MATH3720A - Lecture Notes

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5 System of first order equations

Up until now, we have been focusing on ordinary differential equations, where there is **one** independent variable and **one** dependent variable. However, many interesting problems involve multiple ordinary differential equations, leading to a system of equations, that is still **one** independent variable, but now **many** dependent variables. Below we list some examples:

Example 5.1 (SIR model for disease spreading)**.** *Let $S(t)$ denote the number of susceptible individuals, $I(t)$ be the number of infected individuals, and $R(t)$ be the number of recovered individuals. A person can only be in one of the above three states, and must be susceptible, and then infected, and then recovered. It is not possible to be infected and then susceptible, nor from recovered to infected and susceptible. Hence, we can model this with the following system of ODEs:*

$$S' = -\beta IS, \quad I' = \beta IS - \gamma I, \quad R' = \gamma I,$$

where β is the contact rate and γ is the recovery rate. Note that the number of susceptible people is always decreasing, and the number of recovered people is always increasing. If we sum the three equations we see that

$$\frac{d}{dt}(S + I + R) = 0,$$

*and so the **total** number of people is conserved.*

Example 5.2 (Lotka–Volterra equations)**.** *The Lotka–Volterra equations describe the interaction between two species of population, which we denote by $x(t)$ and $y(t)$, and are given by*

$$x'(t) = x(a - by), \quad y'(t) = cy(x - d),$$

where a and c are the growth rates of species x and y, b can be seen as a predation rate for species x, and d is a loss rate for species y. The interaction between the two species is modelled through the terms proportional to the product xy.

Example 5.3 (Cucker–Smale model for flocking). *Given a collection of N birds, denote by x_i the position of bird i and by v_i the velocity of bird i, $1 \le i \le N$. The birds may fly around in the sky but they can communicate with each other. Let's encode this with a function Ψ that depends only on the relative distance between two birds. Then the Cucker–Smale model is given as*

$$x_i'(t) = v_i, \quad v_i'(t) = \frac{1}{N}\sum_{j=1}^{N} \Psi(|x_i - x_j|)(v_j - v_i).$$

5.1 First order systems

The general system of first order equations involving n dependent variables for an open interval $I \subset \mathbb{R}$ is

$$\begin{aligned} y_1'(t) &= F_1(t, y_1, \ldots, y_n), \\ y_2'(t) &= F_2(t, y_1, \ldots, y_n), \\ &\vdots \\ y_n'(t) &= F_n(t, y_1, \ldots, y_n), \end{aligned}$$

with initial conditions

$$y_1(t_0) = x_1, \quad \ldots, \quad y_n(t_0) = x_n,$$

where $t_0 \in I$, and $x_1, \ldots, x_n \in \mathbb{R}$ are given. A solution to the above system can be convenient written as a vector $\vec{y}(t) = (y_1(t), \ldots, y_n(t))^\top$ where for each $t \in I$, $\vec{y}(t)$ lives in the space $\mathbb{R}^n$. By running through $t \in I$, we then trace out a curve in $\mathbb{R}^n$, which we will call the trajectory, or path. The initial condition $\vec{x} = (x_1, \ldots, x_n)^\top$ determines the starting point of the path.

To ensure that the IVP has exactly one solution, we state the following existence and uniqueness theorem.

Theorem 5.1 (Existence and Uniqueness). *Let I be an open interval in $\mathbb{R}$, and fix constants $\alpha_0, \ldots, \alpha_n, \beta_0, \ldots, \beta_n$, such that $t_0 \in (\alpha_0, \beta_0) \subset I$ Suppose the functions $F_1, \ldots, F_n$ and the partial derivatives $\frac{\partial F_1}{\partial y_1}, \frac{\partial F_1}{\partial y_2}, \ldots, \frac{\partial F_1}{\partial y_n}, \frac{\partial F_2}{\partial y_1}, \ldots, \frac{\partial F_2}{\partial y_n}, \ldots, \frac{\partial F_n}{\partial y_n}$ are all continuous in a region R defined as*

$$R := (\alpha_0, \beta_0) \times (\alpha_1, \beta_1) \times \cdots \times (\alpha_n, \beta_n) \subset \mathbb{R}^{n+1}.$$

Then, for $x_1 \in (\alpha_1, \beta_1), \ldots, x_n \in (\alpha_n, \beta_n)$, there is a constant $h > 0$ such that for all $t \in (t_0 - h, t_0 + h) \cap (\alpha_0, \beta_0)$, there is exactly one solution $\vec{y}(t)$ to the IVP.

Remark 5.1. *(1) If $n = 1$, then we have $y' = F(t, y)$, leading to the same assumptions as in the case of first order equations.*

(2) We only have existence and uniqueness for a small interval $(t_0 - h, t_0 + h)$ around t_0.

Note that we can express a n-th order (nonlinear) equation as a first order system. Indeed, given the ODE

$$y^{(n)} = F(t, y, y', \dots, y^{(n-1)}),$$

setting

$$z_1 = y, \quad z_2 = y', \quad \dots, \quad z_n = y^{(n-1)},$$

then we can write down

$$\begin{aligned} z_1' &= z_2, \\ z_2' &= z_3, \\ &\vdots \\ z_n' &= y^{(n)} = F(t, z_1, \dots, z_{n-1}). \end{aligned}$$

Definition 5.1. *If each F_i, $1 \le i \le n$, is linear with respect to $y_1, \dots, y_n$, then we call the system of ODEs a linear system. Otherwise it is a nonlinear system.*

We now study linear systems is greater detail. The general linear system of first order ODEs is

$$\begin{aligned} y_1'(t) &= P_{11}(t)y_1(t) + P_{12}(t)y_2(t) + \dots + P_{1n}(t)y_n(t) + g_1(t), \\ y_2'(t) &= P_{21}(t)y_1(t) + P_{22}(t)y_2(t) + \dots + P_{2n}(t)y_n(t) + g_2(t), \\ &\vdots \\ y_n'(t) &= P_{n1}(t)y_1(t) + P_{n2}(t)y_2(t) + \dots + P_{nn}(t)y_n(t) + g_n(t), \end{aligned}$$

where $P_{11}(t), \dots, P_{nn}(t), g_1(t), \dots, g_n(t)$ are given functions. It is more convenient to introduce the matrix form. Denoting vectors

$$\vec{y}(t) = (y_1(t), \dots, y_n(t))^\top, \quad \vec{g}(t) = (g_1(t), \dots, g_n(t))^\top,$$

and the matrix

$$\mathbb{P}(t) = \begin{pmatrix} P_{11}(t) & P_{12}(t) & \dots & P_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ P_{n1}(t) & P_{n2}(t) & \dots & P_{nn}(t) \end{pmatrix},$$

the general system can be written as

$$\boxed{\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t) + \vec{g}(t)}.$$

Definition 5.2. *A first order linear system of equations*

$$\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t) + \vec{g}(t)$$

is called ***homogeneous*** *if $\vec{g}(t) = \vec{0}$, i.e., $g_i(t) = 0$ for $1 \le i \le n$. Otherwise it is called non-homogeneous.*

Theorem 5.2 (Existence and uniqueness for linear systems)**.** *Let $I \subset \mathbb{R}$ be an open interval such that functions $P_{11}(t), \dots, P_{nn}(t), g_1(t), \dots, g_n(t)$ are continuous in I. For $t_0 \in I$ and $x_1, \dots, x_n \in \mathbb{R}$, there is exactly one solution $\vec{y}(t) = (y_1(t), \dots, y_n(t))^\top$ to the IVP.*

In order to study first order linear systems, we make use of the convenient matrix form. Therefore we need to recall some basic properties of matrices.

5.2 Review of matrices

A matrix is a **rectangular array** of numbers. We use the notation $\mathbb{A} \in \mathbb{R}^{m \times n}$ to denote a matrix with real entries of size m rows by n columns. If $\mathbb{A}$ is a complex-valued matrix we write $\mathbb{A} \in \mathbb{C}^{m \times n}$. Furthermore, we often write

$$\mathbb{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \text{ or } \mathbb{A} = (a_{ij})_{1 \le i \le m, 1 \le j \le n}.$$

The **transpose** of a matrix $\mathbb{A}$ is denoted as $\mathbb{A}^\top$ which is defined as

$$\mathbb{A}^\top = (a_{ji})_{1 \le j \le n, 1 \le i \le m} \in \mathbb{R}^{n \times m}.$$

For a complex-valued matrix $\mathbb{A} \in \mathbb{C}^{m \times n}$, we define its complex conjugate as

$$\overline{\mathbb{A}} = (\overline{a_{ij}})_{1 \le i \le m, 1 \le j \le n}.$$

For example

$$\mathbb{A} = \begin{pmatrix} 1 & 2 & 3+i \\ i & 4 & -7 \end{pmatrix} \text{ with } \mathbb{A}^\top = \begin{pmatrix} 1 & i \\ 2 & 4 \\ 3+i & -7 \end{pmatrix}, \quad \overline{\mathbb{A}} = \begin{pmatrix} 1 & 2 & 3-i \\ -i & 4 & -7 \end{pmatrix}.$$

For two matrices of the same size $\mathbb{A}, \mathbb{B} \in \mathbb{R}^{m \times n}$ we can define addition and subtraction, as well as scalar multiplication. For products of matrices, we require a matrix $\mathbb{A} \in \mathbb{R}^{m \times p}$ and another $\mathbb{B} \in \mathbb{R}^{p \times n}$, where the number of columns in $\mathbb{A}$ is equal to the number of rows in $\mathbb{B}$. Then

$$\mathbb{A}\mathbb{B} = \left(\sum_{j=1}^{p} a_{ij} b_{jk} \right)_{1 \le i \le m, 1 \le k \le n}.$$

The same also holds for complex-valued matrices. Note that in general

$$\boxed{\mathbb{A}\mathbb{B} \ne \mathbb{B}\mathbb{A}},$$

i.e., product of matrices is **not commutative**.

In order to solve a linear system of equations, we can use the matrix notation to write

$$\begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

as

$$\mathbb{A}\vec{x} = \vec{b}, \qquad \text{where } \mathbb{A} \in \mathbb{R}^{m \times n},\ \vec{x} \in \mathbb{R}^n,\ \vec{b} \in \mathbb{R}^m.$$

To find the solution (if one exists) we can apply **elementary row operations** to the augmented matrix $(\mathbb{A}|\vec{b}) \in \mathbb{R}^{m \times (n+1)}$. Let us briefly recall the elementary row operations applied to a matrix $\mathbb{A} \times \mathbb{R}^{m \times n}$. Denoting the ith rows of $\mathbb{A}$ as $r_i \in \mathbb{R}^n$, we have

(1) interchange two rows $r_i \leftrightarrow r_j$;

(2) non-zero scalar multiple of one row $r_i \mapsto \alpha r_i$, $\alpha \neq 0$;

(3) adding a multiple of one row to another $r_i \mapsto r_i + \alpha r_j$, $\alpha \neq 0$.

Example 5.4. *Solve the linear system*

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 7, \\ -x_1 + x_2 - 2x_3 &= -5, \\ 2x_1 - x_2 - x_3 &= 4. \end{aligned}$$

Writing

$$\mathbb{A} = \begin{pmatrix} 1 & -2 & 3 \\ -1 & 1 & -2 \\ 2 & -1 & -1 \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 7 \\ -5 \\ 4 \end{pmatrix},$$

we now apply elementary row operations to the augmented matrix

$$\left(\mathbb{A}|\vec{b}\right) = \left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ -1 & 1 & -2 & -5 \\ 2 & -1 & -1 & 4 \end{array}\right).$$

First apply $r_2 \mapsto r_1 + r_2$ and $r_3 \mapsto r_3 - 2r_1$ leads to

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & -1 & 1 & 2 \\ 0 & 3 & -7 & -10 \end{array}\right).$$

Then apply $r_2 \mapsto -r_2$ leads to

$$\left(\begin{array}{ccc|c} 1 & -2 & 3 & 7 \\ 0 & 1 & -1 & -2 \\ 0 & 3 & -7 & -10 \end{array}\right).$$

Then apply $r_1 \mapsto r_1 + 2r_2$ and $r_3 \mapsto r_3 - 3r_2$ leads to

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & -4 & -4 \end{array}\right).$$

Then apply $r_3 \mapsto -\frac{1}{4}r_3$ leads to

$$\left(\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{array}\right).$$

Then apply $r_1 \mapsto r_1 - r_3$ and $r_2 \mapsto r_3 + r_2$ leads to

$$\left(\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right),$$

and so the solution is

$$x_1 = 2, \quad x_2 = -1, \quad x_3 = 1.$$

For the case $m = n$, i.e., $\mathbb{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{A} \in \mathbb{C}^{n\times n}$) we call $\mathbb{A}$ a **square matrix**. A special square matrix is the identity matrix $\mathbb{I}$ defined as

$$\mathbb{I}_{kk} = 1, \ 1 \le k \le n, \quad I_{ij} = 0 \ 1 \le i \ne j \le n.$$

In particular, all the diagonal entries are one and all other entries are zero.

Definition 5.3. *We say a square matrix $\mathbb{A} \in \mathbb{R}^{n\times n}$ is* ***invertible*** *if there is a matrix $\mathbb{B} \in \mathbb{R}^{n\times n}$ such that*

$$\boxed{\mathbb{A}\mathbb{B} = \mathbb{B}\mathbb{A} = \mathbb{I}}.$$

In this case we write $\mathbb{B} = \mathbb{A}^{-1}$. Matrices that do not have an inverse are called ***singular*** *or* ***non-invertible***.

For a square matrix $\mathbb{A} \in \mathbb{R}^{n\times n}$ the following statements are equivalent:

(1) $\mathbb{A}$ is invertible;

(2) the determinant $\det\mathbb{A}$ is non-zero;

(3) The only solution to the problem $\mathbb{A}\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.

We can also use the elementary row operations to find the inverse of a square matrix. To do this we consider the augmented matrix $(\mathbb{A}|\mathbb{I})$ and transform this into the matrix $(\mathbb{I}|\mathbb{B})$. It turns out that $\mathbb{B}$ will be the inverse of $\mathbb{A}$.

