MATH3720A - Lecture Notes

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3 Second order linear equations

A second order linear ODE is of the form

$$p(t)y'' + q(t)y' + r(t)y = s(t)$$

for some given functions p, q, r and s. If $p(t) \neq 0$ then we can express the second order ODE alternatively as

$$y'' + a(t)y' + b(t)y = f(t), \quad a(t) := \frac{q(t)}{p(t)}, \, b(t) = \frac{r(t)}{p(t)}, \, f(t) = \frac{s(t)}{p(t)}.$$

In Chapter 1 we discussed that for a second order ODE we require **two** initial conditions

$$y(t_0) = y_0, \quad y'(t_0) = y_1,$$

for given constants t_0, y_0, y_1. Note that we are not only prescribing that the solution $y(t)$ passes through the point (t_0, y_0), but also **its slope** $y'(t)$ passes through the point (t_0, y_1).

The theory for second order linear equations is more involved than the theory for first order linear equations. We first introduce the following classification.

Definition 3.1 (Homogeneous equation)**.** *A second order linear ODE*

$$p(t)y'' + q(t)y' + r(t)y = s(t), \quad p(t) \neq 0,$$

is called ***homogeneous*** *if $s(t) \equiv 0$. Otherwise, if $s(t) \neq 0$, the ODE is called* ***non-homogeneous****.*

Do not be **confused** about homogeneous **first** order ODE $\frac{dy}{dt} = F(y/t)$ and homogeneous **second** order ODE $y'' + a(t)y' + b(y)y = 0$.

Example 3.1 (Reduce to first order)**.** *If a general second order equation*

$$y''(t) = F(t, y, y')$$

for which the variable y does not appear, i.e.,

$$y''(t) = F(t, y').$$

Then, using the substitution $v(t) = y'(t)$, we have

$$v'(t) = y''(t) = F(t, y') = F(t, v) \Rightarrow \boxed{v'(t) = F(t, v)},$$

*that is, we now have a **first order** equation. As an example, solve the ODE*

$$y'' + ay' = 0, \quad a \in \mathbb{R}, \quad a \neq 0.$$

Setting $v = y'$ the ODE satisfied by v is

$$v' + av = 0 \Rightarrow v(t) = c\exp(-at), \quad c \in \mathbb{R}.$$

Hence

$$y'(t) = c\exp(-at) \Rightarrow \boxed{y(t) = \frac{-c}{a}\exp(-at) + c_0}, \quad c_0 \in \mathbb{R}.$$

Note that we need two initial conditions to determine the constants c and c_0.

Before we study the methods to solve second order equations, let us mention an application.

Motivation - vibrations. Consider the motion of a mass on a spring. On one end the spring the fixed to the ceiling and on the other end it is attached to an object with mass $m > 0$. Before hanging the object, the spring has a length $l > 0$, and after hanging the spring is **stretched** by a length $L > 0$ downwards.

If no additional force is acting on the mass-spring system, then there are just two forces acting on the object: (1) the weight that acts downwards $F_g = mg$, and (2) a **restoring** force F_s from the spring that tries to pull the object upwards. We assume that the stretching L is small, so that the force F_s is **proportional** to L. Denoting by the constant of proportionality by k, we now have $F_s = kL$. This is commonly known as **Hooke's law** and k is called the **spring constant**. The net force (pointing downwards) is

$$F = mg - kL,$$

and if the object is in equilibrium, we must have $F = 0$ and so

$$mg = kL.$$

Now suppose we pull on the object and the spring is further extended, and then let go. We want to **measure** the **displacement** $u(t)$ of the object from its equilibrium position. Note that $u(t)$ can take both positive and negative values. Positive values of $u(t)$ means the object at time t is **below** the equilibrium position and negative values of $u(t)$ means the object at time t is **above** the equilibrium position.

Using Newton's second law, and the fact that the acceleration is the second derivative of the displacement, i.e., $a = u''$, we have

$$mu''(t) = f(t),$$

where $f(t)$ is the net force acting on the object comprises of

(1) the weight $F_g = mg$ acting downwards;

(2) a restoring force from the spring pulling the mass upwards $F_s = -k(L+u)$;

(3) a resistance force F_d (air resistance/friction) that acts in the opposite of the motion and is proportional to the speed u'. This is usually referred to as **viscous damping** and $F_d = -\gamma u'$ with constant $\gamma > 0$ (**damping constant**);

(4) an external force $F(t)$ that models the up/down movement of the ceiling.

Altogether we arrive at the ODE

$$\begin{aligned} mu''(t) &= mg - k(L + u(t)) - \gamma u'(t) + F(t) \\ \Rightarrow & \boxed{mu''(t) + \gamma u'(t) + ku(t) = F(t)} \end{aligned}$$

if we also use $mg = kL$. To complete the model we specific two initial conditions

$$\boxed{u(0) = u_0, \quad u'(0) = v_0},$$

where u_0 is the initial position (right before let go of the spring) and v_0 is the initial velocity.

3.1 Existence and Uniqueness

Let us first state the abstract result on the existence and uniqueness of solutions to second order linear equations.

Theorem 3.1. *Consider the IVP*

$$y'' + p(t)y' + q(t)y = r(t), \quad y(t_0) = y_0, \quad y'(t_0) = y_1.$$

Suppose there is an open interval $I \subset \mathbb{R}$ such that $t_0 \in I$, and the functions p, q, r are ***continuous*** *in I. Then, there is* ***exactly one*** *solution $y(t)$ to the IVP for $t \in I$.*

We will not discuss the proof of the theorem. It will be sufficient use the result in this course.

Example 3.2. *The IVP*

$$(t^2 - 3t)y'' + \exp(t)y' - \sin(t+3)y = 0, \quad y(1) = 2, \quad y'(1) = 1,$$

can be written as

$$y'' + \frac{\exp(t)}{t^2 - 3t}y' - \frac{\sin(t+3)}{t^2 - 3t}y = 0, \quad y(1) = 2, \quad y'(1) = 1.$$

The functions $p(t) = \frac{\exp(t)}{t^2-3t}$ *and* $q(t) = \frac{\sin(t+3)}{t^2-3t}$ *are continuous except at the points* $t = 0$ *and* $t = 3$. *Since* $t_0 = 1$ *the largest interval which the functions* p *and* q *are continuous in* $(0,3)$. *Hence, by Theorem 3.1, there exists a unique solution to the IVP for* $t \in (0,3)$.

If we instead consider the initial condition $y(4) = 2$ *and* $y'(4) = 1$, *then the largest interval for which there exists a unique solution to the IVP is* $(3,\infty)$.

