MATH3720A - Lecture Notes

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4 Higher order linear equations

The theory for higher order linear equations is analogous to that of the second order case. Let us give a brief recap:

- For a general second order equation

$$y'' + p(t)y' + q(t)y = g(t).$$

If there is an interval I such that p, q and g are continuous, then for $t_0 \in I$ and given initial conditions $x_0, x_1 \in \mathbb{R}$, the IVP has exactly one solution in I.

- Given two linearly independent solutions y_1, y_2 to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0,$$

they form a fundamental set of solutions if any solution ϕ to the homogeneous ODE can be written as a linear combination of y_1 and y_2. This is equivalent to the condition that the Wronskian $W(y_1, y_2)[t_*] = y_2'(t_*)y_1(t_*) - y_1'(t_*)y_2(t_*) \neq 0$ for some $t_* \in I$.

- Abel's theorem states that

$$W(y_1, y_2)[t] = ce^{-\int p(t)\, dt}$$

for some constant c not depending on t.

- For homogeneous equations with constant coefficients:

$$ay'' + by' + cy = 0,$$

finding two solutions y_1 and y_2 amounts to finding the roots of the characteristic equation

$$ar^2 + br + c = 0.$$

- For non-homogeneous equations we have two methods:

  1. Method of undetermined coefficients: if $g(t)$ is a sum or product of exponentials, polynomials, cosine and sine.
  2. Variation of parameters: for more general linear equations where the solution is of the form $y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$.

4.1 General theory

The general nth order linear ODE is of the form

$$y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t),$$

and for an IVP we prescribe initial conditions

$$y(t_0) = x_0, \quad y'(t_0) = x_1, \quad \ldots, \quad y^{(n-1)}(t_0) = x_{n-1}.$$

We first state the abstract existence and uniqueness theorem.

Theorem 4.1 (Existence and Uniqueness). *Let $I \subset \mathbb{R}$ be an open interval and suppose $g, P_0, P_1, \ldots, P_{n-1}$ are continuous functions in I. For $t_0 \in I$ and $x_0, \ldots, x_{n-1} \in \mathbb{R}$ there is exactly one solution to the IVP*

$$\begin{cases} y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0(t)y = g(t), \\ y(t_0) = x_0, \quad y'(t_0) = x_1, \quad \ldots, \quad y^{(n-1)}(t_0) = x_{n-1}. \end{cases}$$

In the following we will mainly focus on homogeneous equations, i.e., we set $g = 0$.

Definition 4.1. *We say that the functions $f_1, \ldots, f_n$ are* ***linearly independent*** *on the interval I if*

$$\begin{aligned} &\alpha_1 f_1(t) + \cdots + \alpha_n f_n(t) = 0 \quad \forall t \in I \\ \Rightarrow\ &\alpha_1 = \cdots = \alpha_n = 0. \end{aligned}$$

Otherwise, we say that the functions $f_1, \ldots, f_n$ are ***linearly dependent****.*

Example 4.1. *Given functions $f_1(t) = 1$, $f_2(t) = t$, $f_3(t) = t^2$ defined on the interval $I = \mathbb{R}$, suppose there are constants $\alpha_1, \alpha_2, \alpha_3$ such that*

$$\alpha_1 f_1(t) + \alpha_2 f_2(t) + \alpha_3 f_3(t) = \alpha_1 + \alpha_2 t + \alpha_3 t^2 = 0 \quad \forall t \in I. \tag{4.1}$$

Then, in order for the above equality to hold for all $t \in I = \mathbb{R}$, it must be true at any three distinct points in I. It is convenient to choose $t = 0$, $t = 1$, $t = -1$, leading to three equations

$$\alpha_1 = 0, \quad \alpha_1 + \alpha_2 + \alpha_3 = 0, \quad \alpha_1 - \alpha_2 + \alpha_3 = 0.$$

The first equation gives $\alpha_1 = 0$, and the second and third equations then give $\alpha_2 = \alpha_3 = 0$, thus there does not exist a set of non-zero constants $(\alpha_1, \alpha_2, \alpha_3)$ for which the condition (4.1) is satisfied, which then implies that f_1, f_2, f_3 are linearly independent in $I = \mathbb{R}$.