Example 5.5. *Find the inverse of the matrix*

$$\mathbb{A} = \begin{pmatrix} 1 & -1 & -1 \\ 3 & -1 & 2 \\ 2 & 2 & 3 \end{pmatrix}.$$

Let us write the augmented matrix $(\mathbb{A}|\mathbb{I})$:

$$(\mathbb{A}|\mathbb{I}) = \left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 3 & -1 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right).$$

First apply $r_2 \mapsto r_2 - 3r_1$ and $r_3 \mapsto r_3 - 2r_1$ leads to

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 5 & -3 & 1 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right).$$

Then apply $r_2 \mapsto r_2/2$ leads to

$$\left(\begin{array}{ccc|ccc} 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 4 & 5 & -2 & 0 & 1 \end{array}\right).$$

Then apply $r_3 \mapsto r_3 - 4r_2$ and $r_1 \mapsto r_1 + r_2$ leads to

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & -5 & 4 & -2 & 1 \end{array}\right).$$

Then apply $r_3 \mapsto -r_3/5$ leads to

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 3/2 & -1/2 & 1/2 & 0 \\ 0 & 1 & 5/2 & -3/2 & 1/2 & 0 \\ 0 & 0 & 1 & -4/5 & 2/5 & -1/5 \end{array}\right).$$

Then apply $r_1 \mapsto r_1 - 3r_3/2$ and $r_2 \mapsto r_2 - -5r_3/2$ leads to

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 7/10 & -1/10 & 3/10 \\ 0 & 1 & 0 & 1/2 & -1/2 & 1/2 \\ 0 & 0 & 1 & -4/5 & 2/5 & -1/5 \end{array}\right).$$

Hence, the inverse of $\mathbb{A}$ is

$$\mathbb{A}^{-1} = \begin{pmatrix} 7/10 & -1/10 & 3/10 \\ 1/2 & -1/2 & 1/2 \\ -4/5 & 2/5 & -1/5 \end{pmatrix}.$$

In our study of firs order systems, we will deal with the case where the entries of the matrix $\mathbb{A}$ are functions of the independent variable t, hence we can define a matrix function of t as $\mathbb{A}(t)$ where

$$\mathbb{A}(t) = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \dots & a_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}(t) & a_{m2}(t) & \dots & a_{mn}(t) \end{pmatrix}.$$

We say that $\mathbb{A}(t)$ is **continuous** if all the entries $a_{11}(t), \dots, a_{mn}(t)$ are continuous functions of t. Similarly, we say $\mathbb{A}(t)$ is **differentiable** if all its entries are differentiable functions. Then

$$\frac{d}{dt}\mathbb{A}(t) = \begin{pmatrix} a'_{11}(t) & a'_{12}(t) & \dots & a'_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ a'_{m1}(t) & a'_{m2}(t) & \dots & a'_{mn}(t) \end{pmatrix}.$$

We can also define the (indefinite) integral of $\mathbb{A}(t)$ as

$$\int \mathbb{A}(t)\, dt = \left(\int a_{ij}(t)\, dt\right)_{1\le i\le m, 1\le j\le n}.$$

Example 5.6. *For*

$$\mathbb{A}(t) = \begin{pmatrix} \cos t & \sin t \\ e^t & t \end{pmatrix},$$

we have

$$\mathbb{A}'(t) = \begin{pmatrix} -\sin t & \cos t \\ e^t & 1 \end{pmatrix}, \quad \int \mathbb{A}(t)\, dt = \begin{pmatrix} \sin t & -\cos t \\ e^t & \frac{1}{2}t^2 \end{pmatrix}.$$

An important concept of matrix theory involves eigenvalues and eigenvectors.

Definition 5.4. *For a square matrix $\mathbb{A} \in \mathbb{R}^{n\times n}$, if there is a number r and a **non-zero** vector $\vec{x}$ such that*

$$\boxed{\mathbb{A}\vec{x} = r\vec{x}},$$

*then we say r is an **eigenvalue** of $\mathbb{A}$ with **corresponding eigenvector** $\vec{x}$.*

Note that

$$\mathbb{A}\vec{x} = r\vec{x} \Leftrightarrow (\mathbb{A} - r\mathbb{I})\vec{x} = \vec{0}.$$

Therefore we can see that $\vec{x}$ is a **non-zero** solution to the problem

$$\boxed{(\mathbb{A} - r\mathbb{I})\vec{y} = \vec{0}},$$

and so

$$\det(\mathbb{A} - r\mathbb{I}) = 0.$$

In particular, using that fact that the determinant of $(\mathbb{A} - r\mathbb{I})$ can be expressed as a polynomial in r of degree n, which we also term as the **characteristic polynomial of** $\mathbb{A}$, we can find the roots of this polynomial to deduce the eigenvalues.

Definition 5.5. *Let r be an eigenvalue of the matrix $\mathbb{A} \in \mathbb{R}^{n\times n}$. We define the **algebraic multiplicity** of r as the multiplicity of r when treated as a root of the characteristic polynomial $P_{\mathbb{A}}(x) = \det(\mathbb{A} - x\mathbb{I})$. The **geometric multiplicity** of r is the number of linearly independent eigenvectors associated to r (i.e., the dimension of the eigenspace for r).*

Example 5.7. *Let*

$$\mathbb{A} = \begin{pmatrix} 8 & -9 \\ 4 & -4 \end{pmatrix}.$$

Then

$$\det(\mathbb{A} - r\mathbb{I}) = \begin{vmatrix} 8-r & -9 \\ 4 & -4-r \end{vmatrix} = (8-r)(-4-r) + 36 = r^2 - 4r + 4 = (r-2)^2.$$

Hence the eigenvalues are

$$r_1 = r_2 = 2,$$

i.e., the eigenvalue 2 has an algebraic multiplicity of two. To find eigenvectors, we consider non-zero vectors $\vec{x}$ satisfying

$$(\mathbb{A} - 2\mathbb{I})\vec{x} = \vec{0} \Rightarrow \begin{cases} 8x_1 - 9x_2 = 2x_1 \\ 4x_1 - 4x_2 = 2x_2. \end{cases}$$

Solving the equations implies we have $2x_1 = 3x_2$ *and so we can choose*

$$x_1 = 1, \quad x_2 = -2/3 \quad \Rightarrow \quad \vec{x} = \begin{pmatrix} 1 \\ -2/3 \end{pmatrix}.$$

However, we do not have enough information to deduce another linearly independent vector, thus there is only one eigenvector corresponding to the eigenvalue $r = 2$. *Therefore the geometric multiplicity of* $r = 2$ *is one.*

In general, we have

$$\boxed{1 \le \text{ geo. mult.} \le \text{ alg. mult.}}.$$

Example 5.8. *Let*

$$\mathbb{A} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 1 \\ 2 & 0 & -1 \end{pmatrix}.$$

Computing for its eigenvalues, we first solve

$$\det(\mathbb{A} - r\mathbb{I}) = \begin{vmatrix} 1-r & 2 & 1 \\ 1 & -1-r & 1 \\ 2 & 0 & -1-r \end{vmatrix} = -(r-3)(r+1)^2.$$

Hence, we see that $r_1 = 3$ *is an eigenvalue of algebraic multiplicity one while* $r_2 = r_3 = -1$ *is an eigenvalue of algebraic multiplicity two. To find the corresponding eigenvectors, we first compute*

$$\mathbb{A} - 3\mathbb{I} = \begin{pmatrix} -2 & 2 & 1 \\ 1 & -4 & 1 \\ 2 & 0 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}$$

where we have used elementary row operations. Hence, if we want to find a non-zero vector $\vec{x}$ *such that*

$$(\mathbb{A} - 3\mathbb{I})\vec{x} = \vec{0} \Leftrightarrow \begin{cases} x_1 - x_3 = 0, \\ x_2 - x_3/2 = 0, \end{cases}$$

and we can choose

$$x_1 = 1, \quad x_2 = 1/2, \quad x_3 = 1 \quad \Rightarrow \quad \vec{x} = \begin{pmatrix} 1 \\ 1/2 \\ 1 \end{pmatrix}.$$

Meanwhile, for the other eigenvalue $r_2 = r_3 = -1$, *we see that*

$$\mathbb{A} + \mathbb{I} = \begin{pmatrix} 2 & 2 & 1 \\ 1 & 0 & 1 \\ 2 & 0 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}.$$

Hence, for a non-zero vector $\vec{y}$ such that

$$(\mathbb{A}+\mathbb{I})\vec{y}=\vec{0} \Leftrightarrow \begin{cases} y_1+y_3=0 \\ y_2-y_3/2=0, \end{cases}$$

we can choose

$$y_1=-1, \quad y_2=1/2, \quad y_3=1 \quad \Rightarrow \quad \vec{y}=\begin{pmatrix} -1 \\ 1/2 \\ 1 \end{pmatrix}.$$

Unfortunately there is no other choice of $\vec{y}$, and thus we only have one eigenvector for the eigenvalue $r=-1$. Therefore the geometric multiplicity of $r=-1$ is one.

We now look at an example with complex eigenvalues.

Example 5.9. *Let*

$$\mathbb{A}=\begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix}.$$

Computing for the eigenvalues, we solve

$$\det(\mathbb{A}-r\mathbb{I})=r^2+2r+5=0.$$

The quadratic formula gives

$$r_1=-1+2i, \quad r_2=-1-2i.$$

Since both eigenvalues are distinct, the algebraic multiplicity and hence the geometric multiplicity are one. To find the eigenvectors, consider

$$\mathbb{A}-r_1\mathbb{I}=\begin{pmatrix} -2-2i & -2 \\ 4 & 2-2i \end{pmatrix} \to \begin{pmatrix} 1+i & 1 \\ 2 & 1-i \end{pmatrix} \to \begin{pmatrix} 1+i & 1 \\ 1 & 1/2-1/2i \end{pmatrix} \to \begin{pmatrix} 1+i & 1 \\ 0 & 0 \end{pmatrix}.$$

Hence, if

$$(\mathbb{A}-r_1\mathbb{I})\vec{x}_1=\vec{0} \Rightarrow (1+i)x_1+x_2=0,$$

we can choose

$$x_1=-1, \quad x_2=1+i \quad \Rightarrow \quad \vec{x}=\begin{pmatrix} -1 \\ 1+i \end{pmatrix}=\begin{pmatrix} -1 \\ 1 \end{pmatrix}+i\begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

that is the eigenvector corresponding to a complex eigenvalue is also complex-valued. Note that we do not need to repeat the computations in order to deduce the eigenvector corresponding to r_2, since $\mathbb{A}$ is a real-valued matrix we see that

$$\overline{(\mathbb{A}-r_1\mathbb{I})}=\mathbb{A}-r_2\mathbb{I}.$$

Hence, if $\vec{x}$ is an eigenvector corresponding to r_1, we have

$$\vec{0}=\overline{(\mathbb{A}-r_1\mathbb{I})\vec{x}}=(\mathbb{A}-r_2\mathbb{I})\overline{\vec{x}},$$

and we can take the eigenvector $\vec{y}$ corresponding to r_2 as the complex-conjugate of $\vec{x}$, i.e.,

$$\vec{y}=\begin{pmatrix} -1 \\ 1-i \end{pmatrix}=\begin{pmatrix} -1 \\ 1 \end{pmatrix}-i\begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

The takeaway message is that complex eigenvalues and eigenvectors occur in ***conjugate pairs****.*

Exercise: Find the eigenvalues and eigenvectors of these matrices

$$\mathbb{A} = \begin{pmatrix} 5 & 3 \\ -1 & 1 \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} -3 & -5 \\ 3/4 & 1 \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}.$$

5.3 Basic theory of systems of first order linear equations

The general first order linear system is

$$\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t) + \vec{g}(t),$$

for given $\vec{g}(t) = (g_1(t), \ldots, g_n(t))^\top$ and $\mathbb{P}(t)$ is a square matrix of functions

$$\mathbb{P}(t) = \begin{pmatrix} P_{11}(t) & \ldots & P_{1n}(t) \\ \vdots & \ddots & \vdots \\ P_{n1}(t) & \ldots & P_{nn}(t) \end{pmatrix}.$$

Recall that

(a) Second order equations $\to$ 2 L.I. solutions to the homogeneous equation;

(b) nth order equations $\to$ n L.I. solutions to the homogeneous equation;

and so for a system of n first order equations, we expect n L.I. solutions to the homogeneous system.

Let us use the following notation:

$$\boxed{\vec{y}_j(t) = \text{ j-th solution,} \quad y_{ij}(t) = \text{ i-th component of the j-th solution}}.$$

This means that

$$\boxed{\vec{y}_j(t) = \begin{pmatrix} y_{1j}(t) \\ y_{2j}(t) \\ \vdots \\ y_{nj}(t) \end{pmatrix}}.$$

Theorem 5.3 (Principle of superposition). *Let $\vec{y}_1$ and $\vec{y}_2$ be two solutions to the homogeneous system*

$$\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t) \tag{5.1}$$

then any linear combination

$$\vec{\phi}(t) = c_1\vec{y}_1(t) + c_2\vec{y}_2(t)$$

is also a solution for any $c_1, c_2 \in \mathbb{R}$.

The natural question is: Can every solution to the homogeneous system (5.1) be written as a linear combination? The answer is **YES**, with some analogue of Wronskian for system of equations.

Definition 5.6 (Wronskian)**.** *Let $\vec{y}_1(t), \dots, \vec{y}_n(t)$ be n solutions to the homogeneous system* (5.1)*. We define the matrix*

$$\mathbb{X}(t) := \begin{pmatrix} y_{11}(t) & y_{12}(t) & \dots & y_{1n}(t) \\ y_{21}(t) & y_{22}(t) & \dots & y_{2n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(t) & y_{n2}(t) & \dots & y_{nn}(t) \end{pmatrix} = \begin{pmatrix} | & | & \dots & | \\ \vec{y}_1(t) & \vec{y}_2(t) & \dots & \vec{y}_n(t) \\ | & | & \dots & | \end{pmatrix},$$

where the i-th column of $\mathbb{X}$ is the vector $\vec{y}_i(t)$. Then we set the Wronskian $W(\vec{y}_1, \dots, \vec{y}_n)[t]$ to be

$$\boxed{W(\vec{y}_1, \dots, \vec{y}_n)[t] := \det\mathbb{X}(t).}$$

Remark 5.2. *Note that this definition of Wronskian **<u>does not involve derivatives</u>**! Furthermore,*

$$\boxed{W \neq 0 \Leftrightarrow \det\mathbb{X} \neq 0 \Leftrightarrow \{\vec{y}_1, \dots, \vec{y}_n\} \text{ are L.I.}}.$$

Unlike for scalar equations, where we have $W \neq 0 \Rightarrow$ L.I., but the converse is in general not true unless the functions are solutions to a homogeneous equation, in the system case, we have $W \neq 0 \Leftrightarrow$ L.I..

