, \pi/2) \times \mathbb{R},$$

then V is positive definite in D with $V(0,0) = 0$. We saw that

$$\frac{d}{dt} V(y_1(t), y_2(t)) = 0 = W(y_1, y_2).$$

Since the zero function is negative definite on D, we obtain from Thm. 6.2 that the critical point $(0,0)$ is stable.

For the critical point $(2\pi, 0)$ we transform the system to

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} w_2 \\ -\frac{g}{L} \sin w_1 \end{pmatrix}, \qquad \text{for } w_1 = y_1 - 2\pi, \quad w_2 = y_2.$$

The same function

$$V(w_1, w_2) = mgL(1 - \cos w_1) + \frac{1}{2} mL^2 w_2^2$$

satisfies

$$\frac{d}{dt} V(w_1(t), w_2(t)) = 0 = W(w_1(t), w_2(t)),$$

and V is positive definite on the region $D := (-\pi/2, \pi/2) \times \mathbb{R}$. This corresponds to the region $(3\pi/2, 5\pi/2) \times \mathbb{R}$ for the original variables (y_1, y_2). Therefore, by Thm. 6.2, $(2\pi, 0)$ is a stable critical point.

For instability we have the following theorem

Theorem 6.3 (Liapunov's instabiity theorem)**.** *Consider the autonomous system*

$$y_1' = F_1(y_1, y_2), \quad y_2' = F_2(y_1, y_2) \textit{ for } t \in I,$$

with an isolated critical point $(0,0)$*. Suppose there is a function* V *that is continuous with continuous derivatives and* $V(0,0) = 0$*. Suppose in* ***every neighbourhood*** *of* $(0,0)$ *there is at least one point* (z_{1*}, z_{2*}) *such that* $V(z_{1*}, z_{2*})$ *is positive (resp. negative).*

If there is a region D *with* $(0,0) \in D$ *and* $W(y_1, y_2)$ *is positive (resp. negative) definite in* D*, then the origin* $(0,0)$ *is an unstable critical point.*

For the instability theorem there is an additional condition to check, namely in **every neighbourhood** of $(0,0)$ there is at least one point (z_{1*}, z_{2*}) such that $V(z_{1*}, z_{2*})$ is positive (resp. negative). We demonstrate this with an example involving the critical point $(\pi, 0)$ of the undamped pendulum. Recall the equations are

$$\begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \begin{pmatrix} y_2 \\ -\frac{g}{L}\sin y_1 \end{pmatrix},$$

and setting $z_1 = y_1 - \pi$, $z_2 = y_2$ yields

$$\begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \begin{pmatrix} z_2 \\ \frac{g}{L}\sin z_1 \end{pmatrix},$$

so that the critical point $(y_1, y_2) = (\pi, 0)$ is now the critical point $(z_1, z_2) = (0,0)$. Looking at the total energy (now called U)

$$U(z_1, z_2) = mgL(1 - \cos(z_1 + \pi)) + \frac{1}{2}mL^2 z_2^2 = mgL(1 + \cos z_1) + \frac{1}{2}mL^2 z_2^2,$$

we see that $U(0,0) = 2mgL \neq 0$. Therefore we cannot use U as the function V and apply Thm. 6.3. In addition, we can compute

$$\frac{d}{dt}U(z_1(t), z_2(t)) = 0,$$

and Thm. 6.3 requires W to be positive or negative definite (not semidefinite). Thus we need another function. The idea is to try

$$V(z_1, z_2) = z_2 \sin z_1.$$

Then, $V(0,0) = 0$ and

$$\frac{d}{dt}V(z_1(t), z_2(t)) = \frac{g}{L}\sin^2 z_1(t) + z_2(t)^2 \cos z_1(t) =: W(z_1(t), z_2(t)).$$

So for $z_1 \in (-\pi/4, \pi/4)$ and $z_2 \in \mathbb{R}$, the function $W(z_1, z_2)$ is positive definite in $D := (-\pi/4, \pi/4) \times \mathbb{R}$. The only thing remaining is to see if there are points in every neighbourhood of the origin where the function V is positive. Note that V is always positive in the regions on D where $z_1, z_2 > 0$ or $z_1, z_2 < 0$. Hence, this condition is always satisfied and by Thm. 6.3 the critical point $(z_1, z_2) = (0,0)$ is unstable.

Definition 6.8 (Liapunov function)**.** *The function V in Thm. 6.2 and 6.3 is known as a Liapunov function.*

Remark 6.2. *In general, there is no method to construct Liapunov functions, often a lucky guess is needed or intitution from physics.*

6.4.3 Quadratic Liapunov functions

We now study systems of equations that allows us to construct Liapunov functions with quadratic form. I.e., $V(x,y)$ looks like $ax^2+bxy+cy^2$. First let's give a theorem.

Theorem 6.4. *The function*

$$V(x,y) = ax^2 + bxy + cy^2$$

for constants a, b, c satisfies the following properties

(a) V is positive definite if and only if $a > 0$ and $4ac - b^2 > 0$.

(b) V is negative definite if and only if $a < 0$ and $4ac - b^2 > 0$.