Similar to the second order case, we have the following principle of superposition:

Theorem 4.2 (Principle of superposition)**.** *Let $y_1, \dots, y_n$ be solutions to the homogeneous equation*

$$y^{(n)} + P_{n-1}(t)y^{(n-1)} + \dots + P_1(t)y' + P_0(t)y = 0,$$

then, for any constants $c_1, \dots, c_n \in \mathbb{R}$, the function

$$\phi(t) = c_1 y_1(t) + \dots + c_n y_n(t)$$

is also a solution to the homogeneous equation.

We also have an analogue to the Wronskian:

Definition 4.2. *Given functions $f_1, \dots, f_n$ that are differentiable up to order $n-1$, we define the Wronskian W as*

$$\boxed{W(f_1, \dots, f_n)[t] = \det \begin{pmatrix} f_1 & f_2 & \dots & f_n \\ f_1' & f_2' & \dots & f_n' \\ \vdots & \vdots & \vdots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \dots & f_n^{(n-1)} \end{pmatrix} [t]}.$$

The natural question is: given n solutions $y_1, \dots, y_n$ to the homogeneous equation, can every solution ϕ to the homogeneous equation be expressed as a linear combination of $y_1, \dots, y_n$?

Theorem 4.3. *If $P_0, \dots, P_{n-1}$ are continuous functions in I, and $y_1, \dots, y_n$ are solutions to the homogeneous equation satisfying $W(y_1, \dots, y_n)[t_0] \neq 0$ for some $t_0 \in I$, then every solution ϕ to the homogeneous equation can be expressed as a linear combination of $y_1, \dots, y_n$. In this case we call $(y_1, \dots, y_n)$ a* ***fundamental set of solutions*** *to the homogeneous equation.*

Let us remark that the above theorem gives

$$W(y_1, \dots, y_n)[t] \neq 0 \Rightarrow (y_1, \dots, y_n) \text{ are linearly independent.}$$

Again, the converse is not true in general, unless $y_1, \dots, y_n$ are solutions to some homogeneous equation, then the converse is true.

Theorem 4.4. *Let $y_1, \dots, y_n$ be linearly independent solutions to the homogeneous equation*

$$y^{(n)} + P_{n-1}(t)y^{(n-1)} + \dots + P_1(t)y' + P_0(t)y = 0,$$

for $t \in I$. Then, the Wronskian $W(y_1, \dots, y_n)[t]$ is non-zero in I.

Proof. Suppose the conclusion is false, i.e., there is at least one point $t_0 \in I$ where the Wronskian is zero. Then, consider the equation

$$\alpha_1 y_1(t) + \dots + \alpha_n y_n(t) = 0,$$

for constants $\alpha_1, \ldots, \alpha_n$. Differentiating repeatedly leads to

$$\begin{aligned} \alpha_1 y_1'(t) + \cdots + \alpha_n y_n'(t) &= 0, \\ &\vdots \\ \alpha_1 y_1^{(n-1)}(t) + \cdots + \alpha_n y_n^{(n-1)}(t) &= 0. \end{aligned}$$

In particular we obtain after substituting $t = t_0$

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) & \ldots & y_n(t_0) \\ y_1'(t_0) & y_2'(t_0) & \ldots & y_n'(t_0) \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)}(t_0) & y_2^{(n-1)}(t_0) & \ldots & y_n^{(n-1)}(t_0) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

Since the Wronskian is zero at $t = t_0$, there exists a non-zero solution $(\alpha_1^*, \ldots, \alpha_n^*)$ to the above matrix problem. Defining the function

$$\phi(t) = \alpha_1^* y_1(t) + \cdots + \alpha_n^* y_n(t),$$

where thanks to the principle of superposition, ϕ is also a solution to the homogeneous equation. Furthermore, at $t = t_0$, ϕ satisfies the initial conditions

$$\phi(t_0) = 0, \quad \phi'(t_0) = 0, \ldots, \phi^{(n-1)}(t_0) = 0.$$

But the solution $z(t) = 0$ for $t \in I$ is also a solution to the IVP with zero initial conditions. Consequently, by the Uniqueness of solutions to IVP we find that $\phi(t) = 0$ for $t \in I$. Consequently we have found non-zero constants $\alpha_1^*, \ldots, \alpha_n^*$ such that

$$\alpha_1^* y_1(t) + \cdots + \alpha_n^* y_n(t) = 0 \quad \forall t \in I.$$

This contradicts with the linear independence of $y_1, \ldots, y_n$. ☐

Finally, we state an analogous result to Abel's theorem:

Theorem 4.5 (Abel's theorem)**.** *Let $y_1, \ldots, y_n$ be solutions to the homogeneous equation*

$$y^{(n)} + P_{n-1}(t) y^{(n-1)} + \cdots + P_1(t) y' + P_0(t) y = 0,$$

for $t \in I$. Then,

$$W(y_1, \ldots, y_n)[t] = c e^{-\int P_{n-1}(t)\, dt}$$

for a constant c not dependent on $t \in I$.