Theorem 5.4. *Let $\vec{y}_1(t), \dots, \vec{y}_n(t)$ be n solutions to the homogeneous system* (5.1) *defined on an open interval I. Then, $\vec{y}_1(t), \dots, \vec{y}_n(t)$ are linearly independent on the interval I **<u>if and only if</u>** the Wronskian $W(\vec{y}_1, \dots, \vec{y}_n)[t]$ is non-zero for $t \in I$. In such a case we say that $\{\vec{y}_1(t), \dots, \vec{y}_n(t)\}$ forms a fundamental set of solutions, and any solution $\vec{\phi}(t)$ to the homogeneous system* (5.1) *can be expressed as a linear combination:*

$$\vec{\phi}(t) = c_1\vec{y}_1(t) + \dots + c_n\vec{y}_n(t),$$

*for constants $c_1, \dots, c_n \in \mathbb{R}$ in **<u>exactly one way</u>**. That is, the constants $c_1, \dots, c_n$ are **<u>uniquely</u>** determined.*

Proof. The aim is to show if $\vec{y}_1, \dots, \vec{y}_n$ are linearly independent (or equivalently $W(\vec{y}_1, \dots, \vec{y}_n) \neq 0$), then any solution can be written as a linear combination of $\vec{y}_1, \dots, \vec{y}_n$. Let $\vec{\phi}$ be any solution to the homogeneous system (5.1) for $t \in I$, where I is an open interval. Let $t_0 \in I$ and denote the vector

$$\vec{\xi} := \vec{\phi}(t_0) = (\xi_1, \dots, \xi_n)^\top.$$

Then, we find values $c_1, \dots, c_n \in \mathbb{R}$ that satisfies

$$c_1\vec{y}_1(t_0) + \dots + c_n\vec{y}_n(t_0) = \vec{\xi},$$

or equivalently

$$\begin{aligned} c_1 y_{11}(t_0) + \dots + c_n y_{1n}(t_0) &= \xi_1, \\ &\vdots \\ c_1 y_{n1}(t_0) + \dots + c_n y_{nn}(t_0) &= \xi_n \end{aligned}$$

or also equivalently

$$\begin{pmatrix} y_{11}(t_0 & \dots & y_{1n}(t_0) \\ \vdots & \ddots & \vdots \\ y_{n1}(t_0) & \dots & y_{nn}(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix} = \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \end{pmatrix}.$$

As the Wronskian is not zero at t_0, the matrix is invertible and hence there is a unique solution $(c_1^*, \dots, c_n^*)^\top$ to the above problem.

Now define a new function $\vec{\eta}$ by

$$\vec{\eta}(t) = c_1^* \vec{y}_1(t) + \dots + c_n^* \vec{y}_n(t) \quad \forall t \in I.$$

It is clear that $\vec{\eta}(t_0) = \vec{\xi} = \vec{\phi}(t_0)$. Hence, both $\vec{\eta}$ and $\vec{\phi}$ are solutions to the IVP

$$\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t), \quad \vec{y}(t_0) = \vec{\xi}.$$

By uniqueness we must have $\vec{\eta} = \vec{\phi}$ and thus

$$\vec{\phi}(t) = c_1^* \vec{y}_1(t) + \dots + c_n^* \vec{y}_n(t) \quad \forall t \in I.$$

□

Since we have an analogue of the Wronskian for systems of equations, we should expect an analogue of Abel's theorem as well. For systems of equations, this is called <u>**Liouville's formula**</u>.

Theorem 5.5 (Liouville's formula)**.** *Let $\vec{y}_1, \dots, \vec{y}_n$ be n solutions to the homogeneous equation* (5.1) *in the open interval I. Then, the Wronskian is given by*

$$\boxed{W(\vec{y}_1, \dots, \vec{y}_n)[t] = c \exp\left(\int \operatorname{tr}(\mathbb{P}(t))\, dt \right)},$$

where the trace of a matrix $\mathbb{A} \in \mathbb{R}^{n \times n}$ is defined as

$$\operatorname{tr}(\mathbb{A}) := \sum_{i=1}^{n} a_{ii} \quad (\textit{sum of the diagonal entries}),$$

and c is a constant not depending on $t \in I$. Consequently, the Wronskian is either always zero for $t \in I$ or never zero for $t \in I$.

Proof. We will prove this for the case $n = 2$: Let $\vec{y}_1, \vec{y}_2$ be two solutions to the homogeneous system (5.1), i.e.,

$$\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t), \quad \mathbb{P}(t) \in \mathbb{R}^{2\times 2} \text{ for } t \in I.$$

Then, the Wronskian is

$$W(\vec{y}_1, \vec{y}_2)[t] = \begin{vmatrix} y_{11}(t) & y_{12}(t) \\ y_{21}(t) & y_{22}(t) \end{vmatrix} = y_{11}(t)y_{22}(t) - y_{12}(t)y_{21}(t).$$

Taking the derivative leads to

$$\begin{aligned}
\frac{d}{dt}W[t] &= y'_{11}(t)y_{22}(t) - y'_{12}(t)y_{21}(t) + y_{11}(t)y'_{22}(t) - y_{12}(t)y'_{21}(t) \\
&= \begin{vmatrix} y'_{11}(t) & y'_{12}(t) \\ y_{21}(t) & y_{22}(t) \end{vmatrix} + \begin{vmatrix} y_{11}(t) & y_{12}(t) \\ y'_{21}(t) & y'_{22}(t) \end{vmatrix} \\
&= \begin{vmatrix} P_{11}y_{11} + P_{12}y_{21} & P_{11}y_{12} + P_{12}y_{22} \\ y_{21}(t) & y_{22}(t) \end{vmatrix} + \begin{vmatrix} y_{11}(t) & y_{12}(t) \\ P_{21}y_{11} + P_{22}y_{21} & P_{21}y_{12} + P_{22}y_{22} \end{vmatrix} \\
&= (P_{11} + P_{22})(y_{11}y_{22} - y_{12}y_{21}) = (P_{11} + P_{22})W[t],
\end{aligned}$$

where we have used from the fact that $\vec{y}_1, \vec{y}_2$ solve $\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t)$ to deduce

$$\begin{pmatrix} y'_{11} \\ y'_{21} \end{pmatrix} = \begin{pmatrix} P_{11}y_{11} + P_{12}y_{21} \\ P_{21}y_{11} + P_{22}y_{21} \end{pmatrix}, \quad \begin{pmatrix} y'_{12} \\ y'_{22} \end{pmatrix} = \begin{pmatrix} P_{11}y_{12} + P_{12}y_{22} \\ P_{21}y_{12} + P_{22}y_{22} \end{pmatrix}.$$

This implies we have

$$\frac{d}{dt}W[t] = (P_{11} + P_{22})W[t] = \mathrm{tr}(\mathbb{P}(t))W[t].$$

□

Next, the question of "does at least one fundamental set of solutions for systems of equations always exists" is answered.

Theorem 5.6 (Existence of at least one fundamental set of solutions)**.** *Let*

$$\vec{e}_i = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

where the entry 1 *appears in the* i*-th row, and let* $\vec{y}_i$ *be the unique solution to the IVP*

$$\begin{aligned}
\vec{y}'(t) &= \mathbb{P}(t)\vec{y}(t) \text{ for } t \in I, \\
\vec{y}(t_0) &= \vec{e}_i,
\end{aligned}$$

for $t_0 \in I$*. Then, the functions* $\vec{y}_1(t), \ldots, \vec{y}_n(t)$ *form a fundamental set of solutions to the homogeneous system* $\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t)$*.*

Proof. Simply compute the Wronskian at t_0:

$$W(\vec{y}_1, \ldots, \vec{y}_n)[t_0] = \det \mathbb{I} = 1 \neq 0.$$

□

Note that once a fundamental set of solutions has been found, we can construct **other** fundamental set of solutions by forming linear combinations of the vectors from the first set - however one must also ensure that the new set of functions are linearly independent.

Finally, just as for second order equations, a system with real coefficients may give rise to complex-valued solutions.

Theorem 5.7. *If $\vec{y}(t) = \vec{u}(t) + i\vec{v}(t)$ is a complex-valued solution to the homogeneous system* (5.1)*, where the entries of $\mathbb{P}(t)$ are* ***real-valued*** *functions, and the vectors $\vec{u}(t)$ and $\vec{v}(t)$ are also real-valued, then $\vec{u}(t)$ and $\vec{v}(t)$ are both solutions to the homogeneous system.*

Summary: A set of n linearly independent solutions $\vec{y}_1, \ldots, \vec{y}_n$ to $\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t)$ forms a fundamental set of solutions (which at least one set always exists) and any solution $\vec{\phi}$ to the homogeneous system can be written as a **unique** linear combination of $\vec{y}_1, \ldots, \vec{y}_n$.

We now turn to the case of constant coefficients, i.e., $\mathbb{P}(t) = \mathbb{A}$, where $\mathbb{A}$ is a square matrix with real, constant coefficients (not functions of t), and our goal is to derive explicit formulae for $\vec{y}_1, \ldots, \vec{y}_n$.

5.4 Homogeneous system with constant coefficients

We now focus on systems of the form

$$\vec{y}'(t) = \mathbb{A}\vec{y}(t), \quad t \in I, \tag{5.2}$$

where $\mathbb{A} \in \mathbb{R}^{n \times n}$. There are three special cases which we can already deal with.

(1) In the case $n = 1$, then $\mathbb{A}$ is just a scalar, i.e., $\mathbb{A} = a \in \mathbb{R}$, then (5.2) becomes

$$y'(t) = ay(t) \Rightarrow y(t) = ce^{at}, \quad c \in \mathbb{R}.$$

(2) $\mathbb{A}$ is a diagonal matrix:

$$\mathbb{A} = \begin{pmatrix} r_1 & 0 & \ldots & 0 \\ 0 & r_2 & \ldots & 0 \\ 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & r_n \end{pmatrix}$$

for constants $r_1, \ldots, r_n \in \mathbb{R}$. Then (5.2) reduces to

$$\begin{cases} y_1'(t) & = r_1 y_1(t), \\ y_2'(t) & = r_2 y_2(t), \\ & \vdots \\ y_n'(t) & = r_n y_n(t), \end{cases} \Rightarrow y_i(t) = c_i e^{r_i t}, \quad c_i \in \mathbb{R}, \quad 1 \le i \le n.$$

(3) $\mathbb{A}$ is a **Jordan matrix**. A **Jordan block** J is a matrix in $\mathbb{R}^{k\times k}$, where $1 \le k \le n$ is of the form

$$J = \begin{pmatrix} r & 1 & 0 & \dots & 0 \\ 0 & r & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & r & 1 \\ 0 & 0 & 0 & \dots & r \end{pmatrix},$$

that is, the main diagonal is a constant $r \in \mathbb{R}$, and immediately above the main diagonal is a diagonal of ones. The rest of the matrix entries is zero. We say that the matrix $\mathbb{A}$ is in **Jordan normal form** if there are Jordan blocks $J_1, \dots, J_m$, where each $J_i \in \mathbb{R}^{k_i\times k_i}$ and $k_1 + \dots + k_m = n$, such that $\mathbb{A}$ is of the form

$$\mathbb{A} = \begin{pmatrix} \boxed{J_1} & O & O & O & O \\ O & \boxed{J_2} & O & O & O \\ O & O & \boxed{J_3} & O & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & O & O & O & \boxed{J_m} \end{pmatrix}.$$

For example, let

$$J_1 = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}, \quad J_2 = (3), \quad J_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix},$$

then

$$\mathbb{A} = \begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

To solve the system

$$\vec{y}'(t) = J\vec{y}(t), \quad J = \begin{pmatrix} r & 1 & 0 & \dots & 0 \\ 0 & r & 1 & \dots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & r & 1 \\ 0 & 0 & 0 & \dots & r \end{pmatrix} \in \mathbb{R}^{n\times n}$$

writing out the equations gives

$$\begin{cases} y_n'(t) & = ry_n(t), \\ y_{n-1}'(t) & = ry_{n-1}(t) + y_n(t), \\ y_{n-2}'(t) & = ry_{n-2}(t) + y_{n-1}(t), \\ & \vdots \\ y_2'(t) & = ry_2(t) + y_3(t), \\ y_1'(t) & = ry_1(t) + y_2(t). \end{cases}$$

In particular, we can solve in **reverse order**, first compute y_n, then y_{n-1}, then y_{n-2}, and so on.

Exercise. Solve $\vec{y}'(t) = \mathbb{A}\vec{y}(t)$ when $\mathbb{A}$ is of the form

$$\mathbb{A} = \begin{pmatrix} 7 & 1 \\ 0 & 7 \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{pmatrix}.$$

What about a general matrix $\mathbb{A} \in \mathbb{R}^{n\times n}$? The idea is to try

$$\vec{y}(t) = \vec{\xi}e^{rt},$$

where $\vec{\xi}$ is a **constant vector** (not depending on t) and $r \in \mathbb{C}$. We have to determine the constant r and the constant vector $\vec{\xi}$ to obtain a solution.

Substituting this function into the equation yields

$$\vec{0} = \vec{y}'(t) - \mathbb{A}\vec{y}(t) = e^{rt}(r\vec{\xi} - \mathbb{A}\vec{\xi}) = e^{rt}(\mathbb{A} - r\mathbb{I})\vec{\xi}.$$

Since the exponential term is never zero, we see that for $\vec{\xi}e^{rt}$ to be a solution to the homogeneous system, we require

$$\boxed{(\mathbb{A} - r\mathbb{I})\vec{\xi} = \vec{0}},$$

i.e., the constant r should be an **eigenvalue** of the matrix $\mathbb{A}$ with corresponding **eigenvector** $\vec{\xi}$.

5.4.1 Two-by-two matrices

Let $\mathbb{A} \in \mathbb{R}^{2\times 2}$ be a two-by-two matrix with real entries. Then, $\mathbb{A}$ has two eigenvalues. What are the possibilities for the eigenvalues r_1 and r_2?

(1) $r_1, r_2 \in \mathbb{R}$, $r_1 \neq r_2$ - real and distinct;

(2) $r_1, r_2 \in \mathbb{C}$, $r_1 = \lambda + i\mu$, $\lambda, \mu \in \mathbb{R}$ with $r_2 = \lambda - i\mu$ - complex conjugate pair;

(3) $r_1 = r_2 \in \mathbb{R}$ - repeated and real.

Note that it is not possible to have

- $r_1 \in \mathbb{R}$, $r_2 \notin \mathbb{R}$;
- $r_1 = r_2 \notin \mathbb{R}$,

since complex eigenvalues **always** occur in conjugate pairs, and if $r_1 = r_2$ with r_1, r_2 complex, this implies that $\lambda + i\mu = \lambda - i\mu$ and so $\mu = 0$ and $r_1 = r_2 = \lambda \in \mathbb{R}$.