Example 3.3 (Application of uniqueness)**.** *Find the solution to the IVP*

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_1) = 0,$$

for continuous functions p, q, r *and given constants* $t_0, t_1 \in \mathbb{R}$.

Note that $y \equiv 0$ *is a solution to the IVP. By Theorem 3.1 there is exactly one solution, and so the* ***only*** *solution is* $y(t) \equiv 0$.

3.2 Principle of superposition

For second order linear **homogeneous** equations we have the following.

Theorem 3.2 (Principe of superposition)**.** *If* y_1 *and* y_2 *are two solutions of the ODE*

$$a(t)y'' + b(t)y' + c(t)y = 0. \tag{3.1}$$

Then for any constants $c_1, c_2 \in \mathbb{R}$, *the function* $c_1y_1(t) + c_2y_2(t)$ *is also a solution to the ODE.*

A special case is when $c_1 = 0$ we get the solution y_2 and when $c_2 = 0$ we get y_1.

Take away message - From two solutions we can construct an **infinite family** of solutions to the homogeneous linear ODE. That is, we can define a set of (general) solutions

$$\mathcal{S} := \{y := c_1y_1 + c_2y_2 \mid c_1, c_2 \in \mathbb{R}\}$$

to the ODE. Note that we have not included in the initial conditions, and it turns out that if $y = c_1y_1 + c_2y_2$ is to be a solution to the IVP then some condition has to hold.

Given the initial conditions

$$y(t_0) = x_0, \quad y'(t_0) = x_1,$$

can we find the constants c_1 and c_2 so that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ solve the IVP? Plugging in the initial conditions leads to

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= x_0, \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= x_1, \end{aligned}$$

or in a matrix form:

$$\boxed{\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} x_0 \\ x_1 \end{pmatrix}}. \tag{3.2}$$

We can solve for (c_1, c_2) by inverting the matrix, which requires the determinant

$$\boxed{W := y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0)} \tag{3.3}$$

to be non-zero.

Definition 3.2 (Wronskian)**.** *The Wronskian $W(y_1, y_2)[t_0]$ is defined as*

$$\boxed{W(y_1, y_2)[t_0] = y_1(t_0) y_2'(t_0) - y_2(t_0) y_1'(t_0)}.$$

The idea is: if the Wronskian is non-zero, we can solve the matrix-vector problem (3.2) by inverting the matrix and compute for the coefficients c_1 and c_2. This leads to the following result.

Theorem 3.3. *Let y_1 and y_2 be two solutions to the ODE*

$$a(t)y'' + b(t)y' + c(t)y = 0.$$

For any $(x_0, x_1) \in \mathbb{R}^2$, it is ***always possible*** *to choose two constants c_1 and c_2 such that the function*

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

is a solution to the ODE and

$$y(t_0) = x_0, \quad y'(t_0) = x_1$$

if and only if the Wronskian $W(y_1, y_2)$ at t_0 is non-zero.

Remark 3.1. *Notice that y_1 need not be equal to y_2 as we did not specify any initial conditions, i.e., the uniqueness part of Theorem 3.1 does not cause a contradiction here.*

What about the case where the Wronskian is zero? If $W = W(y_1, y_2)[t_0] = 0$, then we cannot solve the matrix-vector problem (3.2) in general. This means that there are many initial conditions (x_0, x_1) for which no pair of constants (c_1, c_2) exists so that $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is a solution to the IVP.

Example 3.4. *One can check that the functions* $y_1(t) = \exp(-2t)$ *and* $y_2(t) = \exp(-3t)$ *are solutions to the ODE*

$$y'' + 5y' + 6y = 0.$$

The Wronskian at t is

$$W(y_1, y_2)[t] = -\exp(-5t)$$

which is non-zero for any $t \in \mathbb{R}$. *Therefore we can use* y_1 *and* y_2 *to construct a family of solutions to the ODE.*

The question that arises now is: "when can any solution to the ODE (3.7) be expressed as a **linear combination** of two solutions y_1 and y_2?" A positive answer would imply that if we know two solutions y_1 and y_2 to the ODE (3.7), then **any** possible solution y to (3.7) can be expressed as $y = c_1 y_1 + c_2 y_2$ for some constants c_1 and c_2. Returning to the example, any solution y to the ODE $y'' + 5y' + 6y = 0$ must be of the form

$$y(t) = c_1 \exp(-2t) + c_2 \exp(-3t)$$

for some constants c_1 and c_2, where are then determined by the initial conditions.

The answer to the question is given below.

Theorem 3.4. *Let I be an open interval, p and q are continuous functions in I. Let* y_1 *and* y_2 *be two solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0$$

for $t \in I$. *Then, any solution y to the ODE can be expressed as*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3.4}$$

for constants c_1 *and* c_2 *if and only if there is a point* $t_0 \in I$ *such that the Wronskian* $W(y_1, y_2)[t_0]$ *is non-zero at* t_0.

The theorem says that if at some point t_0, the Wronskian is non-zero, then a general solution to the ODE is given be the formula (3.4).

Proof. Let us consider the direction ($\Leftarrow$), that is, we assume there is a point $t_0 \in I$ where the Wronskian $W(y_1, y_2)[t_0]$ is non-zero. Let ϕ be any solution to the ODE. We need to show that ϕ can be written as a linear combination of y_1 and y_2.

Consider the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = \phi(t_0), \quad y'(t_0) = \phi'(t_0). \tag{3.5}$$

That is, the initial values are $\phi(t_0)$ and $\phi'(t_0)$. Then, a solution to the IVP is the function ϕ itself. Since the Wronskian is non-zero at t_0, the matrix-vector problem

$$\begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} \phi(t_0) \\ \phi'(t_0) \end{pmatrix}$$

admits a unique solution (c_1^*, c_2^*). This means that the function

$$z(t) := c_1^* y_1(t) + c_2^* y_2(t)$$

is a solution to the IVP (3.5). By Theorem 3.1, there is only one solution to the IVP, therefore

$$\phi(t) = z(t) = c_1^* y_1(t) + c_2^* y_2(t).$$

For the converse ($\Rightarrow$) we prove by contrapositive, which amounts to show that if there is no points where the Wronskian is non-zero, then every solution ϕ to the ODE **cannot** be written as a linear combination of y_1 and y_2. Suppose for all $t_0 \in I$, $W(y_1, y_2)[t_0] = 0$. Then by Theorem 3.3, there exists initial conditions x_0, x_1 such that the matrix-value problem (3.2) has no solution. That is, we cannot find constants c_1, c_2 such that

$$\begin{aligned} c_1 y_1(t_0) + c_2 y_2(t_0) &= x_0, \\ c_1 y_1'(t_0) + c_2 y_2'(t_0) &= x_1. \end{aligned}$$

In particular, if ϕ is a solution to the IVP with initial conditions (x_0, x_1). Then, we see that it is not possible to write ϕ as a linear combination of y_1 and y_2. □

Theorem 3.4 says that once we know two solutions y_1 and y_2 to the ODE, and if the Wronskian W is non-zero at some point $t_0 \in I$, then we know what a general solution to the ODE looks like. In particular we can express every solution to the ODE as a linear combination of y_1 and y_2. In this regard, we say that (y_1, y_2) form a **fundamental set of solutions** to the ODE.