Proof. The idea is to derive an equation satisfied by the Wronskian. From properties of matrix determinants, we see that

$$\frac{d}{dt} \begin{vmatrix} a & b \\ c & d \end{vmatrix} = \frac{d}{dt}(ad - bc) = ad' + a'd - bc' - b'c = \begin{vmatrix} a' & b' \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ c' & d' \end{vmatrix}.$$

Hence, we can deduce

$$\frac{d}{dt}W(y_1,\dots,y_n)[t] = \frac{d}{dt}\begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \dots & y_n^{(n)} \end{vmatrix}.$$

Technically, we should have

$$\frac{d}{dt}W[t] = \begin{vmatrix} y_1' & y_2' & \dots & y_n' \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix} + \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1'' & y_2'' & \dots & y_n'' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_n^{(n-1)} \end{vmatrix}$$
$$+\dots+\begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ y_1^{(n)} & y_2^{(n)} & \dots & y_n^{(n)} \end{vmatrix}.$$

But noting that in the first $n-1$ determinants, there is always a repeated row, hence the the first $n-1$ determinants are zero and only the last determinant survives. Using that for each $1 \le k \le n$,

$$y_k^{(n)} = -P_{n-1}y_k^{(n-1)} - \dots - P_1 y_k' - P_0 y_k,$$

then applying elementary column operations we find that

$$\frac{d}{dt}W[t] = \begin{vmatrix} y_1 & y_2 & \dots & y_n \\ y_1' & y_2' & \dots & y_n' \\ \vdots & \vdots & \ddots & \vdots \\ -P_{n-1}y_1^{(n-1)} & -P_{n-1}y_2^{(n-1)} & \dots & -P_{n-1}y_n^{(n-1)} \end{vmatrix} = -P_{n-1}W[t].$$

□

4.2 Homogeneous equation with constant coefficients

Our aim is to study, for constants $a_n \neq 0, a_{n-1},\dots,a_0 \in \mathbb{R}$ the equation

$$a_n y^{(n)} + a_{n-1}y^{(n-1)} + \dots + a_1 y' + a_0 y = 0.$$

From the theory of second order equations, we consider a trial function $\phi = e^{rt}$ for $r \in \mathbb{R}$. Substituting this into the above equation gives the **characteristic equation**

$$\boxed{a_n r^n + \dots + a_1 r + a_0 = 0}.$$

The **characteristic polynomial** is

$$\boxed{Z(r) = a_n r^n + \dots + a_1 r + a_0}.$$

From the fundamental theorem of algebra, every polynomial with real coefficients of degree n has n complex roots. Hence

$$Z(r) = a_n(r - r_1)(r - r_2)\dots(r - r_n),$$

where $r_1, \dots, r_n$ are complex numbers. Note that real numbers are also complex numbers.

Definition 4.3. *Let $P_k(x)$ be a polynomial of degree k in the variable x. A root r has* ***multiplicity*** *$m \in \mathbb{N}$, $m \geq 1$, if there is another polynomial $S_{k-m}(x)$ of degree $k - m$ such that $S_{k-m}(r) \neq 0$ and*

$$P_k(x) = S_{k-m}(x)(x - r)^m.$$

The idea is to solve the characteristic equation to obtain roots $r_1, \dots, r_n$. Similar to the second order equations, it is possible that roots are repeated (e.g., for $ay'' + by' + cy = 0$ a repeated root of multiplicity two is $r = -\frac{b}{2a}$). We divide the analysis into the following cases:

Case 1. If the roots of $Z(r) = 0$ are all **real** and **distinct**, i.e., $r_1 \neq r_2 \neq \dots \neq r_n$, then we have the solutions

$$y_1(t) = e^{r_1 t}, \quad \dots, \quad y_n(t) = e^{r_n t},$$

and they form a fundamental set of solutions.