Case 1 - Real distinct eigenvalues. Let $\vec{\xi}_1$ and $\vec{\xi}_2$ be the corresponding eigenvectors to r_1 and r_2. Note that $\vec{\xi}_1$ and $\vec{\xi}_2$ are linearly independent. Then, we can compute the Wronskian to see that for the functions $\vec{y}_1(t) = \vec{\xi}_1 e^{r_1 t}$ and $\vec{y}_2(t) = \vec{\xi}_2 e^{r_2 t}$,

$$W(\vec{y}_1, \vec{y}_2)[t] = \begin{vmatrix} \xi_{11}e^{r_1 t} & \xi_{12}e^{r_2 t} \\ \xi_{21}e^{r_1 t} & \xi_{22}e^{r_2 t} \end{vmatrix} = e^{r_1 t} \begin{vmatrix} \xi_{11} & \xi_{12}e^{r_2 t} \\ \xi_{21} & \xi_{22}e^{r_2 t} \end{vmatrix} = e^{(r_1+r_2)t} \begin{vmatrix} \xi_{11} & \xi_{12} \\ \xi_{21} & \xi_{22} \end{vmatrix} \neq 0.$$

for any $t \in I$. Hence, by Theorem 5.4, the general solution to the homogeneous system (5.2) is

$$\boxed{\vec{y}(t) = c_1 e^{r_1 t}\vec{\xi}_1 + c_2 e^{r_2 t}\vec{\xi}_2}.$$

Example 5.10. *For*

$$\mathbb{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}$$

with eigenvalues and corresponding eigenvectors

$$r_1 = 3, \quad \xi_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad r_2 = -1, \quad \xi_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix},$$

the general solution is

$$\vec{y}(t) = c_1 e^{3t}\begin{pmatrix} 1 \\ 2 \end{pmatrix} + c_2 e^{-t}\begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Case 2 - Comples conjugate eigenvalues. Let $r_1 = \lambda + i\mu$, with $\lambda, \mu \in \mathbb{R}$ and corresponding eigenvector $\vec{\xi}_1 = \vec{u} + i\vec{v}$. We expect that

$$\vec{x}_1(t) = (\vec{u} + i\vec{v})e^{(\lambda+i\mu)t}, \quad \vec{x}_2(t) = (\vec{u} - i\vec{v})e^{(\lambda-i\mu)t}$$

are solutions to the homogeneous system (5.2). But the disadvantage of using $\vec{x}_1$ and $\vec{x}_2$ is that they are complex-valued. Therefore, we rewrite

$$\begin{aligned} \vec{x}_1 &= (\vec{u} + i\vec{v})e^{\lambda t}(\cos(\mu t) + i\sin(\mu t)) \\ &= e^{\lambda t}[\vec{u}\cos(\mu t) - \vec{v}\sin(\mu t)] + ie^{\lambda t}[\vec{u}\sin(\mu t) + \vec{v}\cos(\mu t)]. \end{aligned}$$

Using Theorem 5.7 we infer that the real and imaginary parts of $\vec{x}_1$ are also solutions. Hence, we define

$$\vec{y}_1(t) = e^{\lambda t}(\vec{u}\cos(\mu t) - \vec{v}\sin(\mu t)), \quad \vec{y}_2(t) = e^{\lambda t}(\vec{u}\sin(\mu t) + \vec{v}\cos(\mu t)).$$

Since the real and imaginary parts of a complex eigenvector are linearly independent, we can check that the Wronskian for $\vec{y}_1$ and $\vec{y}_2$ is non-zero. Then, by Theorem 5.4 we see that the general solution to the homogeneous system (5.2) is

$$\boxed{\vec{y}(t) = e^{\lambda t}\left[\cos(\mu t)[c_1\vec{u} + c_2\vec{v}] + \sin(\mu t)[c_2\vec{u} - c_1\vec{v}]\right]}.$$

Example 5.11. *For*

$$\mathbb{A} = \begin{pmatrix} -3 & -2 \\ 4 & 1 \end{pmatrix}$$

with eigenvalues $r_1 = \overline{r_2}$ *and corresponding eigenvectors* $\vec{x}_1 = \overline{\vec{x}_2}$*:*

$$r_1 = -1 + 2i, \quad \vec{\xi}_1 = \begin{pmatrix} -1 \\ 1 \end{pmatrix} + i \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

the general solution is

$$\vec{y}(t) = c_1 e^{-t} \left(\cos(2t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \sin(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right) \\ + c_2 e^{-t} \left(\sin(2t) \begin{pmatrix} -1 \\ 1 \end{pmatrix} + \cos(2t) \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right).$$

Case 3 - Repeated real eigenvalues. If $r_1 = r_2$, then we have an eigenvalue with algebraic multiplicity of two. We need to divide our analysis into two subcases:

(1) The geometric multiplicity is also two, which implies there are two linearly independent eigenvectors $\vec{\xi}_1, \vec{\xi}_2$ corresponding to $r_1 = r_2 =: q$. Then, going back to Case 1, the general solution to the homogeneous system (5.2) is

$$\vec{y}(t) = c_1 e^{qt} \vec{\xi}_1 + c_2 e^{qt} \vec{\xi}_2.$$

(2) If the geometric multiplicity is one, then there is only one eigenvector $\vec{\xi}$ corresponding to the eigenvalue q. We know one solution is

$$\vec{y}_1 = \vec{\xi} e^{qt},$$

what about a second solution that is linearly independent? As with second order equations, let's try

$$\vec{z}(t) = t\vec{\xi} e^{qt}.$$

Differentiating and plugging this into the homogeneous system (5.2) leads to

$$\vec{z}\,'(t) - \mathbb{A}\vec{z}(t) = \vec{\xi}(qte^{qt} + e^{qt}) - \mathbb{A}\vec{\xi} te^{qt} = (\vec{\xi} q - \mathbb{A}\vec{\xi}) te^{qt} + \vec{\xi} e^{qt}.$$

We observe there are two terms: one involving the coefficient te^{qt} and the other involving just the coefficient e^{qt}. Since we want $\vec{z}$ to be a solution, both terms must vanish. Hence, we require

$$\mathbb{A}\vec{\xi} = q\vec{\xi}, \quad \vec{\xi} = \vec{0}.$$

The first condition amounts to saying $\vec{\xi}$ is an eigenvector for q, which we already have by definition, but the second condition leads to a contradiction.

Therefore, we deduce that a solution to the homogeneous system (5.2) cannot be of the form $t\vec{\xi}e^{qt}$.

To remedy this, since computing $\vec{z}'(t) - \mathbb{A}\vec{z}$ leads to an expression involving te^{qt} and e^{qt}, we should try

$$\vec{w}(t) = \vec{\xi}te^{qt} + \vec{\eta}e^{qt},$$

for some constant vector $\vec{\eta}$ to be determined. Then, computing $\vec{w}'(t) - \mathbb{A}\vec{w}(t)$ gives

$$\vec{w}'(t) - \mathbb{A}\vec{w}(t) = te^{qt}(q\vec{\xi} - \mathbb{A}\vec{\xi}) + e^{qt}(q\vec{\eta} - \mathbb{A}\vec{\eta} + \vec{\xi}).$$

Hence, for $\vec{w}$ to be a solution we need

$$\mathbb{A}\vec{\xi} = q\vec{\xi}, \quad \boxed{(\mathbb{A} - q\mathbb{I})\vec{\eta} = \vec{\xi}}.$$

Since $\vec{\xi}$ is an eigenvector corresponding to q, the first condition is satisfied. Now, suppose the vector $\vec{\eta}$ exists such that

$$(\mathbb{A} - q\mathbb{I})\vec{\eta} = \vec{\xi},$$

then we have two solutions

$$\boxed{\vec{y}_1(t) = \vec{\xi}e^{qt}, \quad \vec{y}_2(t) = t\vec{\xi}e^{qt} + \vec{\eta}e^{qt}}.$$

Computing the Wronskian gives

$$W(\vec{y}_1, \vec{y}_2)[t] = e^{qt}\begin{vmatrix} \xi_1 & t\xi_1 + \eta_1 \\ \xi_2 & t\xi_2 + \eta_2 \end{vmatrix} = e^{qt}\begin{vmatrix} \xi_1 & \eta_1 \\ \xi_2 & \eta_2 \end{vmatrix},$$

and so the Wronskian is non-zero if and only if $\vec{\xi}$ and $\vec{\eta}$ are linearly independent. If so, then by Theorem 5.4, the general solution is

$$\boxed{\vec{y}(t) = c_1\vec{\xi}e^{qt} + c_2(t\vec{\xi}e^{qt} + \vec{\eta}e^{qt})}.$$

In the above second case for repeated real eigenvalues, there are still two unresolved issues:

- Does $\vec{\eta}$ such that $(\mathbb{A} - q\mathbb{I})\vec{\eta}) = \vec{\xi}$ exists?
- Are $\vec{\xi}$ and $\vec{\eta}$ linearly independent?

First, let us answer the second question. Suppose there are constants α_1, α_2 such that $\alpha_1\vec{\xi} + \alpha_2\vec{\eta} = \vec{0}$. Since $\mathbb{A} \neq q\mathbb{I}$ (otherwise $\vec{\eta}$ would not exist), applying $\mathbb{A} - q\mathbb{I}$ leads to

$$\vec{0} = \alpha_1(\mathbb{A} - q\mathbb{I})\vec{\xi} + \alpha_2(\mathbb{A} - q\mathbb{I})\vec{\eta} = \alpha_2\vec{\xi},$$

since $\vec{\xi}$ is an eigenvector corresponding to q. This implies that $\alpha_2 = 0$, since $\vec{\xi}$ is non-zero. Then, going back we see that

$$\alpha_1\vec{\xi} = \vec{0} \Rightarrow \alpha_1 = 0.$$

Hence, if $\vec{\eta}$ exists, we see that $\vec{\xi}$ and $\vec{\eta}$ are linearly independent.

For the first question, take another vector $\vec{v}$ that is not a constant multiple of the eigenvector $\vec{\xi}$. Then, since $\vec{\xi}$ is a vector in $\mathbb{R}^2$, we see that $\vec{v}$ and $\vec{\xi}$ must be linearly independent (if $\vec{v}$ is not a constant multiple of $\vec{\xi}$), and hence they also form a basis of $\mathbb{R}^2$. So every vector $\vec{x} \in \mathbb{R}^2$ can be written as a linear combination of $\vec{v}$ and $\vec{\xi}$.

Define the vector $\vec{w} = (\mathbb{A} - q\mathbb{I})\vec{v}$. Then, we can find constants $\alpha, \beta \in \mathbb{R}$ such that

$$\vec{w} = \alpha\vec{v} + \beta\vec{\xi}.$$

Now apply $\mathbb{A} - q\mathbb{I}$ to both sides gives

$$(\mathbb{A} - q\mathbb{I})\vec{w} = \alpha(\mathbb{A} - q\mathbb{I})\vec{v} + \beta(\mathbb{A} - q\mathbb{I})\vec{\xi} = \alpha(\mathbb{A} - q\mathbb{I})\vec{v} = \alpha\vec{w},$$

since $\vec{\xi}$ is an eigenvector of $\mathbb{A}$. Rearranging gives

$$\mathbb{A}\vec{w} = (q + \alpha)\vec{w},$$

and so $\vec{w}$ is an eigenvector corresponding to eigenvalue $\alpha + q$. But, since $\mathbb{A}$ has a repeated eigenvalue q, there is no other possible eigenvalues and hence <u>α **must be zero**</u>. From this, we see that

$$\vec{w} = \beta\vec{\xi}, \quad \beta \neq 0.$$

that is $\vec{w}$ is parallel to $\vec{\xi}$. Recalling the definition of $\vec{w}$, we see that

$$\vec{w} = (\mathbb{A} - q\mathbb{I})\vec{v} = \beta\vec{\xi},$$

and if we set $\vec{\eta} = \frac{1}{\beta}\vec{v}$, we see that

$$(\mathbb{A} - q\mathbb{I})\vec{\eta} = \vec{\xi}$$

as required.

Example 5.12. *For*

$$\mathbb{A} = \begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix},$$

the eigenvalues are $r_1 = r_2 = q = 2$*, i.e., algebraic multiplicity is two, while the eigenvector corresponding to* q *is*

$$\vec{\xi} = \begin{pmatrix} 1 \\ -1 \end{pmatrix},$$

and so the geometric multiplicity is one. We now need to find a vector $\vec{\eta}$ *such that*

$$(\mathbb{A} - 2\mathbb{I})\vec{\eta} = \vec{\xi}.$$

Computing $\mathbb{A} - 2\mathbb{I}$ *gives*

$$\begin{pmatrix} -1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix} \Rightarrow -\eta_1 - \eta_2 = 1.$$

We can take $\eta_1 = 0$ *and* $\eta_2 = -1$*, leading to the general solution*

$$\vec{y}(t) = c_1 e^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2\left(te^{2t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + e^{2t}\begin{pmatrix} 0 \\ -1 \end{pmatrix}\right).$$

Definition 5.7. *Given a real square matrix* $\mathbb{A} \in \mathbb{R}^{n\times n}$ *with eigenvalue* q *and corresponding eigenvector* $\vec{\xi}$*. We say a vector* $\vec{\eta}$ *is a* ***generalized eigenvector*** *(of rank 1) corresponding to the eigenvalue* q *if*

$$\boxed{(\mathbb{A} - q\mathbb{I})\vec{\eta} = \vec{\xi}},$$

or equivalently

$$\boxed{(\mathbb{A} - q\mathbb{I})^2\vec{\eta} = \vec{0}}.$$

With this the theory for the case $\mathbb{A} \in \mathbb{R}^{2\times 2}$ is complete. But what about for the general case $\mathbb{A} \in \mathbb{R}^{n\times n}$. There are some specific subcases that are easily generalizable.