Definition 3.3 (Fundamental set of solutions)**.** *A pair of functions (y_1, y_2) is called a fundamental set of solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0$$

if any solution y to the ODE can be written as

$$y(t) = c_1 y_1(t) + c_2 y_2(t)$$

for some constants c_1, c_2.

Example 3.5. *For the ODE*

$$2t^2 y'' + 3ty' - y = 0, \quad t > 0,$$

the functions $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions. Let us compute the Wronskian

$$W(y_1, y_2)[t] = -\frac{3}{2} t^{-3/2},$$

which is non-zero for $t > 0$. Therefore we can deduce that (y_1, y_2) form a fundamental set of solutions for the ODE, and a general solution y to the ODE can be expressed as

$$y(t) = c_1 t^{1/2} + c_2 t^{-1},$$

for some constants c_1, c_2.

Does a fundamental set of solutions always exists? This is answered in the next theorem.

Theorem 3.5 (Existence of fundamental set of solutions). *Let I be an open interval of $\mathbb{R}$, p and q are continuous functions in I. For any $t_0 \in I$, let $y_1(t)$ be the (unique) solution to the IVP*

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 1, \quad y'(t_0) = 0,$$

and $y_2(t)$ be the (unique) solution to the IVP

$$y'' + p(t)y' + q(t)y = 0, \quad y(t_0) = 0, \quad y'(t_0) = 1.$$

Then, (y_1, y_2) forms a fundamental set of solutions to the ODE.

Note that the existence of y_1 and y_2 to the corresponding IVPs is guaranteed by Theorem 3.1. By Theorem 3.4 it suffices to show that the Wronskian $W(y_1, y_2)[t_0]$ is non-zero. Computing gives

$$W(y_1, y_2)[t_0] = 1.$$

The idea behind the special choice of initial conditions for y_1 and y_2 is closely related to a concept in **linear algebra**. Our goal is to find an expression for the general solution of a second order linear ODE in terms of two (known) solutions y_1 and y_2. However the method we studied above does not work if y_1 is **not different** from y_2. Borrowing an idea from linear algebra we make the following definition.

Definition 3.4 (Linear independence). *Let $n \in \mathbb{N}$ be fixed, and consider n functions $x_1(t), \dots, x_n(t)$ defined on an interval $I \subset \mathbb{R}$. We say that $x_1, \dots, x_n$ are **linearly independent** if the only solution of*

$$\alpha_1 x_1(t) + \cdots + \alpha_n x_n(t) = 0 \quad \forall t \in I$$

is $\alpha_1 = \cdots = \alpha_n = 0$.

For two functions, linear independence means one function is **not a constant multiple** of the other. Since, if x_1 and x_2 are proportional on I then for some constant $c \neq 0$ we have

$$x_1(t) = cx_2(t) \Rightarrow x_1(t) - cx_2(t) = 0.$$

Conversely, if x_1 and x_2 are linearly dependent, then there are constants $a_1, a_2 \neq 0$ such that

$$a_1 x_1(t) + a_2 x_2(t) = 0 \Rightarrow x_1(t) = \frac{-a_2}{a_1} x_2(t).$$

Example 3.6 (Two fundamental sets of solutions). *Consider the ODE*

$$y'' - y = 0.$$

Note that $y_1(t) = \exp(t)$ and $y_2(t) = \exp(-t)$ are solutions to the ODE. The Wronskian $W(y_1, y_2)[t] = -2 \neq 0$ and so by Theorem 3.4 they form a fundamental set of solutions.

However, the pair $(\exp(t), \exp(-t))$ does not satisfy the conditions in Theorem 3.5 at $t_0 = 0$. Since by Theorem 3.4, any solution y to $y'' - y = 0$ must be of the form

$$y(t) = c_1 \exp(t) + c_2 \exp(-t),$$

plugging in the initial condition $y(0) = 1$ and $y'(0) = 0$ gives a solution

$$z_1(t) = \frac{1}{2}\exp(t) + \frac{1}{2}\exp(-t) = \cosh(t).$$

Similarly, plugging in the initial condition $y(0) = 0$ and $y'(0) = 1$ gives a solution

$$z_2(t) = \frac{1}{2}\exp(t) - \frac{1}{2}\exp(-t) = \sinh(t).$$

Furthermore, the Wronskian $W(z_1, z_2)[t] = \cosh^2(t) - \sinh^2(t) = 1$, and so (z_1, z_2) forms a fundamental set of solutions as stated by Theorem 3.5. This means that any solution y to the ODE $y'' - y = 0$ can also be expressed as

$$y(t) = d_1 \cosh(t) + d_2 \sinh(t)$$

for constants d_1, d_2.

From the above example, we see that there are **more than one** fundamental set of solutions for a given ODE. Indeed, it has **infinitely many** fundamental sets. But Theorem 3.5 says that there is always one.

A large part of the above results rely on having at hand two solutions y_1 and y_2 to the second order linear ODE. Although we have not discussed how to find them, we can actually compute an expression for the Wronskian without any knowledge of the explicit forms for y_1 and y_2. This is summarised in the next theorem.

Theorem 3.6 (Abel's theorem)**.** *Let I be an open interval, p and q are continuous in I. Suppose y_1 and y_2 are two non-zero solutions to the ODE*

$$y'' + p(t)y' + q(t)y = 0.$$

Then, the Wronskian is given as

$$W(y_1, y_2)[t] = c \exp\left(-\int p(t)\, dt\right),$$

*where the constant c depends on y_1 and y_2, **but not on t**. Consequently, $W(y_1, y_2)[t] = 0$ if and only if $c = 0$.*

In particular, if we know that $W(y_1, y_2)[t_*] = 0$ for some $t_* \in I$, then it holds that $W(y_1, y_2)[t] = 0$ for all $t \in I$.