Exercise. Compute the Wronskian $W(y_1, \dots, y_n)[t]$ to show that $(y_1, \dots, y_n)$ do indeed form a fundamental set of solutions.

Case 2. If $Z(r) = 0$ has n repeated real roots, i.e., $r_1 = r_2 = \dots = r_n =: q$ and so

$$Z(r) = a_n(r - q)^n.$$

Then q is a root of multiplicity n, and the solutions

$$y_1(t) = e^{qt}, \quad y_2(t) = te^{qt}, \quad \dots, \quad y_n(t) = t^{n-1}e^{qt}$$

forms a fundamental set of solutions.

Case 3. If $Z(r) = 0$ has k distinct real roots $r_1 \neq r_2 \neq \dots \neq r_k$, and one real root q with multiplicity $n - k$, i.e.,

$$Z(r) = a_n(r - r_1)(r - r_2)\dots(r - r_k)(r - q)^{n-k}.$$

Then, the solutions

$$\begin{aligned} y_1(t) = e^{r_1 t}, \quad \dots, \quad y_k(t) = e^{r_k t}, \\ y_{k+1}(t) = e^{qt}, \quad y_{k+2}(t) = te^{qt}, \quad \dots, \quad y_n(t) = t^{n-k-1}e^{qt}, \end{aligned}$$

form a fundamental set of solutions.

By consideration of the above three cases, we can formulate the general rule: If $Z(r) = 0$ has real roots $r_1, \dots, r_k$ with multiplicity $m_1, \dots, m_k$, respectively. Then $m_1 + \dots + m_k = n$ and

$$Z(r) = a_n(r - r_1)^{m_1}(r - r_2)^{m_2} \dots (r - r_k)^{m_k}.$$

Furthermore, the functions

$$\begin{gathered} y_1 = e^{r_1 t},\ y_2 = te^{r_1 t},\ \dots,\ y_{m_1} = t^{m_1 - 1} e^{r_1 t}, \\ y_{m_1+1} = e^{r_2 t},\ \dots,\ y_{m_1+m_2} = t^{m_2-1} e^{r_2 t},\ \dots,\ y_n = t^{m_k - 1} e^{r_k t} \end{gathered}$$

form a fundamental set of solutions.

Case 4. One pair of complex conjugate roots. Suppose $Z(r) = 0$ has $n - 2$ real roots $r_1, \dots, r_{n-2}$ and a pair of complex conjugate roots $r_{n-1}, r_n \in \mathbb{C}$ with $r_{n-1} = \overline{r_n}$. Setting

$$r_{n-1} = \lambda + i\mu, \quad r_n = \lambda - i\mu,$$

the functions

$$y_1 = e^{r_1 t}, \quad \dots, \quad y_{n-2} = e^{r_{n-2} t}, \quad y_{n-1} = e^{\lambda t} \cos(\mu t), \quad y_n = e^{\lambda t} \sin(\mu t)$$

form a fundamental set of solutions.

Case 5. Repeated pairs of complex conjugate roots. Suppose

$$Z(r) = a_n(r - r_1) \dots (r - r_k)(r - (\lambda + i\mu))^s (r - (\lambda - i\mu))^s,$$

where $k + 2s = n$. Note that if a complex root $\lambda + i\mu$ is repeated s times, then its conjugate $\lambda - i\mu$ is also repeated s times. This means we need $2s$ linearly independent solutions:

$$\begin{gathered} e^{\lambda t} \cos(\mu t), \quad te^{\lambda t} \cos(\mu t), \quad \dots, \quad t^{s-1} e^{\lambda t} \cos(\mu t), \\ e^{\lambda t} \sin(\mu t), \quad te^{\lambda t} \sin(\mu t), \quad \dots, \quad t^{s-1} e^{\lambda t} \sin(\mu t), \end{gathered}$$

together with $y_1 = e^{r_1 t}, \dots, y_k = e^{r_k t}$ we obtain a fundamental set of solutions.