(a) If all the eigenvalues are **real and distinct**, $r_1 \neq r_2 \neq \cdots \neq r_n$, then there are n linearly independent eigenvectors $\vec{\xi}_1, \ldots, \vec{\xi}_n$ corresponding to $r_1, \ldots, r_n$. We see that the general solution is

$$\boxed{\vec{y}(t) = c_1\vec{\xi}_1 e^{r_1 t} + \cdots + c_n\vec{\xi}_n e^{r_n t}}.$$

(b) If there are k pairs of complex conjugate eigenvalues $r_1 = \overline{r_2}, \ldots, r_{2k-1} = \overline{r_{2k}}$, and the rest $r_{2k+1}, \ldots, r_n$ are real and distinct, then we still have n linearly independent eigenvectors, where $(\vec{u}_j, \vec{v}_j)$ for $1 \leq j \leq k$ are the real and imaginary parts of the eigenvectors corresponding to $r_1, \ldots, r_{2k}$, and $\vec{\xi}_{2k+1}, \ldots, \vec{\xi}_n$ are the eigenvectors corresponding to $r_{2k+1}, \ldots, r_n$. Then, the general solution is

$$\begin{aligned}\vec{y}(t) &= c_{2k+1}\vec{\xi}_{2k+1}e^{r_{2k+1}t} + \cdots + c_n\vec{\xi}_n e^{r_n t} \\ &\quad + c_1 e^{\lambda_1 t}(\vec{u}_1\cos(\mu_1 t) - \vec{v}_1\sin(\mu_1 t)) \\ &\quad + c_2 e^{\lambda_1 t}(\vec{u}_1\sin(\mu_1 t) + \vec{v}_1\cos(\mu_1 t)) \\ &\quad + \ldots.\end{aligned}$$

(c) If all repeated eigenvalues have geometric multiplicity = algebraic multiplicity, then a total of n linearly independent eigenvectors can be found. In this case the general solution is the same as in Case (a).

If there is a repeated eigenvalue q with geometric multiplicity **strictly less** than its algebraic multiplicity, then the theory is more complicated. To illustrate this, we now study the case where $\mathbb{A} \in \mathbb{R}^{3\times 3}$.

5.4.2 Three-by-three matrices

When $n = 3$, there are more combinations of possible eigenvalues than in the case $n = 2$. We just focus on the case where we have a repeated eigenvalue q with geo. mult. < alg. mult., since for the case geo. mult. = alg. mult. we know what to do. Then the following situations are possible:

(1) geo. mult. = 1 < alg. mult. = 3;

(2) geo. mult. = 2 < alg. mult. = 3;

(3) geo. mult. = 1 < alg. mult. = 2;

Note that Case 3 can be treated as in Section 5.4.1 as the alg. mult. is two, and so there remaining eigenvalue must be distinct from q. We will omit this and focus on the first two cases.

Case 1. Since geo. mult. is one, there is only one eigenvector $\vec{\xi} \in \mathbb{R}^3$ corresponding to the repeated eigenvalue q with alg. mult. three. As we have seen in the 2×2 theory, one solution is $\vec{y}_1(t) = \vec{\xi}e^{qt}$. Let's consider a generalized eigenvector $\vec{\eta} \in \mathbb{R}^3$ satisfying $(A - q\mathbb{I})\vec{\eta} = \vec{\xi}$, then we have a second solution $\vec{y}_2(t) = t\vec{\xi}e^{qt} + \vec{\eta}e^{qt}$. What about a third solution?

Let's try

$$\vec{w}(t) = \frac{t^2}{2}\vec{\xi}e^{qt} + t\vec{\eta}e^{qt} + \vec{\theta}e^{qt},$$

for some constant vector $\vec{\theta} \in \mathbb{R}^3$. If $\vec{w}$ solves the homogeneous system (5.2) then we must have

$$\vec{w}'(t) - \mathbb{A}\vec{w}(t) = e^{qt}((q\mathbb{I} - \mathbb{A})\vec{\theta} + \vec{\eta}) = \vec{0},$$

that is

$$\boxed{(\mathbb{A} - q\mathbb{I})\vec{\theta} = \vec{\eta}}.$$

Hence, to obtain three linearly independent solutions to the homogeneous system (5.2) we have to compute for vectors $\vec{\xi}, \vec{\eta}, \vec{\theta}$ such that

$$(\mathbb{A} - q\mathbb{I})\vec{\xi} = \vec{0}, \quad (\mathbb{A} - q\mathbb{I})\vec{\eta} = \vec{\xi}, \quad (\mathbb{A} - q\mathbb{I})\vec{\theta} = \vec{\eta}.$$

Note that the latter two conditions are equivalent to

$$(A - q\mathbb{I})^2\vec{\eta} = \vec{0}, \quad (\mathbb{A} - q\mathbb{I})^3\vec{\theta} = \vec{0},$$

and so we see that $\vec{\theta}$ is a generalized eigenvector of rank 2 corresponding to the eigenvalue q.

We now check if $\vec{\xi}, \vec{\eta}, \vec{\theta}$ are linearly independent. Suppose there are constants α, β, γ such that

$$\alpha\vec{\xi} + \beta\vec{\eta} + \gamma\vec{\theta} = \vec{0}.$$

Applying $(\mathbb{A} - q\mathbb{I})$ gives

$$\beta\vec{\xi} + \gamma\vec{\eta} = \vec{0},$$

and applying $(\mathbb{A} - q\mathbb{I})$ once more gives

$$\gamma\vec{\xi} = \vec{0}.$$

Since $\vec{\xi}$ is a non-zero eigenvector, we see that γ must be zero, and from this we can also deduce that $\alpha = \beta = 0$.

By Theorem 5.4 we have that the general solution is

$$\boxed{\vec{y}(t) = c_1\vec{\xi}e^{qt} + c_2\left(t\vec{\xi}e^{qt} + \vec{\eta}e^{qt}\right) + c_3\left(\frac{t^2}{2}\vec{\xi}e^{qt} + t\vec{\eta}e^{qt} + \vec{\theta}e^{qt}\right)}.$$

Case 2. Since the geo. mult. is two, we have two linearly independent eigenvectors $\vec{\xi}_1$ and $\vec{\xi}_2$ corresponding to the repeated eigenvalue q. This leads to two solutions

$$\vec{y}_1(t) = \vec{\xi}_1 e^{qt}, \quad \vec{y}_2(t) = \vec{\xi}_2 e^{qt}.$$

We need to construct a third solution that is linearly independent to $\vec{y}_1$ and $\vec{y}_2$. If we have $\vec{w}(t) = \vec{\theta}te^{qt} + \vec{\eta}e^{qt}$ is a solution, then we need

$$(\mathbb{A} - q\mathbb{I})\vec{\theta} = \vec{0}, \quad (\mathbb{A} - q\mathbb{I})\vec{\eta} = \vec{\theta}.$$

Here, we can choose $\vec{\theta}$ as a **linear combination** of the eigenvectors $\vec{\xi}_1$ and $\vec{\xi}_2$, such that there is a solution $\vec{\eta}$ to $(\mathbb{A} - q\mathbb{I})\vec{\eta} = \vec{\theta}$. We now demonstrate in an example why sometimes choosing $\vec{\theta} = \vec{\xi}_1$ or $\vec{\theta} = \vec{\xi}_2$ **may not yield** the existence of the generalized eigenvector $\vec{\eta}$.

Example 5.13. *For*

$$\mathbb{A} = \begin{pmatrix} 4 & 6 & -15 \\ 1 & 3 & -5 \\ 1 & 2 & -4 \end{pmatrix},$$

the characteristic equation is

$$P_{\mathbb{A}}(r) = (r-1)^3 \Rightarrow r_1 = r_2 = r_3 = q = 1,$$

i.e., we have a repeated eigenvalue $q = 1$ with alg. mult. three. Computing $\mathbb{A} - \mathbb{I}$ gives

$$\mathbb{A} - \mathbb{I} = \begin{pmatrix} 3 & 6 & -15 \\ 1 & 2 & -5 \\ 1 & 2 & -5 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -5 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

after applying elementary row operations. So if there is a vector $\vec{x}$ such that $(\mathbb{A}-\mathbb{I})\vec{x} = \vec{0}$, this is equivalent to solving

$$x_1 + 2x_2 - 5x_3 = 0.$$

We can choose

$$x_1 = -2, \quad x_2 = 1, \quad x_3 = 0 \Rightarrow \vec{\xi}_1 = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix},$$
$$x_1 = 5, \quad x_2 = 0, \quad x_3 = 1 \Rightarrow \vec{\xi}_2 = \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix}.$$

We have two linearly independent eigenvectors and so geo. mult. is two. We now choose $\vec{\theta} = \vec{\xi}_1$ *and try to compute for* $\vec{\eta}$*:*

$$(\mathbb{A} - \mathbb{I})\vec{\eta} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 3 & 6 & -15 \\ 1 & 2 & -5 \\ 1 & 2 & -5 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix}.$$

Writing the matrix problem as a system of linear equations, we see that there is no solution that will satisfy the above equation. Since we would have

$$\eta_1 + 2\eta_2 - 5\eta_3 = 1,$$
$$\eta_1 + 2\eta_2 - 5\eta_3 = 0.$$

Similarly, choose $\vec{\theta} = \vec{\xi}_2$*, we also find that there is no solution* $\vec{\eta}$ *to the problem* $(\mathbb{A} - \mathbb{I})\vec{\eta} = \vec{\xi}_2$*.*

So how do we fix this? Recall, $\vec{\theta}$ *and* $\vec{\eta}$ *satisfy the equations*

$$(\mathbb{A} - \mathbb{I})\vec{\theta} = \vec{0}, \quad (\mathbb{A} - q\mathbb{I})\vec{\eta} = \vec{\theta} \Rightarrow (\mathbb{A} - q\mathbb{I})^2\vec{\eta} = \vec{0}.$$

Let's compute for $\vec{\eta}$ *first and then try and determine* $\vec{\theta}$*. We see that*

$$(\mathbb{A} - \mathbb{I})^2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and so we can choose $\vec{\eta}$ *as any non-zero vector, for example*

$$\vec{\eta} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then,

$$\vec{\theta} = (\mathbb{A} - \mathbb{I})\vec{\eta} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}.$$

We now check if $(\mathbb{A} - \mathbb{I})\vec{\theta} = \vec{0}$*:*

$$\begin{pmatrix} 3 & 6 & -15 \\ 1 & 2 & -5 \\ 1 & 2 & -5 \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix},$$

and so the desired vectors are

$$\vec{\theta} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{\eta} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Let us note that

$$\vec{\theta} = \vec{\xi}_1 + \vec{\xi}_2$$

and so $\vec{\theta}$ is a linear combination of $\vec{\xi}_1$ and $\vec{\xi}_2$. Then, the general solution is

$$\vec{y}(t) = c_1 \begin{pmatrix} -2 \\ 1 \\ 0 \end{pmatrix} e^t + c_2 \begin{pmatrix} 5 \\ 0 \\ 1 \end{pmatrix} e^t + c_3 \left(\begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix} te^t + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} e^t \right).$$

Remark 5.3. *The above example shows that choosing $\vec{\theta} = \vec{\xi}_1$ or $\vec{\xi}_2$ may not give the existence of $\vec{\eta}$ needed to construct the third linearly independent solution. We may have to consider $\vec{\theta}$ as a linear combination of $\vec{\xi}_1$ and $\vec{\xi}_2$. Furthermore, it may be easier by first computing $(\mathbb{A} - q\mathbb{I})^2$ and finding $\vec{\eta}$, then set $\vec{\theta} := (\mathbb{A} - q\mathbb{I})\vec{\eta}$. Afterwards we must check that $(\mathbb{A} - q\mathbb{I})\vec{\theta} = \vec{0}$.*

Example 5.14. *For*

$$\mathbb{A} = \begin{pmatrix} -7 & -5 & -3 \\ 2 & -2 & -3 \\ 0 & 1 & 0 \end{pmatrix}$$

the characteristic polynomial is

$$P_{\mathbb{A}}(r) = (r+3)^3 \Rightarrow r_1 = r_2 = r_3 = -3.$$

Furthermore,

$$\mathbb{A} + 3\mathbb{I} = \begin{pmatrix} -4 & -5 & -3 \\ 2 & 1 & -3 \\ 0 & 1 & 3 \end{pmatrix} \to \begin{pmatrix} 2 & 1 & -3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{pmatrix}$$

after applying elementary row operations. Then

$$(\mathbb{A} + 3\mathbb{I})\vec{x} = \vec{0} \Rightarrow 2x_1 = 6x_3, \quad x_2 + 3x_3 = 0.$$

We can choose

$$x_1 = 3, \quad x_2 = 3, \quad x_3 = 1 \quad \Rightarrow \quad \vec{\xi} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix}.$$

Hence the alg. mult. is three but the geo. mult. is one. We have to find vectors $\vec{\eta}$ and $\vec{\theta}$ such that

$$(\mathbb{A} + 3\mathbb{I})\vec{\eta} = \vec{\xi}, \quad (\mathbb{A} + 3\mathbb{I})\vec{\theta} = \vec{\eta}.$$

Computing:

$$\begin{pmatrix} -4 & -5 & -3 \\ 2 & 1 & -3 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \\ \eta_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix},$$

we obtain simplified equations

$$\eta_1 - 3\eta_3 = -2, \quad \eta_1 + \eta_2 = -1.$$

Hence, we can take

$$\vec{\eta} = \begin{pmatrix} 0 \\ -1 \\ 2/3 \end{pmatrix}.$$

Then, we now seek $\vec{\theta}$ such that

$$\begin{pmatrix} -4 & -5 & -3 \\ 2 & 1 & -3 \\ 0 & 1 & 3 \end{pmatrix} \begin{pmatrix} \theta_1 \\ \theta_2 \\ \theta_3 \end{pmatrix} = \begin{pmatrix} 0 \\ -1 \\ 2/3 \end{pmatrix}.$$

We again get simplified equations

$$2\theta_1 - 6\theta_3 = -5/3, \quad 2\theta_1 + 2\theta_2 = -1/3.$$

We can take

$$\vec{\theta} = \begin{pmatrix} -1/6 \\ 0 \\ 2/9 \end{pmatrix}.$$

Then, we see that

$$\begin{vmatrix} | & | & | \\ \vec{\xi} & \vec{\eta} & \vec{\theta} \\ | & | & | \end{vmatrix} = -\frac{1}{2} \neq 0,$$

and so $\vec{\xi}, \vec{\eta}, \vec{\theta}$ are linearly independent, and the general solution i

$$\vec{y}(t) = e^{-3t} \begin{pmatrix} 3 \\ -3 \\ 1 \end{pmatrix} (c_1 + c_2 t + c_3 t^2/2) + e^{-3t} \begin{pmatrix} 0 \\ -1 \\ 2/3 \end{pmatrix} (c_2 + c_3 t) + e^{-3t} c_3 \begin{pmatrix} -1/6 \\ 0 \\ 2/9 \end{pmatrix}. \tag{5.3}$$