Proof. The idea is to derive an ODE for the Wronskian W. Going back to the ODE, as y_1 is a solution we have

$$y_1'' + p(t)y_1' + q(t)y_1 = 0 \Rightarrow y_2y_1'' + y_2p(t)y_1' + y_2q(t)y_1 = 0.$$

Similarly, as y_2 is a solution,

$$y_1y_2'' + y_1p(t)y_2' + y_1q(t)y_2 = 0.$$

Subtracting one from another gives

$$(y_1y_2'' - y_2y_1'') + p(t)(y_1y_2' - y_2y_1') = 0. \tag{3.6}$$

Noting that

$$\begin{aligned} &W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ \Rightarrow\ &W'(y_1, y_2)[t] = y_1(t)y_2''(t) - y_2(t)y_1''(t), \end{aligned}$$

from (3.6) we have

$$W' + p(t)W = 0,$$

which is a linear first order equation. By integrating factors we find the general solution

$$W(y_1, y_2)[t] = c\exp\left(-\int p(t)\, dt\right)$$

for some constant $c \in \mathbb{R}$. As a constant of integration, c does not depend on t. □

Example 3.7. *Previously we verified that the functions $y_1(t) = t^{1/2}$ and $y_2(t) = t^{-1}$ are solutions to*

$$2t^2y'' + 3ty' - y = 0, \quad t > 0.$$

We computed the Wronskian as $W(y_1, y_2)[t] = -(3/2)t^{-3/2}$. We check this with Abel's theorem. Writting the ODE in standard form

$$y'' + \frac{3}{2t}y' - \frac{1}{2t^2}y = 0 \Rightarrow p(t) = \frac{3}{2t}, \quad q(t) = -\frac{1}{2t^2}.$$

Then,

$$W(y_1, y_2)[t] = c\exp\left(-\int \frac{3}{2t}\, dt\right) = c\exp\left(-\frac{3}{2}\ln(t)\right) = ct^{-3/2}.$$

Then, on comparison we have $c = -3/2$.

Example 3.8. *Given $y_1 = t^{1/2}$ is a solution to*

$$2t^2y'' + 3ty' - y = 0, \quad t > 0,$$

find the other solution y_2 to the ODE.

Although we are missing a second solution y_2, let us assume $y_2(t)$ exists and compute the Wronskian:

$$W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_2(t)y_1' = t^{1/2}y_2' - \frac{1}{2}t^{-1/2}y_2.$$

On the other hand, by Abel's theorem

$$W(y_1, y_2)[t] = c\exp\Big(-\int \frac{3}{2t}\,dt\Big) = ct^{-3/2}.$$

Choosing $c = 1$, the equating yields an ODE for y_2:

$$t^{1/2}y_2' - \frac{1}{2}t^{-1/2}y_2 = t^{-3/2},$$

which is a ***first order linear*** *equation. Computing the integrating factor $\mu(t) = \exp(-\int \frac{1}{2t}\,dt) = t^{-1/2}$ we see that*

$$y_2(t) = t^{1/2}\Big(\int t^{-5/2}\,dt + d\Big) = -\frac{2}{3}t^{-1} + d\,t^{1/2}.$$

In particular, if $d = 0$, we see that the function t^{-1} is also a solution to the ODE. Hence, even if we only have one solution y_1, we can find another solution y_2 if the Wronskian is non-zero.

Exercise. Check that if y_1 solves $y'' + p(t)y' + q(t)y = 0$, then the function z which is a solution to

$$y_1(t)z' - y_1'(t)z = \exp\Big(-\int p(t)\,dt\Big)$$

satisfies $z'' + p(t)z' + q(t)z = 0$.

Based on the above results, the strategy to solve

$$y'' + p(t)y' + q(t)y = 0, \quad t \in I,$$

can be summarised as follows:

(1) Find two solutions y_1, y_2 satisfying the ODE.

(2) Find $t_* \in I$ such that the Wronskian $W(y_1, y_2)[t_*]$ is non-zero. Then, the general solution to the ODE is

$$y(t) = c_1y_1(t) + c_2y_2(t)$$

for some constants c_1, c_2.

(3) If initial conditions are prescribed at some $t_0 \in I$, compute c_1 and c_2 to determine the particular solution.

We now discuss how to find y_1 and y_2 for constant coefficients, i.e., $p(t)$ and $q(t)$ are constants.

3.3 Homogeneous equations with constant coefficients

The objective of this section is to study the solutions to the ODE

$$ay'' + by' + cy = 0 \tag{3.7}$$

for fixed constants $a, b, c \in \mathbb{R}$ with $a \neq 0$. In Example 3.1 we saw how to solve the equation

$$y'' + \frac{b}{a}y' = 0,$$

which has a general solution involving the exponential function. Hence, let us consider substituting a **trial** function $y(t) = \exp(rt)$ for some constant r into the ODE. This yields

$$(ar^2 + br + c)\exp(rt) = 0.$$

Since $\exp(rt)$ is positive, we obtain that

$$\boxed{ar^2 + br + c = 0}. \tag{3.8}$$

The equation (3.8) is known as the **characteristic equation** for the ODE (3.7). If we can find the roots of the characteristic equation, then we know that $\exp(rt)$, where r is a root, is a solution to (3.7).

Since (3.8) is a quadratic equation, by the well-known quadratic formula, we see that

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Notice that for different values of a, b, c we encounter the possibilities:

(1) Two distinct real roots r_1, r_2 if $b^2 > 4ac$.

(2) Two complex roots (complex conjugate pairs) $r_1, \overline{r_1}$ if $b^2 < 4ac$.

(3) A repeated real root r if $b^2 = 4ac$.

Immediately we see that the explicit formula for the solution $y(t)$ to (3.7) will depend heavily on the **discriminant** $b^2 - 4ac$.

3.3.1 Two distinct real roots

In the case $b^2 - 4ac > 0$, we obtain two real roots

$$\boxed{r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \quad r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}}.$$

This gives us two functions

$$y_1(t) = \exp(r_1 t), \quad y_2(t) = \exp(r_2 t).$$

Let us now check if they are **linearly independent** as in Definition 3.4. Let $\alpha_1, \alpha_2 \in \mathbb{R}$ be constants such that

$$\alpha_1 y_1(t) + \alpha_2 y_2(t) = 0 \quad \forall t \in I$$
$$\Rightarrow \alpha_1 \exp(r_1 t) + \alpha_2 \exp(r_2 t) = 0.$$

Since $r_1 \neq r_2$, the only possible way for the above equality to hold is if $\alpha_1 = \alpha_2 = 0$ (simply plugging in two different values of t). So y_1 and y_2 are linearly independent.