Let's look at some examples:

Example 4.2. *(1) Characteristic equation* $(r^2+1)(r-1)^2(r+2) = 0$, *and so* $r_1 = i$, $r_2 = -i$, $r_3 = r_4 = 1$ *and* $r_5 = -2$. *Therefore, we have*

$$y_1 = \cos t,\ y_2 = \sin t,\ y_3 = e^t,\ y_4 = te^t,\ y_5 = e^{-2t}.$$

(2) Characteristic equation $(r^2 + 1)^2(r - 1)^3 = 0$, *and so* $r_1 = r_2 = i$, $r_3 = r_4 = -i$, $r_5 = r_6 = r_7 = 1$. *Therefore, we have*

$$y_1 = \cos t,\ y_2 = t\cos t,\ y_3 = \sin t,\ y_4 = t\sin t,\ y_5 = e^t,\ y_6 = te^t,\ y_7 = t^2 e^t.$$

4.3 Non-homogeneous equations

Consider the non-homogeneous equation

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0 y = g(t). \tag{4.2}$$

If Y_1 and Y_2 are both solutions to the non-homogeneous problem, then $Y_1 - Y_2$ is a solution to the homogeneous equation. Given a fundamental set of solutions $(y_1, \ldots, y_n)$ to the homogeneous equation, we see that a general solution to the non-homogeneous equation is

$$y(t) = c_1 y_1(t) + \cdots + c_n y_n(t) + Y(t),$$

where $Y(t)$ is a solution to the non-homogeneous equation.

4.3.1 Method of undetermined coefficients

Similar to second order equations, we now find a particular solution Y to the non-homogeneous equation (4.2) if $g(t)$ is a sum/product of exponentials, cosine, sine and polynomials. But the **main difference** is that the multiplicity of roots to the characteristic equation can be **greater** than two. Therefore, **higher powers** of t need to be multiplied to get the solution to the non-homogeneous equation.

We again investigate the cases:

(1) $g(t) = e^{\alpha t} P_m(t)$,

(2) $g(t) = e^{\alpha t} P_m(t) \cos(\beta t)$

(3) $g(t) = e^{\alpha t} P_m(t) \sin(\beta t)$.

The particular solutions are

(1) $Y(t) = t^s e^{\alpha t} Q_m(t)$, where

$$Q_m(t) = A_m t^m + \cdots + A_1 t + A_0$$

for undetermined coefficients $A_m, \ldots, A_0$, and s is the multiplicity of α if α is a root of the characteristic equation, zero otherwise.

(2,3) $Y(t) = t^s e^{\alpha t}[Q_m(t) \cos(\beta t) + R_m(t) \sin(\beta t)]$, where Q_m, R_m are polynomials of degree m with undetermined coefficients, and s is the multiplicity of $\alpha + i\beta$ if $\alpha + i\beta$ is a root of the characteristic equation, zero otherwise.

Example 4.3. *Solve*

$$y''' - 3y'' + 3y' - y = 4e^t.$$

For the homogeneous equation, the associated characteristic equation is

$$r^3 - 3r^2 + 3r - 1 = (r-1)^3 = 0,$$

and so $r_1 = r_2 = r_3 = 1$, *i.e., a repeated eigenvalue of multiplicity three. So we set*

$$y_1 = e^t, \quad y_2 = te^t, \quad y_3 = t^2e^t,$$

and the complementary solution (to the homogeneous equation) is

$$y_c(t) = c_1e^t + c_2te^t + c_3t^2e^t.$$

Since $g(t) = 4e^t$ *and so* $\alpha = 1$ *is a root of the characteristic equation. Therefore we have to consider* $s = 3$ *and a trial solution*

$$Y(t) = At^3e^t.$$

Computing gives

$$Y''' - 3Y'' + 3Y' - Y = 6Ae^t = 4e^t \Rightarrow A = \frac{2}{3},$$

and so the general solution to the non-homogeneous ODE is

$$y(t) = c_1e^t + c_2te^t + c_3t^2e^t + \frac{2}{3}t^3e^t.$$

Another example involving sine:

Example 4.4. *Solve*

$$y^{(4)} + 2y'' + y = 3\sin t.$$

The characteristic equation corresponding to the homogeneous equation is

$$r^4 + 2r^2 + 1 = (r^2+1)(r^2+1) = 0$$

and so $r_1 = r_3 = i$, $r_2 = r_4 = -i$, *i.e, a repeated pair of complex conjugate roots (multiplicity is two). Then we see that*

$$y_1 = \cos t, \quad y_2 = \sin t, \quad y_3 = t\cos t, \quad y_4 = t\sin t,$$

and the complementary solution to the homogeneous equation is

$$y_c(t) = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t.$$

For the non-homogeneous term $g(t) = 3\sin t$, *we have* $\alpha = 0$, $\beta = 1$, *and so* $s = 2$. *Thus we consider a trial solution*

$$Y(t) = At^2\sin t + Bt^2\cos t.$$

Then,

$$Y^{(4)} + 2Y'' + Y = -8A\sin t - 8B\cos t = 3\sin t \Rightarrow B = 0, \quad A = -\frac{3}{8}.$$

Hence, the general solution to the non-homogeneous equation is

$$y(t) = c_1\cos t + c_2\sin t + c_3t\cos t + c_4t\sin t - \frac{3}{8}t^2\sin t.$$

4.4 Variation of parameters

Analogous to second order equations, there is also a method to treat rather general high order equations

$$y^{(n)} + P_{n-1}(t)y^{(n-1)} + \cdots + P_1(t)y' + P_0 y = g(t), \quad t \in I.$$

Suppose we have solutions $y_1, \dots, y_n$ to the homogeneous equation, which forms a fundamental set of solutions. Then, the complementary solution is

$$y_c(t) = c_1 y_1(t) + \cdots + c_n y_n(t).$$

Now, we consider a trial solution for the non-homogeneous equation of the form

$$Y(t) = u_1(t)y_1(t) + \cdots + u_n(t)y_n(t)$$

for unknown functions $u_1, \dots, u_n$. Differentiating gives

$$Y'(t) = u_1(t)y_1'(t) + \cdots + u_n(t)y_n'(t) + u_1'(t)y_1(t) + \cdots + u_n'(t)y_n(t).$$

As before we set the constraint

$$\boxed{u_1'(t)y_1(t) + u_2'(t)y_2(t) + \cdots + u_n'(t)y_n(t) = 0},$$

so that the expression for Y' simplifies to

$$Y'(t) = u_1(t)y_1'(t) + u_2(t)y_2'(t) + \cdots + u_n(t)y_n'(t).$$

Computing Y'' and setting

$$\boxed{u_1'(t)y_1'(t) + \dots u_n'(t)y_n'(t) = 0}$$

leads to the simplified expression for the second derivative

$$Y''(t) = u_1(t)y_1''(t) + \cdots + u_n(t)y_n''(t).$$

Repeating this procedure (differentiating and then setting the sum of terms involving the derivatives of $u_1, \dots, u_n$ to zero) leads to the $n-1$ equations

$$u_1'(t)y_1^{(m)}(t) + \cdots + u_n'(t)y_n^{(m)}(t) = 0 \quad \forall 1 \le m \le n-2,$$

as well as a simplified expression for $Y^{(m)}$:

$$\begin{aligned} Y^{(m)}(t) &= u_1(t)y_1^{(m)}(t) + \cdots + u_n(t)y_n^{(m)}(t), \quad m = 1, \dots, n-1, \\ Y^{(n)}(t) &= u_1(t)y_1^{(n)}(t) + \cdots + u_n(t)y_n^{(n)}(t) + u_1'(t)y_1^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t). \end{aligned}$$

So if Y is a particular solution to the non-homogeneous equation, substituting all the expressions for Y and its derivative into the equation, and using that $y_1, \dots, y_n$ solve the homogeneous equation, we are lead to

$$\boxed{u_1'(t)y_1^{(n-1)}(t) + \cdots + u_n'(t)y_n^{(n-1)}(t) = g(t)}.$$

Collecting all the expressions involving the first derivative of $u_1, \dots, u_n$, we obtain

$$\begin{pmatrix} y_1 & y_2 & \dots & y_{n-1} & y_n \\ y_1' & y_2' & \dots & y_{n-1}' & y_n' \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ y_1^{(n-2)} & y_2^{(n-2)} & \dots & y_{n-1}^{(n-2)} & y_n^{(n-2)} \\ y_1^{(n-1)} & y_2^{(n-1)} & \dots & y_{n-1}^{(n-1)} & y_n^{(n-1)} \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_{n-1}' \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}.$$