In the above example, we computed for $\vec{\xi}$, and then $\vec{\eta}$ and then $\vec{\theta}$ in this order. Now we present another method for find three linearly independent vectors $\vec{\xi}_*, \vec{\eta}_*, \vec{\theta}_*$ satisfying

$$(\mathbb{A} + 3\mathbb{I})\vec{\xi}_* = \vec{0}, \quad (\mathbb{A} + 3\mathbb{I})\vec{\eta}_* = \vec{\xi}_*, \quad (\mathbb{A} + 3\mathbb{I})\vec{\theta}_* = \vec{\eta}_*.$$

Example 5.15. *We begin by computing*

$$(\mathbb{A}+3\mathbb{I})^2 = \begin{pmatrix} 6 & 12 & 18 \\ -6 & -12 & -18 \\ 2 & 4 & 6 \end{pmatrix}, \quad (\mathbb{A}+3\mathbb{I})^3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

and since

$$(\mathbb{A}+3\mathbb{I})^3\vec{\theta}_* = \vec{0},$$

we can choose $\vec{\theta}_*$ *to be any non-zero vector, e.g.*

$$\vec{\theta}_* = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.$$

Then,

$$\vec{\eta}_* = (\mathbb{A}+3\mathbb{I})\vec{\theta}_* = \begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}, \quad \vec{\xi}_* = (\mathbb{A}+3\mathbb{I})\vec{\eta}_* = \begin{pmatrix} 6 \\ -6 \\ 2 \end{pmatrix}.$$

To check, we again compute

$$(\mathbb{A}+3\mathbb{I})\vec{\xi}_* = \vec{0},$$

and so the general solution is

$$\vec{y}_*(t) = e^{-3t}\begin{pmatrix} 6 \\ -6 \\ 2 \end{pmatrix}(c_1 + c_2 t + c_3 t^2/2) + e^{-3t}\begin{pmatrix} -4 \\ 2 \\ 0 \end{pmatrix}(c_2 + c_3 t) + e^{-3t}c_3\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \tag{5.4}$$

Now, are the two solutions (5.3) *and* (5.4) *equivalent? Indeed, since we can write* $\vec{\xi}, \vec{\eta}, \vec{\theta}$ *as linear combinations of* $\vec{\xi}_*, \vec{\eta}_*, \vec{\theta}_*$*:*

$$\vec{\xi} = 2\vec{\xi}_*, \quad \vec{\eta} = \frac{1}{3}\vec{\xi}_* + \frac{1}{2}\vec{\eta}_*, \quad \vec{\theta} = \frac{1}{9}\vec{\xi}_* + \frac{1}{3}\vec{\eta}_* + \frac{1}{2}\vec{\theta}_*.$$

5.5 Non-homogeneous linear systems

We now study for $\mathbb{A} \in \mathbb{R}^{n\times n}$ the non-homogeneous system

$$\vec{y}'(t) = \mathbb{A}\vec{y}(t) + \vec{g}(t), \tag{5.5}$$

and if $\vec{y}_1, \dots, \vec{y}_n$ are n linearly independent solutions to the homogeneous system $\vec{y}'(t) = \mathbb{A}\vec{y}(t)$, and $\vec{v}(t)$ is a particular solution to the non-homogeneous system, then the general solution is

$$\vec{y}(t) = c_1\vec{y}_1(t) + \cdots + c_n\vec{y}_n(t) + \vec{v}(t).$$

There are certain special cases where the theory simplifies.

(1) $\mathbb{A} \in \mathbb{R}^{n \times n}$ is a diagonal matrix, i.e.,

$$\mathbb{A} = \begin{pmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r_n \end{pmatrix},$$

then (5.5) becomes

$$\begin{aligned} y_1'(t) &= r_1 y_1(t) + g_1(t), \\ &\vdots \\ y_n'(t) &= r_n y_n(t) + g_n(t), \end{aligned}$$

i.e., we obtain n first order linear equations where the solution is

$$y_i(t) = e^{r_i t} \int e^{-r_i s} g_i(s)\, ds + c_i e^{r_i t}.$$

(2) $\mathbb{A}$ is a Jordan matrix, i.e.,

$$\mathbb{A} = \begin{pmatrix} r & 1 & 0 & \dots \\ 0 & r & 1 & \dots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & r \end{pmatrix},$$

then (5.5) becomes

$$\begin{aligned} y_1'(t) &= r y_1(t) + r_2(t) + g_1(t), \\ y_2'(t) &= r y_2(t) + y_3(t) + g_2(t), \\ &\vdots \\ y_{n-1}'(t) &= r y_{n-1}(t) + y_n(t) + g_{n-1}(t), \\ y_n'(t) &= r y_n(t) + g_n(t). \end{aligned}$$

We can solve in reverse order:

$$y_n(t) = e^{rt} \int e^{-rs} g_n(s)\, ds + c_n e^{rt},$$

and then solve for y_{n-1}, and so on.

Since having the matrix $\mathbb{A}$ in a diagonal or Jordan form is rather useful, can we transform a general matrix $\mathbb{A}$ into something equivalent?

Definition 5.8. *A matrix $\mathbb{A} \in \mathbb{R}^{n \times n}$ is **diagonalizable** if there is an invertible matrix $\mathbb{P}$ and a diagonal matrix $\mathbb{D}$ such that*

$$\boxed{\mathbb{P}^{-1} \mathbb{A} \mathbb{P} = \mathbb{D}}.$$

From linear algebra, if $\mathbb{A} \in \mathbb{R}^{n\times n}$ has n linearly independent eigenvectors $\vec{x}_1, \dots, \vec{x}_n$ with eigenvalues $r_1, \dots, r_n$, then

$$\mathbb{P} = (\vec{\xi}_1\ \vec{\xi}_2\ \dots\ \vec{\xi}_n), \quad \mathbb{D} = \begin{pmatrix} r_1 & 0 & \dots & 0 \\ 0 & r_2 & \dots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \dots & r_n \end{pmatrix},$$

i.e., the columns of $\mathbb{P}$ are the eigenvectors and $\mathbb{D}$ is the diagonal matrix where the entries of the main diagonal are the eigenvalues. Hence, if $\mathbb{A}$ is diagonalizable, we define the new vector $\vec{x} := \mathbb{P}^{-1}\vec{y}$, which is well-defined since $\mathbb{P}$ is invertible. Then,

$$\begin{aligned} &\vec{x}'(t) = \mathbb{P}^{-1}\vec{y}'(t) = \mathbb{P}^{-1}(\mathbb{A}\mathbb{P}\vec{x}(t) + \vec{g}(t)) \\ \Rightarrow &\boxed{\vec{x}'(t) = \mathbb{D}\vec{x}(t) + \vec{h}(t), \quad \vec{h}(t) := \mathbb{P}^{-1}\vec{g}(t)}. \end{aligned}$$

In particular, we know how to solve for $\vec{x}$:

$$x_i(t) = e^{r_i t} \int e^{-r_i s} h_i(s)\, ds + c_i e^{r_i t}.$$

Then, the solution to the non-homogeneous system (5.5) when $\mathbb{A}$ is a diagonalizable matrix can be computed from the expression $\vec{y}(t) = \mathbb{P}\vec{x}(t)$.

Example 5.16. *For*

$$\mathbb{A} = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix},$$

the eigenvalues are $r_1 = 3$ *and* $r_2 = -1$ *with corresponding eigenvectors*

$$\vec{\xi}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}.$$

Setting

$$\mathbb{P} = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix}, \quad \mathbb{P}^{-1} = \begin{pmatrix} 1/2 & 1/4 \\ 1/2 & -1/4 \end{pmatrix},$$

we see that

$$\mathbb{P}^{-1}\mathbb{A}\mathbb{P} = \begin{pmatrix} 3 & 0 \\ 0 & -1 \end{pmatrix} = \mathbb{D}.$$

Sadly, **not every** matrix can be diagonalized. However, we can turn every square matrix into a Jordan matrix (so-called its **Jordan normal form**).

Let $\mathbb{A} \in \mathbb{R}^{n\times n}$ be a matrix with eigenvalues $r_1, \dots, r_m$ where $m \in \mathbb{N}$, and each eigenvalue r_i has an alg. mult. of $k_i \in \mathbb{N}$. This implies that the characteristic equation looks like

$$P_{\mathbb{A}}(r) = (r - r_1)^{k_1}(r - r_2)^{k_2} \dots (r - r_m)^{k_m},$$

where $k_1 + \cdots + k_m = n$. Now suppose each eigenvalue has a geo. mult. of l_i, where for each $1 \leq i \leq m$, $1 \leq l_i \leq k_i$ (recall $1 \leq$ geo. mult. $\leq$ alg. mult.). Then, $\mathbb{A}$ has the following Jordan normal form $\mathbb{J}$:

$$\mathbb{J} = \begin{pmatrix} \boxed{J_1} & O & O & \dots & O \\ O & \boxed{J_2} & O & \dots & O \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ O & O & \dots & O & \boxed{J_l} \end{pmatrix}$$

where each Jordan block J_i, $1 \leq i \leq l$ is a Jordan matrix. Furthermore,

(a) $l = l_1 + \cdots + l_m$ is the total number of Jordan blocks. I.e., the sum of the geo. mult. is the total number of Jordan matrices in the Jordan normal form $\mathbb{J}$.

(b) For each eigenvalue r_i, the number of Jordan blocks with values r_i on its diagonal is equal to l_i, the geo. mult. corresponding to r_i.

(c) The eigenvalue r_i appears on the main diagonal of $\mathbb{J}$ exactly k_i times (the alg. mult. of r_i).

In addition, up to reordering of the blocks, the Jordan normal form of a matrix is unique.

Example 5.17. *Let*

$$\mathbb{A} = \begin{pmatrix} 2 & -3 \\ 3 & 4 \end{pmatrix},$$

then the eigenvalues are $r_1 = r_2 = -1$, i.e., the alg. mult. is two. It turns out that the geo. mult. is only one. Therefore, $m = 1$, $l_1 = 1$ and $k_1 = 2$. From the above we expect that the Jordan normal form $\mathbb{J}$ for $\mathbb{A}$ has the following:

- *1 Jordan block ($l_1 = 1$) for the eigenvalue $r = -1$, and $r = -1$ appears twice ($k_1 = 2$) on the main diagonal of $\mathbb{J}$.*

Therefore, the Jordan normal form should look like

$$\mathbb{J} = \begin{pmatrix} -1 & 1 \\ 0 & -1 \end{pmatrix}.$$

Example 5.18. *Let*

$$\mathbb{A} = \begin{pmatrix} 3 & 1 & 0 \\ -1 & 1 & 0 \\ 3 & 2 & 2 \end{pmatrix},$$

then the eigenvalues are $r_1 = r_2 = r_3 = 3$, and it turns out that the geo. mult. is one. Hence, we expect that the Jordan normal form $\mathbb{J}$ for $\mathbb{A}$ has the following:

- *1 Jordan block ($l_1 = 1$) for the eigenvalue $r = 3$, and $r = 3$ appears three times ($k_1 = 3$) on the main diagonal of $\mathbb{J}$.*

Therefore, the Jordan normal form should look like

$$\mathbb{J} = \begin{pmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{pmatrix}.$$

Example 5.19. *Let*

$$\mathbb{A} = \begin{pmatrix} 2 & 4 & -8 \\ 0 & 0 & 4 \\ 0 & -1 & 4 \end{pmatrix},$$

then the eigenvalues are $r_1 = r_2 = r_3 = 2$, but this time the geo. multi. is two. Hence, we expect

- *2 Jordan blocks ($l_1 = 2$) and $r = 2$ appears three times ($k_1 = 3$) on the main diagonal of $\mathbb{J}$.*

Therefore, the Jordan normal form should look like

$$\mathbb{J} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix} \text{ or } \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

Similar to the concept of diagonalizable matrices, if we can find an invertible matrix $\mathbb{Q}$ such that

$$\mathbb{Q}^{-1}\mathbb{A}\mathbb{Q} = \mathbb{J},$$

where $\mathbb{J}$ is the Jordan normal form of $\mathbb{A}$. Then, by setting $\vec{z} = \mathbb{Q}^{-1}\vec{y}$, we find that

$$\begin{aligned} &\vec{z}'(t) = \mathbb{Q}^{-1}\vec{y}'(t) = \mathbb{Q}^{-1}(\mathbb{A}\mathbb{Q}\vec{z}(t) + \vec{g}(t)) \\ \Rightarrow &\boxed{\vec{z}'(t) = \mathbb{J}\vec{z}(t) + \vec{h}(t), \quad \vec{h}(t) := \mathbb{Q}^{-1}\vec{g}(t)}. \end{aligned}$$

Solving the above system when we have a Jordan matrix is considerably easier compared to the original matrix $\mathbb{A}$. Then, the solution to the original non-homogeneous system can be obtained from the expression $\vec{y}(t) = \mathbb{Q}\vec{z}(t)$.

The question is how we can find such a matrix $\mathbb{Q}$? It turns out that the columns of $\mathbb{Q}$ are the **generalized eigenvectors** of $\mathbb{A}$.

Let's start with an example. Suppose $\mathbb{A}$ is a 4-by-4 matrix with Jordan normal form

$$\mathbb{J} = \begin{pmatrix} r_1 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 \\ 0 & 0 & r_3 & 1 \\ 0 & 0 & 0 & r_3 \end{pmatrix},$$

that is, we have one Jordan block of size 1×1 for r_1, one Jordan block of size 1×1 for r_2 and one Jordan block of size 2×2 for r_3. Let the columns of $\mathbb{Q}$ be denoted as $\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4$. Using that if $\mathbb{Q}^{-1}\mathbb{A}\mathbb{Q} = \mathbb{J}$, we have $\mathbb{A}\mathbb{Q} = \mathbb{Q}\mathbb{J}$ and so

$$\begin{aligned}
\mathbb{A}\begin{pmatrix} | & | & | & | \\ \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \vec{q}_4 \\ | & | & | & | \end{pmatrix} &= \begin{pmatrix} | & | & | & | \\ \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \vec{q}_4 \\ | & | & | & | \end{pmatrix}\mathbb{J} \\
&= \begin{pmatrix} | & | & | & | \\ r_1\vec{q}_1 & r_2\vec{q}_2 & r_3\vec{q}_3 & r_3\vec{q}_4 + \vec{q}_3 \\ | & | & | & | \end{pmatrix}
\end{aligned}$$

Comparing the columns we have

$$\begin{aligned}
(\mathbb{A} - r_1\mathbb{I})\vec{q}_1 &= \vec{0}, \\
(\mathbb{A} - r_2\mathbb{I})\vec{q}_2 &= \vec{0}, \\
(\mathbb{A} - r_3\mathbb{I})\vec{q}_3 &= \vec{0}, \\
(\mathbb{A} - r_3\mathbb{I})\vec{q}_4 &= \vec{q}_3.
\end{aligned}$$

In particular, $\vec{q}_1, \vec{q}_2, \vec{q}_3$ are eigenvectors corresponding to eigenvalues r_1, r_2, r_3, and $\vec{q}_4$ is a generalized eigenvector of rank 1 corresponding to r_3.