Let now check the Wronskian:

$$W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_2(t)y_1'(t) = r_2 \exp((r_1 + r_2)t) - r_1 \exp((r_1 + r_2)t)$$
$$= (r_2 - r_1)\exp((r_1 + r_2)t).$$

Since $r_1 \neq r_2$ and the exponential is never zero, we see that the Wronskian $W(y_1, y_2)[t]$ is positive for all $t \in \mathbb{R}$.

Then, by Theorem 3.4, any solution y to the ODE $ay'' + by' + cy = 0$ where $b^2 - 4ac > 0$ can be expressed as a linear combination of y_1 and y_2. More precisely, any solution $y(t)$ to the ODE is of the form

$$\boxed{y(t) = c_1 \exp(r_1 t) + c_2 \exp(r_2 t)} \tag{3.9}$$

for some constants c_1 and c_2.

Exercise: Check by differentiating to see that (3.9) is a solution to the ODE.

To determine the values of c_1 and c_2, suppose we have the IVP

$$ay'' + by' + cy = 0, \quad y(t_0) = x_0, \quad y'(t_0) = x_1.$$

Then, a simple calculation shows that

$$c_1 \exp(r_1 t_0) + c_2 \exp(r_2 t_0) = x_0,$$
$$r_1 c_1 \exp(r_1 t_0) + r_2 c_2 \exp(r_2 t_0) = x_1.$$

Upon rearranging leads to

$$c_1 = \frac{x_1 - x_0 r_2}{r_1 - r_2}\exp(-r_1 t_0), \quad c_2 = \frac{x_0 r_1 - x_1}{r_1 - r_2}\exp(-r_2 t_0).$$

As $r_1 \neq r_2$, the above expressions always make sense.

Example 3.9. *Find the general solution to the ODE*

$$y'' + 9y' + 20y = 0.$$

As before we consider a trial function $y(t) = \exp(rt)$ *and after substituting, we obtain the characteristic equation*

$$r^2 + 9r + 20 = (r + 4)(r + 5) = 0.$$

This means that we have two real roots $r_1 = -4$ and $r_2 = -5$. Hence, the general solution is

$$y(t) = c_1 \exp(-4t) + c_2 \exp(-5t), \quad c_1, c_2 \in \mathbb{R}.$$

Let us note that, as $t \to \infty$, the solution $y(t)$ will tend to zero. This behaviour does not depend on the sign of c_1 and c_2, since the exponents are both negative in this case.

Example 3.10. *Find the general solution to the ODE*

$$y'' - y' - 42y = 0.$$

We obtain as the characteristic equation

$$r^2 - r - 42 = (r-7)(r+6) = 0.$$

This gives $r_1 = 7$ and $r_2 = -6$ and the general solution is

$$y(t) = c_1 \exp(7t) + c_2 \exp(-6t), \quad c_1, c_2 \in \mathbb{R}.$$

Note that as the function $c_2 \exp(-6t) \to 0$ as $t \to \infty$, and so we have

$$y(t) \to \begin{cases} \infty & \text{if } c_1 > 0, \\ -\infty & \text{if } c_1 < 0, \\ 0 & \text{if } c_1 = 0. \end{cases}$$

Of course the value of c_1 and c_2 are determined by the initial conditions. But it is important to point out that it is possible for the solution y to go to $\pm\infty$ with one of the exponent is positive.

3.3.2 Complex roots

We now consider the case $b^2 - 4ac < 0$. Then, the roots to the characteristic equation $ar^2 + br + c = 0$ is a **complex-conjugate pair**:

$$\boxed{r_1 = \lambda + i\mu, \quad \lambda = \frac{-b}{2a}, \quad \mu = \sqrt{4ac - b^2}, \quad i := \sqrt{-1}, \quad r_2 = \overline{r_1} = \lambda - i\mu}.$$

Since the characteristic equation is obtained by substituting the trial function $y(t) = \exp(rt)$, we obtain two functions

$$\boxed{y_1(t) = \exp(r_1 t) = \exp((\lambda + i\mu)t), \quad y_2(t) = \exp(r_2 t) = \exp((\lambda - i\mu)t)}.$$

Euler's formula: Due to the imaginary number i appearing in the formulae for y_1 and y_2, we use the well-known **Euler's formula**: For any real number $x \in \mathbb{R}$,

$$\boxed{\exp(ix) = \cos(x) + i\sin(x)}.$$

As a consequence of the symmetries of cos and sin, we also have

$$\exp(-ix) = \cos(x) - i\sin(x).$$

Furthermore, applying the general rule

$$\exp(a+b) = \exp(a)\exp(b) \quad \text{for } a, b \in \mathbb{C},$$

we now arrive at

$$\boxed{y_1(t) = \exp(\lambda t)(\cos(\mu t) + i\sin(\mu t)), \quad y_2(t) = \exp(\lambda t)(\cos(\mu t) - i\sin(\mu t))}.$$

Note that there is a **common factor** $\exp(\lambda t)$ appearing in both solutions. One can also check that

$$\overline{y_1(t)} = \exp(\lambda t)(\cos(\mu t) - i\sin(\mu t)) = y_2(t),$$

so that y_2 is the **complex conjugate** of y_1.

Let us first check if y_1 and y_2 are linearly independent. Suppose there are constants $\alpha_1, \alpha_2 \in \mathbb{R}$ such that

$$\begin{aligned} &\alpha_1 y_1(t) + \alpha_2 y_2(t) = 0 \quad \forall t \in I \\ \Rightarrow\ & e^{\lambda t}((\alpha_1 + \alpha_2)\cos(\mu t) + i(\alpha_1 - \alpha_2)\sin(\mu t)) = 0. \end{aligned}$$

The exponential is non-zero for all $t \in \mathbb{R}$, and so to make the above expression zero, we need

$$\alpha_1 + \alpha_2 = 0, \quad \alpha_1 - \alpha_2 = 0 \Rightarrow \alpha_1 = \alpha_2 = 0.$$

So y_1 and y_2 are linearly independent.

Let us now check the Wronskian. Using the differentiation formula

$$\frac{d}{dt}\exp(iqt) = iq\exp(iqt) \quad \text{for } q \in \mathbb{R},$$

we find that

$$\begin{aligned} W(y_1, y_2)[t] &= y_1(t)y_2'(t) - y_2(t)y_1'(t) \\ &= e^{(\lambda+i\mu)t}(\lambda - i\mu)e^{(\lambda-i\mu)t} - (\lambda + i\mu)e^{(\lambda+i\mu)t}e^{(\lambda-i\mu)t} \\ &= e^{2\lambda t}(\lambda - i\mu - \lambda - i\mu) = -2i\mu e^{2\lambda t}. \end{aligned}$$

Since the exponential is never zero for $t \in \mathbb{R}$, and μ is non-zero (otherwise we will not have $b^2 - 4ac < 0$), the Wronskian is non-zero for all $t \in \mathbb{R}$.