Example 6.11. *Consider the system*

$$y_1' = -y_1 - y_1 y_2^2 = F_1(y_1,y_2), \quad y_2' = -y_2 - y_1^2 y_2 = F_2(y_1,y_2).$$

Then, $F_1(0,0) = F_2(0,0) = 0$ and so $(0,0)$ is a critical point. If V is a Liapunov function then

$$\frac{d}{dt}V(y_1(t),y_2(t)) = \frac{\partial V}{\partial y_1}(-y_1 - y_1 y_2^2) + \frac{\partial V}{\partial y_2}(-y_2 - y_1^2 y_2).$$

We now assume V is of the form $V(x,y) = ax^2 + bxy + cy^2$. Then

$$\frac{\partial V}{\partial y_1} = 2ay_1 + by_2, \quad \frac{\partial V}{\partial y_2} = by_1 + 2cy_2,$$

so that

$$\frac{d}{dt}V(y_1(t),y_2(t)) = -\left[2a(y_1^2 + y_1^2 y_2^2) + b(2y_1y_2 + y_1 y_2^3 + y_1^3 y_2) + 2c(y_2^2 + y_1^2 y_2^2)\right].$$

Looking at the above expression, we should set $b = 0$ to remove the cubic terms (which can be positive or negative for different vales of y_1 and y_2). Then, choosing for example $a = c = 0.5$, we obtain

$$\frac{dV}{dt} = -(y_1^2 + 2y_1^2 y_2^2 + y_2^2) =: W(y_1,y_2).$$

Now, it is easy to check that $W(0,0) = 0$ and $W(x,y) < 0$ for all $(x,y) \neq (0,0)$. This shows that W is negative definite on $D = \mathbb{R}^2$. By Thm. 6.2 we have that $(0,0)$ is an asym. stable critical point.

If we use the method of locally linear systems, writing

$$\vec{f}(\vec{y}) = \begin{pmatrix} -y_1 - y_1 y_2^2 \\ -y_2 - y_1^2 y_2 \end{pmatrix},$$

we can show that $\vec{y}'(t) = \vec{f}(\vec{y}(t))$ is locally linear near the critical point $(0,0)$. This is due to the fact that the entires of $\vec{f}$ are twice continuously differentiable functions. Computing the Jacobian matrix:

$$\mathbb{A} = \mathrm{D}\vec{f}(\vec{0}) = \begin{pmatrix} -1-y_2^2 & -2y_1y_2 \\ -2y_1y_2 & -1-y_1^2 \end{pmatrix}\Big|_{(0,0)} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix},$$

we see that the eigenvalues of $\mathbb{A}$ are $r_1 = r_2 = -1$. By Thm. 6.1 we deduce that the critical point $\vec{0}$ is asymptotically stable, which is consistent with our previous analysis with Liapunov's second method.

We present one more example involving stability.

Example 6.12. *Consider*

$$y_1' = -y_1^3 + 2y_1y_2^2 = F_1(y_1, y_2), \quad y_2' = -2y_1^2y_2 - y_2^3 = F_2(y_1, y_2).$$

Note that $F_1(0,0) = F_2(0,0) = 0$ and so $(0,0)$ is a critical point. Assuming V is of the form $V(x,y) = ax^2 + bxy + cy^2$, computing

$$\begin{aligned}\frac{d}{dt}V(y_1(t), y_2(t)) &= (2ay_1 + by_2)(2y_1y_2^2 - y_1^3) + (by_1 + 2cy_2)(-2y_1^2y_2 - y_2^3) \\ &= 4ay_1^2y_2^2 + 2by_1y_2^3 - 2ay_1^4 - by_1^3y_2 - 2by_1^3y_2 - 4cy_1^2y_2^2 - by_1y_2^3 - 2cy_2^4.\end{aligned}$$

We again set $b = 0$ to remove the cubic terms, and choose $a = c = 1$, so that

$$\frac{dV}{dt} = -2y_1^4 - 2y_1^4 = W(y_1, y_2).$$

It is clear that W is negative definite on $D = \mathbb{R}^2$, and by Thm. 6.2 $(0,0)$ is a stable critical point.

The last example is about instability.

Example 6.13. *Consider*

$$x' = 2x^3 - y^3, \quad y' = 2xy^2 + 4x^2y + 2y^3,$$

where $(0,0)$ is a critical point. Consider $V(x,y) = ax^2 + cy^2$, then

$$\frac{d}{dt}V(x(t), y(t)) = 4ax^4 + 4cy^4 + 4cxy^3 - 2axy^3 + 8cx^2y^2.$$

Choosing $4c = 2a$ to remove the term involving xy^3, for example $a = 1, c = 0.5$, leads to

$$\frac{dV}{dt} = 4x^4 + 2y^4 + 4x^2y^2 = W(x,y).$$

It is clear that $W(0,0) = 0$ and $W(x,y)$ is positive for all $(x,y) \neq (0,0)$. So W is positive definite on $D = \mathbb{R}^2$. However, to apply Thm. 6.3 we still need to check that for every neighbourhood of $(0,0)$ there is a point (x_*, y_*) where the function $V(x,y) = x^2 + \frac{1}{2}y^2$ is positive at (x_*, y_*). However, since $V(x,y)$ is strictly positive for $(x,y) \neq (0,0)$, this is fulfilled. Thus, by Thm. 6.3 $(0,0)$ is an unstable critical point.