Example 5.20. *For*

$$\mathbb{A} = \begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix},$$

the eigenvalues are $r_1 = r_2 = 3$. Furthermore,

$$\mathbb{A} - 3\mathbb{I} = \begin{pmatrix} -1 & 1 \\ -1 & 1 \end{pmatrix},$$

and so we only have one eigenvector $\vec{\xi}$ for $r = 3$, which we take as

$$\vec{\xi} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Since the geo. mult. of $r = 3$ is one, we have to obtain a generalized eigenvector $\vec{\eta}$ of rank 1. Note that

$$(\mathbb{A} - 3\mathbb{I})^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix},$$

and so for $\vec{\eta}$ we can choose any arbitrary vector, e.g.,

$$\vec{\eta} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \Rightarrow \vec{\theta} = (\mathbb{A} - 3\mathbb{I})\vec{\eta} = \begin{pmatrix} -1 \\ -1 \end{pmatrix}.$$

Hence, we can set

$$\mathbb{Q} = \begin{pmatrix} -1 & 1 \\ -1 & 0 \end{pmatrix} \text{ with } \mathbb{Q}^{-1} = \begin{pmatrix} 0 & -1 \\ 1 & -1 \end{pmatrix},$$

and one can check that

$$\mathbb{Q}^{-1}\mathbb{A}\mathbb{Q} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}.$$

Notice, we did not use the eigenvector $\vec{\xi}$ with the generalized eigenvector $\vec{\eta}$!

5.5.1 Method of undetermined coefficients

If we have a non-homogeneous term $\vec{g}(t)$ where each component has a sum or product of exponentials, cosine, sine and polynomials, then we can use the method of undetermined coefficients to obtain a particular solution to the non-homogeneous system

$$\vec{y}'(t) = \mathbb{A}\vec{y}(t) + \vec{g}(t).$$

One difference compared to second order equations and n-th order equations is that now the undetermined coefficients are **vectors**. We now list the trial solutions for specific examples of $\vec{g}$:

$\vec{g}(t)$	Solution form	value of s
$\vec{P}_m(t)$	$\vec{Q}_{m+s}(t)$	alg. mult. of 0
$\vec{P}_m(t)e^{\alpha t}$	$\vec{Q}_{m+s}(t)e^{\alpha t}$	alg. mult. of α
$\vec{P}_m(t)e^{\alpha t}\cos(\beta t)$	$\vec{Q}_{m+s}(t)e^{\alpha t}\cos(\beta t) + \vec{R}_{m+s}(t)e^{\alpha t}\sin(\beta t)$	alg. mult. of $\alpha + i\beta$
$\vec{P}_m(t)e^{\alpha t}\sin(\beta t)$	$\vec{Q}_{m+s}(t)e^{\alpha t}\cos(\beta t) + \vec{R}_{m+s}(t)e^{\alpha t}\sin(\beta t)$	alg. mult. of $\alpha + i\beta$

Here, we use the notation

$$\vec{P}_m(t) = \vec{a}_n t^n + \vec{a}_{n-1}t^{n-1} + \cdots + \vec{a}_1 t + \vec{a}_0,$$

where $\vec{a}_0, \ldots, \vec{a}_n$ are constant vectors, so that $\vec{P}_m(t)$ is a vector-valued polynomial of degree m.

Remark 5.4. *In contrast to n-th order linear equations, where the form of the trial solution is $t^s Q_m(t)$, so that the lowest order term is t^s, for linear systems we have to use a trial solution of the form $Q_{m+s}(t)$, which is a polynomial of degree $m+s$ which includes all lower order terms $t^{s-1}, t^{s-2}, \ldots, t^1, t^0$.*

Example 5.21. *Consider*

$$\vec{y}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{y}(t) + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

We set

$$\vec{g}(t) = \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}, \quad \mathbb{A} = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix},$$

and first find the solution to the homogeneous system. The characteristic equation for $\mathbb{A}$ is

$$\det(\mathbb{A} - r\mathbb{I}) = (r+3)(r+1) = 0 \Rightarrow r_1 = -3, \quad r_2 = -1.$$

Computing

$$\mathbb{A} + 3\mathbb{I} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \quad \mathbb{A} + \mathbb{I} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix},$$

and so we can take as eigenvectors

$$\vec{\xi}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Therefore, the complementary solution to the homogeneous system $\vec{y}'(t) = \mathbb{A}\vec{y}(t)$ is

$$\vec{y}_c(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

Next, observe that

$$\vec{g}(t) = \underbrace{\begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t}}_{\vec{g}_1(t)} + \underbrace{\begin{pmatrix} 0 \\ 3 \end{pmatrix} t}_{\vec{g}_2(t)}.$$

Since we have a term $\vec{g}_1(t)$ involving e^{-t}, which forms part of the complementary solution, recalling the theory for second order equations - where if we encounter a non-homogeneous equation $ay'' + by' + cy = e^{\alpha t}$ and α is a root of the characteristic equation $ar^2 + br + c = 0$ we should try $Y(t) = Ate^{\alpha t}$, let's try a trial solution to the non-homogeneous system with $\vec{g}_1(t)$ of the form

$$\vec{x}(t) = \vec{a}te^{t}$$

for some undetermined vector $\vec{a}$. Substituting this into the equation gives

$$\vec{x}'(t) - \mathbb{A}\vec{x}(t) = -te^{-t}(\mathbb{A}\vec{a} + \vec{a}) + \vec{a}e^{-t} = \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{-t}.$$

Comparing the coefficients, naturally we choose $\vec{a} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$. But, we also need to ensure that $\mathbb{A}\vec{a} + \vec{a} = \vec{0}$. A short computation shows that

$$\mathbb{A}\vec{a} + \vec{a} = \begin{pmatrix} -2 \\ 2 \end{pmatrix} \neq \vec{0}.$$

Therefore, the solution cannot be of the form $\vec{a}te^{-t}$.

To remedy this let's try

$$\vec{x}(t) = \vec{a}te^{-t} + \vec{b}e^{-t},$$

and then

$$\vec{x}'(t) - \mathbb{A}\vec{x}(t) - te^{-t}(\mathbb{A}\vec{a} + \vec{a}) - e^{-t}(\vec{b} + \mathbb{A}\vec{b} - \vec{a}) = \vec{g}_1(t).$$

This means we should have

$$\mathbb{A}\vec{a} + \vec{a} = \vec{0}, \quad \vec{b} + \mathbb{A}\vec{b} - \vec{a} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

That is, $\vec{a}$ should be an eigenvector to the eigenvalue $r = -1$, and so we take $\vec{a} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then

$$\mathbb{A}\vec{b} + \vec{b} = \begin{pmatrix} -1 \\ 1 \end{pmatrix} \Rightarrow -b_1 + b_2 = -1.$$

We can take $b_1 = 0$, $b_2 = -1$ and so a particular solution to $\vec{y}'(t) = \mathbb{A}\vec{y}(t) + \vec{g}_1(t)$ is

$$\vec{x}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-t}.$$

For a particular solution to $\vec{y}'(t) = \mathbb{A}\vec{y}(t) + \vec{g}_2(t)$, we try a trial solution of the form

$$\vec{z}(t) = \vec{c}t + \vec{d}.$$

Then,

$$\vec{z}'(t) - \mathbb{A}\vec{z}(t) = (\vec{c} - \mathbb{A}\vec{d}) - t\mathbb{A}\vec{c} = \begin{pmatrix} 0 \\ 3 \end{pmatrix} t.$$

Hence, we require

$$\mathbb{A}\vec{c} = \begin{pmatrix} 0 \\ -3 \end{pmatrix}, \quad \mathbb{A}\vec{d} = \vec{c}.$$

Solving these equations gives

$$\vec{c} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \vec{d} = \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix} \quad \Rightarrow \vec{z}(t) = \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix},$$

and so a particular solution to the non-homogeneous system is

$$\vec{Y}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \begin{pmatrix} 0 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t + \begin{pmatrix} -4/3 \\ -5/3 \end{pmatrix}.$$

Example 5.22. *Find a particular solution to*

$$\vec{y}'(t) = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \vec{y}(t) + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}.$$

The eigenvalues of the matrix $\mathbb{A}$ *are* $r_1 = -3$, $r_2 = 2$ *with corresponding eigenvectors*

$$\vec{\xi}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

So the general solution to the homogeneous system is

$$\vec{y}_c(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 4 \\ 1 \end{pmatrix} e^{2t}, \quad c_1, c_2 \in \mathbb{R}.$$

Writing the term $\vec{g}(t)$ *as*

$$\vec{g}(t) = e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{t} \begin{pmatrix} 0 \\ -2 \end{pmatrix},$$

and since neither -2 *nor* 1 *are eigenvalues of* $\mathbb{A}$, *we try a trial solution of the form*

$$\vec{z}(t) = \vec{a}e^{-2t} + \vec{b}e^{t}.$$

Then, computing

$$\vec{z}'(t) - \mathbb{A}\vec{z}(t) = e^{-2t}(-2\vec{a} - \mathbb{A}\vec{a}) + e^{t}(\vec{b} - \mathbb{A}\vec{b}) = e^{-2t} \begin{pmatrix} 1 \\ 0 \end{pmatrix} + e^{t} \begin{pmatrix} 0 \\ -2 \end{pmatrix},$$

and upon comparing coefficients we need

$$(-2\mathbb{I} - \mathbb{A}) \ veca = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (\mathbb{I} - \mathbb{A})\vec{b} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}.$$

Solving these equations gives

$$\vec{a} = \begin{pmatrix} 0 \\ -0.25 \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} 2 \\ 0 \end{pmatrix},$$

and so a particular solution is

$$\vec{Y}(t) = \begin{pmatrix} 0 \\ -0.25 \end{pmatrix} e^{-2t} + \begin{pmatrix} 2 \\ 0 \end{pmatrix} e^{t}.$$

Exercise: Find a particular solution to

$$\vec{y}'(t) = \begin{pmatrix} 1 & 5 \\ -1 & 1 \end{pmatrix} \vec{y}(t) + \begin{pmatrix} e^{2t} \\ \sin(2t) \end{pmatrix}.$$

5.5.2 Variation of parameters

We now consider more general non-homogeneous first order systems of the form

$$\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t) + \vec{g}(t),$$

where the matrix $\mathbb{P}(t)$ is not constant and is not diagonalizable. For the moment, we neglect the non-homogeneous term and study the homogeneous system $\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t)$.

Definition 5.9 (Fundamental matrix)**.** *Let $\vec{y}_1(t), \ldots, \vec{y}_n(t)$ be a fundamental set of solutions to the homogeneous system $\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t)$. The matrix $\mathbb{F}$ defined as*

$$\mathbb{F}(t) = \begin{pmatrix} | & | & \cdots & | \\ \vec{y}_1(t) & \vec{y}_2(t) & \cdots & \vec{y}_n(t) \\ | & | & \cdots & | \end{pmatrix} = \begin{pmatrix} y_{11}(t) & y_{12}(t) & \cdots & y_{1n}(t) \\ \vdots & \vdots & \ddots & \vdots \\ y_{n1}(t) & y_{n2}(t) & \cdots & y_{nn}(t) \end{pmatrix}$$

is called a ***fundamental matrix*** *for the system $\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t)$.*

Note that the fundamental matrix $\mathbb{F}$ is invertible since its columns are linearly independent. The general solution to the homogeneous system is of the form

$$\vec{y}_c(t) = c_1\vec{y}_1(t) + \cdots + c_n\vec{y}_n(t) = \mathbb{F}(t)\vec{c}, \quad \vec{c} = \begin{pmatrix} c_1 \\ \vdots \\ c_n \end{pmatrix}.$$

If we are also given initial conditions

$$\vec{y}(t_0) = \vec{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix},$$

then we see that

$$\vec{y}(t_0) = \mathbb{F}(t_0)\vec{c} \Rightarrow \vec{c} = \mathbb{F}(t_0)^{-1}\vec{y}(t_0) \Rightarrow \boxed{\vec{y}_c(t) = \mathbb{F}(t)(\mathbb{F}(t_0)^{-1}\vec{y}(t_0)) = \mathbb{F}(t)\mathbb{F}(t_0)^{-1}\vec{v}}.$$

This gives an expression for the unique solution to the IVP.

Theorem 5.8. *Let $I \subset \mathbb{R}$ be an open interval, then $\mathbb{F}(t)$ is a fundamental matrix for the homogeneous system $\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t)$ for $t \in I$ if and only if*

$$\frac{d}{dt}\mathbb{F}(t) = \mathbb{P}(t)\mathbb{F}(t) \qquad \textit{for } t \in I,$$

and $\mathbb{F}(t_0)$ is non-singular (i.e., invertible) for some $t_0 \in I$.