Then, by Theorem 3.4, any solution y to the ODE $ay'' + by' + cy = 0$ where $b^2 - 4ac < 0$ can be expressed as a linear combination of y_1 and y_2. More precisely, any solution $y(t)$ to the ODE is of the form

$$\begin{aligned} &\boxed{y(t) = e^{\lambda t}\left((c_1 + c_2)\cos(\mu t) + i(c_1 - c_2)\sin(\mu t)\right)} \\ \text{or } &\boxed{y(t) = e^{\lambda t}\left(d_1\cos(\mu t) + d_2 i\sin(\mu t)\right)}. \end{aligned} \tag{3.10}$$

for some constants d_1 and d_2.

Exercise: Check by differentiating to see that (3.10) is a solution to the ODE.

Although we have a solution expressed in (3.10) it is a complex-valued function. Since the coefficients of the ODE are real numbers, it would be better for us to obtain a real-valued function as a solution. It turns out that we can do such a thing with the following observation/theorem.

Theorem 3.7. *Given an ODE*

$$y'' + p(t)y' + q(t)y = 0$$

with p and q are continuous ***real-valued*** *functions. If $y(t) = u(t) + iv(t)$ is a* ***complex-valued*** *solution to the ODE with u and v* ***real-valued*** *functions, then its real part $u(t)$ and its imaginary part $v(t)$ are also solutions to the ODE.*

Proof. Substituting the complex-valued solution into the ODE gives

$$\begin{aligned} 0 &= u''(t) + iv''(t) + p(t)u'(t) + ip(t)v'(t) + q(t)u(t) + iq(t)v(t) \\ &= (u''(t) + p(t)u'(t) + q(t)u(t)) + i(v''(t) + p(t)v'(t) + q(t)v(t)). \end{aligned}$$

A complex number is zero if and only if its real part and imaginary part are both zero. On the LHS we have zero and on the RHS we have a complex number for every $t \in I$. Therefore we must have

$$u'' + p(t)u' + q(t)u = 0, \quad v'' + p(t)v' + q(t)v = 0.$$

□

So, from y_1 and y_2, we get the real-valued functions

$$u(t) = e^{\lambda t}\cos(\mu t), \quad v(t) = e^{\lambda t}\sin(\mu t).$$

It is clear that u and v are linearly independent, and computing the Wronskian

$$\begin{aligned} W(u,v)[t] &= u(t)v'(t) - v(t)u'(t) \\ &= e^{\lambda t}\cos(\mu t)e^{\lambda t}(\lambda\sin(\mu t) + \mu\cos(\mu t)) - e^{\lambda t}\sin(\mu t)e^{\lambda t}(\lambda\cos(\mu t) - \mu\sin(\mu t)) \\ &= e^{2\lambda t}\mu(\cos^2(\mu t) + \sin^2(\mu t)) = \mu e^{2\lambda t} \end{aligned}$$

which is non-zero for all $t \in I$ as $\mu \neq 0$.

Thus by Theorem 3.4 we see that any solution y to the ODE $ay'' + by' + cy = 0$ with $b^2 - 4ac < 0$ can be expressed as

$$\boxed{y(t) = c_1 e^{\lambda t}\cos(\mu t) + c_2 e^{\lambda t}\sin(\mu t)}. \tag{3.11}$$

The advantage of this expression over (3.10) is that y is a real-valued function.

Example 3.11. *Solve the IVP*

$$y'' + y' + 9.25y = 0, \quad y(0) = 2, \quad y'(0) = 8.$$

The characteristic equation is

$$r^2 + r + 9.25 = 0,$$

with roots

$$r_1 = -\frac{1}{2} + 3i, \quad r_2 = -\frac{1}{2} - 3i \quad \Rightarrow \quad \lambda = -\frac{1}{2}, \quad \mu = 3.$$

The general solution is

$$y(t) = e^{-\frac{1}{2}t}(c_1 \cos(3t) + c_2 \sin(3t))$$

for some constants $c_1, c_2 \in \mathbb{R}$*. Using the initial conditions we have*

$$y(0) = c_1 = 2, \quad y'(0) = -\frac{1}{2}c_1 + 3c_2 = 8 \quad \Rightarrow \quad c_1 = 2, \quad c_2 = 3.$$

Hence the particular solution is

$$y(t) = e^{-\frac{1}{2}t}(2\cos(3t) + 3\sin(3t)).$$

Similar to the case of two real roots, we now investigate the possible behaviour of the solution (3.11) as $t \to \infty$.

(1) If $\lambda = 0$, then (3.11) becomes

$$y(t) = c_1 \cos(\mu t) + c_2 \sin(\mu t).$$

In this case, the solution y is an **oscillation** with **constant amplitude**. The amplitude will depend on the values of c_1 and c_2, which is worked out by the initial conditions.

(2) If $\lambda > 0$, then due to the factor $e^{\lambda t}$ the amplitude of the oscillation with **grow** (exponentially) in time.

(3) If $\lambda < 0$, then due to the factor $e^{-\lambda t}$, the amplitude of the oscillation with **decay** (exponentially) in time, in this case we can say that $y(t) \to 0$ as $t \to \infty$.

3.3.3 One repeated real root

The last case is when $b^2 - 4ac = 0$ and we have a **repeated** root to the characteristic equation. The quadratic formula yields

$$r_1 = r_2 = -\frac{b}{2a}$$

as solutions to the characteristic equation $ar^2+br+c = 0$. The problem is immediately apparent: both roots gives the same function

$$y_1(t) = y_2(t) = \exp\left(-\frac{b}{2a}t\right).$$

But for our developed theory we **require** at least two linearly independent solutions to the ODE. It is **not obvious** how to find a solution solution that is linearly independent to $y_1(t) = \exp(-\frac{b}{2a}t)$.