Thus, the derivatives of the unknown functions $u_1, \dots, u_n$ can be found by inverting the matrix of derivatives. The determinant of the matrix is the Wronskian, which is non-zero thanks to the fact that $(y_1, \dots, y_n)$ forms a fundamental set of solutions. Setting $M(t)$ as the matrix, we solve

$$M(t) \begin{pmatrix} u_1' \\ u_2' \\ \vdots \\ u_{n-1}' \\ u_n' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g(t) \end{pmatrix}.$$

To invert $M(t)$, we use Cramer's rule, by setting

$$M_i(t) = \begin{pmatrix} y_1 & \dots & 0 & \dots & y_n \\ y_1' & \dots & 0 & \dots & y_n' \\ \vdots & & \vdots & & \vdots \\ y_1^{(n-2)} & \dots & 0 & \dots & y_n^{(n-2)} \\ y_1^{(n-1)} & \dots & 1 & \dots & y_n^{(n-1)} \end{pmatrix},$$

i.e., replace the ith column of $M(t)$ with the vector $(0, \dots, 0, 1)^\intercal$. Then Cramer's rule gives

$$\boxed{u_i'(t) = \frac{g(t)\det M_i(t)}{\det M(t)}},$$

and by integrating we get an expression for $u_i(t)$. The particular solution to the non-homogeneous equation is therefore

$$\boxed{Y(t) = y_1(t) \int \frac{g(t)\det M_1(t)}{\det M(t)}\, dt + \dots + y_n(t) \int \frac{g(t)\det M_n(t)}{\det M(t)}\, dt}.$$

However, in general the evaluation of the integrals can be difficult, but we can always use Abel's theorem to simplify, since

$$\det M(t) = W(y_1, \dots, y_n)[t] = ce^{-\int P_{n-1}(t)\, dt}.$$

We finish with one example.

Example 4.5. *Solve*

$$y''' + y' = \sec^2(t) \text{ for } t \in (-\pi/2, \pi/2).$$

The characteristic equation for the homogeneous problem is $r^3 + r = 0$ *and so* $r_1 = 0$, $r_2 = i$ *and* $r_3 = -i$. *Hence the complementary solution is*

$$y_c(t) = c_1 + c_2 \cos t + c_3 \sin t.$$

By variation of parameters we look for a particular solution of the form

$$Y(t) = u_1 y_1 + u_2 y_2 + u_3 y_3 = u_1(t) + u_2(t) \cos t + u_3(t) \sin t,$$

with

$$\begin{aligned} u_1' + u_2' \cos t + u_3' \sin t &= 0, \\ -u_2' \sin t + u_3' \cos t &= 0, \\ -u_2' \cos t - u_3' \sin t &= \sec^2(t), \end{aligned}$$

or equivalently

$$M(t) \begin{pmatrix} u_1' \\ u_2' \\ u_3' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sec^2(t) \end{pmatrix}, \quad M(t) = \begin{pmatrix} 1 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 0 & -\cos t & -\sin t \end{pmatrix}.$$

Computing the determinant of M, *we see that* $\det M(t) = 1$. *Now, define*

$$M_1(t) = \begin{pmatrix} 0 & \cos t & \sin t \\ 0 & -\sin t & \cos t \\ 1 & -\cos t & -\sin t \end{pmatrix}, \quad M_2(t) = \begin{pmatrix} 1 & 0 & \sin t \\ 0 & 0 & \cos t \\ 0 & 1 & -\sin t \end{pmatrix}, \quad M_3(t) = \begin{pmatrix} 1 & \cos t & 0 \\ 0 & -\sin t & 0 \\ 0 & -\cos t & 1 \end{pmatrix},$$

it is easy to compute that

$$\det M_1(t) = 1, \quad \det M_2(t) = -\cos t, \quad \det M_3(t) = -\sin t,$$

and so

$$\begin{aligned} u_1 &= \int \sec^2(t)\, dt = \tan(t), \\ u_2 &= \int -\sec^2(t) \cos(t)\, dt = -\ln(|\sec(t) + \tan(t)|), \\ u_3 &= \int -\sec^2(t) \sin(t)\, dt = -\sec(t). \end{aligned}$$

Hence, the particular solution is

$$Y(t) = \tan(t) - \cos(t) \ln(|\sec(t) + \tan(t)|) - \sin(t) \sec(t) = -\cos(t) \ln(|\sec(t) + \tan(t)|).$$