Proof. For the direction ($\Rightarrow$), if $\mathbb{F}$ is a fundamental matrix for the system $\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t)$, then

$$\begin{aligned} \frac{d}{dt}\mathbb{F}(t) = \frac{d}{dt}\begin{pmatrix} | & | & \cdots & | \\ \vec{y}_1(t) & \vec{y}_2(t) & \cdots & \vec{y}_n(t) \\ | & | & \cdots & | \end{pmatrix} &= \begin{pmatrix} | & | & \cdots & | \\ \frac{d}{dt}\vec{y}_1(t) & \frac{d}{dt}\vec{y}_2(t) & \cdots & \frac{d}{dt}\vec{y}_n(t) \\ | & | & \cdots & | \end{pmatrix} \\ &= \begin{pmatrix} | & | & \cdots & | \\ \mathbb{P}(t)\vec{y}_1(t) & \mathbb{P}(t)\vec{y}_2(t) & \cdots & \mathbb{P}(t)\vec{y}_n(t) \\ | & | & \cdots & | \end{pmatrix} = \mathbb{P}(t)\mathbb{F}(t). \end{aligned}$$

For the direction ($\Leftarrow$), if a matrix function $\mathbb{F}(t)$ satisfies $\frac{d}{dt}\mathbb{F}(t) = \mathbb{P}(t)\mathbb{F}(t)$ for $t \in I$ with $\mathbb{F}(t_0)$ non-singular at some $t_0 \in I$, then writing

$$\mathbb{F}(t) = \begin{pmatrix} | & | & \cdots & | \\ \vec{x}_1(t) & \vec{x}_2(t) & \cdots & \vec{x}_n(t) \\ | & | & \cdots & | \end{pmatrix},$$

for some vectors $\vec{x}_1(t), \ldots, \vec{x}_n(t)$ denoting the columns of the matrix $\mathbb{F}$, we have to verify that $\vec{x}_1(t), \ldots, \vec{x}_n(t)$ is a fundamental set of solutions to the homogeneous system. Firstly,

$$\frac{d}{dt}\mathbb{F}(t) = \mathbb{P}(t)\mathbb{F}(t) \Rightarrow \frac{d}{dt}\vec{x}_i(t) = \mathbb{P}(t)\vec{x}_i(t),$$

and so $\vec{x}_1(t), \ldots, \vec{x}_n(t)$ are all solutions to the homogeneous system. Furthermore, the Wronskian at t_0 is

$$W(\vec{x}_1, \ldots, \vec{x}_n)[t_0] = \det\mathbb{F}(t_0) \neq 0,$$

by assumption. Hence $(\vec{x}_1, \ldots, \vec{x}_n)$ forms a fundamental set of solutions to the homogeneous system and so $\mathbb{F}$ is a fundamental matrix. $\square$

Theorem 5.9. *If $\mathbb{F}(t)$ is a fundamental matrix and $\mathbb{A}$ is a non-singular constant matrix, then $\mathbb{F}(t)\mathbb{A}$ is a fundamental matrix.*

Proof. Let

$$\mathbb{F}(t) = \begin{pmatrix} | & | & \ldots & | \\ \vec{y}_1(t) & \vec{y}_2(t) & \ldots & \vec{y}_n(t) \\ | & | & \ldots & | \end{pmatrix}.$$

Then,

$$\mathbb{F}(t)\mathbb{A} = \begin{pmatrix} | & | & \ldots & | \\ \vec{y}_1(t) & \vec{y}_2(t) & \ldots & \vec{y}_n(t) \\ | & | & \ldots & | \end{pmatrix}\mathbb{A} = \begin{pmatrix} | & | & \ldots & | \\ \vec{x}_1(t) & \vec{x}_2(t) & \ldots & \vec{x}_n(t) \\ | & | & \ldots & | \end{pmatrix},$$

where

$$\vec{x}_j(t) = \sum_{i=1}^{n} a_{ij}\vec{y}_i(t) \text{ for } 1 \leq j \leq n.$$

Here, we use the notation

$$\mathbb{A} = (a_{ij})_{1\leq i,j\leq n} = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{pmatrix}.$$

Since $\vec{y}_1, \ldots, \vec{y}_n$ solve $\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t)$, we see that

$$\vec{x}_i'(t) = \sum_{j=1}^{n} a_{ij}\vec{y}_j'(t) = \sum_{j=1}^{n} a_{ij}\mathbb{P}(t)\vec{y}_j(t) = \mathbb{P}(t)\vec{x}_i(t) \text{ for } 1 \leq i \leq n.$$

Hence, $\vec{x}_1, \ldots, \vec{x}_n$ also solve the homogeneous system. Moreover,

$$W(\vec{x}_1, \ldots, \vec{x}_n)[t] = \det(\mathbb{F}(t)\mathbb{A}) = \det(\mathbb{F}(t))\det(\mathbb{A}) = \det(\mathbb{A})W(\vec{y}_1, \ldots, \vec{y}_n)[t] \neq 0.$$

This implies that the matrix function $\mathbb{F}(t)\mathbb{A}$ is also a fundamental matrix. $\square$

Remark 5.5. *In general, $\mathbb{A}\mathbb{F}(t)$ is **not** a fundamental matrix!*

With this in mind, we can choose a special non-singular matrix for $\mathbb{A}$ in order to simplify the expression

$$\vec{y}_c(t) = \mathbb{F}(t)\mathbb{F}(t_0)^{-1}\vec{v}$$

for the unique solution to the IVP

$$\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t) \text{ for } t \in I, \quad \vec{y}(t_0) = \vec{v}.$$

In particular, choosing

$$\mathbb{A} = \mathbb{F}^{-1}(t_0)$$

and setting

$$\mathbb{G}(t) = \mathbb{F}(t)\mathbb{F}^{-1}(t_0),$$

then Theorem 5.9 implies that $\mathbb{G}(t)$ is also a fundamental matrix, since by assumption $\mathbb{A} = \mathbb{F}^{-1}(t_0)$ is non-singular. Since $\mathbb{G}$ is a fundamental matrix, then by Theorem 5.8 we see that $\mathbb{G}$ satisfies $\frac{d}{dt}\mathbb{G}(t) = \mathbb{P}(t)\mathbb{G}(t)$ but now

$$\mathbb{G}(t_0) = \mathbb{F}(t_0)\mathbb{F}^{-1}(t_0) = \mathbb{I},$$

where $\mathbb{I}$ is the identity matrix. In particular, if $\mathbb{P}(t)$ is a constant matrix, then it is easy to solve the system

$$\mathbb{G}'(t) = \mathbb{P}(t)\mathbb{G}(t), \quad \mathbb{G}(t_0) = \mathbb{I}.$$

Example 5.23. *For*

$$\vec{y}'(t) = \begin{pmatrix} 1 & 1 \\ 4 & 1 \end{pmatrix}\vec{y}(t),$$

find $\mathbb{G}(t)$ such that $\mathbb{G}(0) = \mathbb{I}$. We proceed in three steps. The first is to find a fundamental set of solutions, which is

$$\vec{y}_1(t) = \begin{pmatrix} e^{3t} \\ 2e^{3t} \end{pmatrix}, \quad \vec{y}_2(t) = \begin{pmatrix} e^{-t} \\ -2e^{-t} \end{pmatrix}.$$

The second step is to write the fundamental matrix $\mathbb{F}$ and compute $\mathbb{F}^{-1}(0)$:

$$\mathbb{F}(t) = \begin{pmatrix} e^{3t} & e^{-t} \\ 2e^{3t} & -2e^{-t} \end{pmatrix} \Rightarrow \mathbb{F}(0) = \begin{pmatrix} 1 & 1 \\ 2 & -2 \end{pmatrix} \Rightarrow \mathbb{F}^{-1}(0) = -\frac{1}{4}\begin{pmatrix} -2 & -1 \\ -2 & 1 \end{pmatrix}.$$

The third step is to compute $\mathbb{G}(t) = \mathbb{F}(t)\mathbb{F}^{-1}(0)$:

$$\mathbb{G}(t) = -\frac{1}{4}\begin{pmatrix} -2e^{3t} - 2e^{-t} & -e^{3t} + e^{-t} \\ -4e^{-3t} + 4e^{-t} & -2e^{3t} - 2e^{-t} \end{pmatrix}$$

which satisfies

$$\mathbb{G}(0) = -\frac{1}{4}\begin{pmatrix} -4 & 0 \\ 0 & -4 \end{pmatrix} = \mathbb{I}.$$

Returning to the non-homogeneous system

$$\vec{y}'(t) = \mathbb{P}(t)\vec{y}(t) + \vec{g}(t).$$

Assume we have a fundamental matrix $\mathbb{F}(t)$ to the homogeneous system with complementary solution

$$\vec{y}_c(t) = \mathbb{F}(t)\vec{c},$$

where $\vec{c}$ is a constant vector. The method of variation of parameters is to consider a trial solution

$$\vec{z}(t) = \mathbb{F}(t)\vec{u}(t),$$

where $\vec{u}(t)$ is a vector of functions. Then, if $\vec{z}$ is a solution to the non-homogeneous system, we find that

$$\mathbb{P}(t)\mathbb{F}(t)\vec{u}(t) + g(t) = \vec{z}'(t) = \mathbb{F}(t)\vec{u}'(t) + \mathbb{F}'(t)\vec{u}(t).$$

Since $\mathbb{F}(t)$ is a fundamental matrix, i.e., $\mathbb{F}'(t) = \mathbb{P}(t)\mathbb{F}(t)$, we see that

$$\mathbb{F}(t)\vec{u}'(t) = \vec{g}(t) \Rightarrow \boxed{\vec{u}'(t) = \mathbb{F}^{-1}(t)\vec{g}(t)}.$$

Integrating this gives

$$\boxed{\vec{u}(t) = \int \mathbb{F}^{-1}(t)\vec{g}(t)\ dt + \vec{a}},$$

where $\vec{a}$ is an arbitrary constant vector. Therefore the general solution to the non-homogeneous system is

$$\begin{aligned}\vec{y}(t) &= \mathbb{F}(t)\vec{c} + \mathbb{F}(t)\left[\int \mathbb{F}^{-1}(t)\vec{g}(t)\ dt + \vec{a}\right]\\ &= \mathbb{F}(t)(\vec{c} + \vec{a}) + \mathbb{F}(t)\left[\int \mathbb{F}^{-1}(t)\vec{g}(t)\ dt\right].\end{aligned}$$

In particular, we could have taken $\vec{a} = \vec{0}$, leading to the expression

$$\boxed{\vec{y}(t) = \mathbb{F}(t)\vec{c} + \mathbb{F}(t)\left[\int \mathbb{F}^{-1}(t)\vec{g}(t)\ dt\right]}.$$

If we are also given initial conditions $\vec{y}(t_0) = \vec{x}$, then in the integral we write

$$\int_{t_0}^{t} \mathbb{F}^{-1}(s)\vec{g}(s)\ ds$$

so that

$$\vec{x} = \vec{y}(t_0) = \mathbb{F}(t_0)\vec{c} \Rightarrow \vec{c} = \mathbb{F}^{-1}(t_0)\vec{x}.$$

Hence, the unique solution to the IVP in the interval I is

$$\boxed{\vec{y}(t) = \mathbb{F}(t)\mathbb{F}^{-1}(t_0)\vec{x} + \mathbb{F}(t)\left[\int_{t_0}^{t} \mathbb{F}^{-1}(s)\vec{g}(s)\ ds\right]}.$$

Example 5.24. *Find a particular solution to*

$$\vec{y}'(t) = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{y}(t) + \begin{pmatrix} 2e^{-t} \\ 3t \end{pmatrix}.$$

Using the method of undetermined coefficients, we have that one particular solution is

$$\vec{Y}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Recalling that the complementary solution to the homogeneous system is

$$\vec{y}_c(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t}.$$

Computing the fundamental matrix

$$\mathbb{F}(t) = \begin{pmatrix} e^{-3t} & e^{-t} \\ -e^{-3t} & e^{-t} \end{pmatrix},$$

its determinant $\det \mathbb{F}(t) = 2e^{-4t}$ *and its inverse*

$$\mathbb{F}^{-1}(t) = \frac{1}{2} \begin{pmatrix} e^{3t} & -e^{3t} \\ e^{t} & e^{t} \end{pmatrix}$$

we can then compute for the unknown coefficients by solving

$$\vec{u}'(t) = \mathbb{F}^{-1}(t)\vec{g}(t) \Rightarrow \begin{cases} u_1'(t) = e^{2t} - \frac{3}{2}te^{3t}, \\ u_2'(t) = 1 + \frac{3}{2}te^{t}. \end{cases}$$

This gives

$$u_1(t) = \frac{1}{2}e^{2t} - \frac{1}{2}te^{3t} + \frac{1}{6}e^{3t}, \quad u_2(t) = t + \frac{3}{2}te^{t} - \frac{3}{2}e^{t},$$

where we used

$$\int te^{\alpha t}\, dt = \frac{\alpha t - 1}{\alpha^2} e^{\alpha t}.$$

Hence, a particular solution is

$$\vec{Z}(t) = \mathbb{F}(t)\vec{u}(t) = te^{-t} \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + t \begin{pmatrix} 1 \\ 2 \end{pmatrix} - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

Note that $\vec{Y}(t)$ *obtained from the method of undetermined coefficients is different from the particular solution* $\vec{Z}(t)$ *obtained from the variation of parameters:*

$$\vec{Y}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} - \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix},$$

$$\vec{Z}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} te^{-t} + \frac{1}{2} \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + \begin{pmatrix} 1 \\ 2 \end{pmatrix} t - \frac{1}{3} \begin{pmatrix} 4 \\ 5 \end{pmatrix}.$$

One can check that both $\vec{Y}$ and $\vec{Z}$ are particular solutions, but the corresponding general solutions to the non-homogeneous system are equivalent:

$$\vec{y}(t) = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \vec{Y}(t) = d_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{3t} + d_2 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{-t} + \vec{Z}(t)$$

if we choose

$$c_1 = d_1, \quad c_2 = d_2 + \frac{1}{2}.$$

Example 5.25. *Find a particular solution to*

$$\vec{y}'(t) = \begin{pmatrix} 1 & 4 \\ 1 & -2 \end{pmatrix} \vec{y}(t) + \begin{pmatrix} e^{-2t} \\ -2e^{t} \end{pmatrix}.$$

From before, the eigenvalues of $\mathbb{A}$ are $r_1 = -3$ and $r_2 = 2$ with eigenvectors

$$\vec{\xi}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \quad \vec{\xi}_2 = \begin{pmatrix} 4 \\ 1 \end{pmatrix}.$$

From this we can write down the fundamental matrix

$$\mathbb{F}(t) = \begin{pmatrix} e^{-3t} & 4e^{2t} \\ -e^{-3t} & e^{2t} \end{pmatrix}.$$

The determinant is $\det \mathbb{F}(t) = 5e^{-t}$, *with inverse*

$$\mathbb{F}(t)^{-1} = \frac{1}{5} \begin{pmatrix} e^{3t} & -4e^{3t} \\ e^{-2t} & e^{-2t} \end{pmatrix}.$$

Then, for the unknown coefficients, we solve

$$\vec{u}'(t) = \mathbb{F}^{-1}(t)\vec{g}(t) \Rightarrow \begin{cases} u_1'(t) = \frac{1}{5}(e^{t} + 8e^{4t}), \\ u_2'(t) = \frac{1}{5}(e^{-4t} - 2e^{-t}). \end{cases}$$

This gives

$$u_1(t) = \frac{1}{5}e^{t} + \frac{2}{5}e^{4t}, \quad u_2(t) = \frac{-1}{20}e^{-4t} + \frac{2}{5}e^{-t},$$

and the particular solution is

$$\vec{Z}(t) = \mathbb{F}(t)\vec{u}(t) = \begin{pmatrix} 2e^{t} \\ -0.25e^{-2t} \end{pmatrix},$$

which coincides with the particular solution obtained from the method of undetermined coefficients.