Idea: Use the Wronskian (see Example 3.8). By Abel's theorem, if $y_1 = \exp(-\frac{b}{2a}t)$ and y_2 are two solutions to the ODE $ay'' + by' + cy = 0$ with $b^2 = 4ac$, then we know that the Wronskian is

$$W(y_1, y_2)[t] = d\exp\left(-\int \frac{b}{a}\,dt\right) = d\exp\left(-\frac{b}{a}t\right)$$

for some constant $d \in \mathbb{R}$. On the other hand we have

$$W(y_1, y_2)[t] = y_1(t)y_2'(t) - y_1'(t)y_2(t) = e^{-\frac{b}{2a}t}y_2'(t) + \frac{b}{2a}e^{-\frac{b}{2a}t}y_2(t).$$

Choosing $d = 1$, and putting things together we have

$$e^{-\frac{b}{2a}t}y_2'(t) + \frac{b}{2a}e^{-\frac{b}{2a}t}y_2(t) = e^{-\frac{b}{a}t} \Rightarrow \boxed{y_2'(t) + \frac{b}{2a}y_2(t) = e^{-\frac{b}{2a}t}}.$$

This is a first order linear ODE for y_2, and using the method of integrating factors we have

$$\boxed{y_2(t) = te^{-\frac{b}{2a}t}}$$

where we have neglected any constants of integration.

Let us now check the linear independence for $y_1 = e^{-\frac{b}{2a}t}$ and $y_2 = te^{-\frac{b}{2a}t}$: Suppose α_1 and α_2 are two constants such that

$$\begin{aligned} \alpha_1 y_1(t) + \alpha_2 y_2(t) = 0 \quad \forall t \in I \\ \Rightarrow e^{-\frac{b}{2a}t}(\alpha_1 + t\alpha_2) = 0. \end{aligned}$$

Since the exponential is never zero, for $\alpha_1 + t\alpha_2$ to be zero for all $t \in I$, we must have $\alpha_1 = \alpha_2 = 0$.

For the Wronskian we compute and see that

$$W(y_1, y_2)[t] = e^{-\frac{b}{2a}t}(e^{-\frac{b}{2a}t} - \frac{b}{2a}te^{-\frac{b}{2a}t}) + \frac{b}{2a}te^{-\frac{b}{a}t} = e^{-\frac{b}{a}t} \neq 0.$$

Thus by Theorem 3.4 we see that any solution y to the ODE $ay'' + by' + cy = 0$ with $b^2 - 4ac = 0$ can be expressed as

$$\boxed{y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 te^{-\frac{b}{2a}t}} \tag{3.12}$$

for constants $c_1, c_2 \in \mathbb{R}$.

Example 3.12. *Solve the IVP*

$$y'' + 4y' + 4y = 0, \quad y(0) = 2, \quad y'(0) = 1.$$

The characteristic equation is

$$r^2 + 4r + 4 = (r+2)^2 = 0.$$

This gives us a repeated root $r = 2$. The general solution is

$$y(t) = c_1 e^{-2t} + c_2 t e^{-2t}.$$

Using the initial conditions we find that

$$y(0) = c_1 = 2, \quad y'(0) = -2c_1 + c_2 = 1 \quad \Rightarrow \quad c_1 = 2, \quad c_2 = 5.$$

Hence the particular solution is

$$y(t) = 2e^{-2t} + 5te^{-2t}.$$

We now investigate the behaviour of the solution as $t \to \infty$. Note that if $\frac{b}{2a} > 0$, then

$$e^{-\frac{b}{2a}t}, \quad te^{-\frac{b}{2a}t} \to 0 \text{ as } t \to \infty.$$

Meanwhile, if $\frac{b}{2a} < 0$, then

$$e^{-\frac{b}{2a}t}, \quad te^{-\frac{b}{2a}t} \to \infty \text{ as } t \to \infty.$$

Roughly speaking we can summarise

$$y(t) = (c_1 + c_2 t)e^{-\frac{b}{2a}t} \to \begin{cases} 0 & \text{if } \frac{b}{2a} > 0, \\ \infty & \text{if } \frac{b}{2a} < 0, \quad c_2 > 0, \\ -\infty & \text{if } \frac{b}{2a} < 0, \quad c_2 < 0. \end{cases}$$

Summary. For the second order linear ODE

$$ay'' + by' + cy = 0$$

with constants a, b, c. Let r_1 and r_2 be the roots to the characteristic equation

$$ar^2 + br + c = 0.$$

- If $b^2 > 4ac$, then r_1 and r_2 are real numbers, and the general solution is given as

$$y(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

- If $b^2 < 4ac$, then r_1 and r_2 are complex numbers such that $r_1 = \lambda + i\mu$ and $r_2 = \overline{r_1} = \lambda - i\mu$ for real numbers λ, μ. Then, the general solution is given as

$$y(t) = e^{\lambda t}(c_1 \cos(\mu t) + c_2 \sin(\mu t)).$$

- If $b^2 = 4ac$, then $r_1 = r_2 = r$. Then the general solution is given as

$$y(t) = c_1 e^{-\frac{b}{2a}t} + c_2 t e^{-\frac{b}{2a}t}.$$

3.4 Reduction of order

In the above we used the Wronskian to deduce that $y_2 = te^{-\frac{b}{2a}t}$ is another solution to the ODE $ay'' + by' + cy = 0$ when $b^2 = 4ac$. There is also another method, called **reduction of order**, which actually can be applied to a second order homogeneous ODE with non-constant coefficient.

Consider the ODE

$$y'' + p(t)y' + q(t)y = 0.$$

Suppose we know $y_1(t)$ is a **non-zero** solution to the ODE. To find a second solution, consider the function

$$y(t) = v(t)y_1(t).$$

Then, product rule entails

$$y'(t) = v'(t)y_1(t) + v(t)y_1'(t), \quad y''(t) = v''(t)y_1(t) + 2v'(t)y_1'(t) + v(t)y_1''(t).$$

If y is a solution to the ODE, we find that

$$\begin{aligned} 0 &= y'' + p(t)y' + q(t)y \\ &= v''y_1 + 2v'y_1' + vy_1'' + p(t)(v'y_1 + vy_1') + q(t)vy_1 \\ \Rightarrow & \boxed{0 = y_1v'' + (2y_1' + p(t)y_1)v'}. \end{aligned}$$

This gives us a second order ODE for v that only involves v'' and v'. Recalling Example 3.1, we define a new function $z := v'$ leading to

$$y_1(t)z' + (2y_1'(t) + p(t)y_1(t))z = 0.$$

Here we treat y_1 and y_1' as given functions. Note that this is a first order linear ODE

$$\boxed{\frac{dz}{dt} + \frac{2y_1' + py_1}{y_1}z = 0},$$

since $y_1 \neq 0$. Solving this gives us

$$\begin{aligned} v'(t) = z(t) &= \exp\left(-\int \frac{2y_1' + py_1}{y_1}\,dt\right) \\ &= \exp\left(-\int p(t)\,dt - 2\ln(y_1(t))\right) = \frac{1}{y_1^2(t)}\exp\left(-\int p(t)\,dt\right). \end{aligned}$$

Integrating once more leads to

$$\boxed{v(t) = \int (y_1(t))^{-2}e^{-\int p(t)\,dt}\,dt}$$

and the second solution to the ODE is given as

$$\boxed{y_2(t) = y_1(t)\int (y_1(t))^{-2}e^{-\int p(t)\,dt}\,dt}.$$

Example 3.13. *Consider the ODE*

$$ay'' + by' + cy = 0,$$

with $b^2 = 4ac$. *We know that* $y_1 = e^{-\frac{b}{2a}t}$ *is a solution. Now written in standard form we see that*

$$y'' + p(t)y' + q(t)y = 0 \quad \textit{with} \quad p = \frac{b}{a}, \quad q = \frac{c}{a}.$$

So from the above formula for v, *we have*

$$v(t) = \int e^{-\frac{b}{a}t} e^{\frac{b}{a}t}\, dt = t \quad \Rightarrow \quad y_2(t) = ty_1(t) = te^{-\frac{b}{2a}t}.$$

Remark 3.2. *This method (Reduction of Order) can be used to find a second solution to the ODE* ***if you already have one solution****. The difficulty actually lies in finding a first solution to the ODE.*

3.5 Non-homogeneous equations

3.5.1 Method of undetermined coefficients

We now turn our attention to ODE of the form

$$y'' + p(t)y' + q(t)y = r(t), \tag{3.13}$$

for given functions p, q and r that are continuous in an interval I. The corresponding **homogeneous** equation is

$$y'' + p(t)y' + q(t)y = 0. \tag{3.14}$$

Immediately we have the following observation. Let Z_1 and Z_2 be solutions to the non-homogeneous problem (3.13). Then, the difference $Z := Z_1 - Z_2$ satisfies

$$Z'' + p(t)Z' + q(t)Z = r - r = 0.$$

That is, the **difference** Z satisfies the **homogeneous** equation (3.14). If (y_1, y_2) are a fundamental set of solutions to the homogeneous problem (3.14), then we can write $Z = Z_1 - Z_2$ as

$$\boxed{Z_1(t) - Z_2(t) = c_1 y_1(t) + c_2 y_2(t)}$$

for some constants c_1, c_2.

From the above we actually derive a general expression for the solution to the non-homogeneous equation (3.13). Let $Y(t)$ denote a solution to (3.13), then **any solution** y to (3.13) can be expressed as

$$\boxed{y(t) = Y(t) + c_1 y_1(t) + c_2 y_2(t)},$$

where (y_1, y_2) is a fundamental set of solutions to the homogeneous problem (3.14).

Definition 3.5. *For a solution expression*

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

to the ODE

$$y'' + p(t)y' + q(t)y = r(t),$$

we call the function

$$y_c(t) := c_1 y_1(t) + c_2 y_2(t)$$

the ***complementary solution****, which is a solution to the homogeneous equation, and the function $Y(t)$ the* ***particular solution****, which is a solution to the non-homogeneous equation.*

This gives us a strategy to solving non-homogeneous second order linear ODEs:

(1) Obtain a fundamental set of solutions (y_1, y_2) to the homogeneous problem (3.14).

(2) Find a solution $Y(t)$ to the non-homogeneous problem (3.13).

(3) The general solution to (3.13) is then given as

$$y(t) = Y(t) + c_1 y_1(t) + c_2 y_2(t).$$

However, several difficulties remain:

- How do we find y_1 and y_2?
- How do we find $Y(t)$?

In Section 3.3 we saw how to find y_1 and y_2 for equations with constant coefficients:

$$ay'' + by' + cy = 0.$$

Therefore, in this section we now show how to obtain a solution Y to the ODE

$$ay'' + by' + cy = r(t)$$

for some specific forms of g. The idea is called the **method of undetermined coefficients**.

The idea is to make a **guess** on what the solution $Y(t)$ could look like. There are only certain classes of functions for $r(t)$ which we have an idea of the solution $Y(t)$ could look like. In particular we consider the non-homogeneous term $r(t)$ to be a mixture of **polynomials**, **exponentials**, **sine** and **cosine**. Although this does not solve the general problem, the method of undetermined coefficients is straightforward to use.

Let us begin with an example.

Example 3.14. *Solve*

$$y'' - 3y' - 4y = 3e^{2t}.$$

In the standard form (3.13) *we have*

$$r(t) = 3e^{2t}.$$

Since the exponential function reproduces itself through differentiation. A possible choice for the particular solution Y would involve exponentials. Before that let us solve the homogeneous problem:

$$y'' - 3y' - 4y = 0$$

and determine the complementary solution. The characteristic equation to the homogeneous ODE is

$$r^2 - 3r - 4 = (r-4)(r+1) = 0.$$

The roots are $r_1 = 4$, $r_2 = -1$, and so a general solution to the homogeneous problem is

$$\boxed{y_c(t) = c_1 e^{4t} + c_2 e^{-t}}.$$

Returning to the non-homogeneous problem, assume $Y(t)$ is of the form

$$Y(t) = Ae^{qt}$$

for some coefficients A and q that are ***not determined yet****, (hence the name method of undetermined coefficients). Plugging into the non-homogeneous equations gives*

$$Y'' - 3Y' - 4Y = Aq^2e^{qt} - 3Aqe^{qt} - 4Ae^{qt} = A(q^2 - 3q - 4)e^{qt} = 3e^{2t}.$$

Therefore, it makes sense to choose

$$q = 2, \quad A(q^2 - 3q - 4) = 3 \Rightarrow A = -\frac{1}{2} \Rightarrow Y(t) = -\frac{1}{2}e^{2t}.$$

Hence, the general solution y to the ODE $y'' - 3y' - 4 = 3e^{2t}$ can be expressed as

$$\boxed{y(t) = c_1e^{4t} + c_2e^{-t} - \frac{1}{2}e^{2t}}.$$

Example 3.15. *This time, solve*

$$y'' - 3y' - 4y = 2sin(t).$$

We know from above that the complementary solution is $y_c = c_1e^{4t} + c_2e^{-t}$. Since the non-homogeneous term $r(t) = 2\sin(t)$, a possible solution would involve sine and cosine, so consider

$$Y(t) = a\sin(\alpha t) + b\cos(\beta t)$$

for undetermined coefficients a, b, α, β. Then, plugging the formula into the non-homogeneous equations gives

$$\begin{aligned}
&Y'' - 3Y' - 4Y \\
&= -a\alpha^2 \sin(\alpha t) - b\beta^2 \cos(\beta t) - 3(a\alpha \cos(\alpha t) - b\beta \sin(\beta t)) - 4(a \sin(\alpha t) + b \cos(\beta t)) \\
&= \sin(\alpha t)[-a\alpha^2 - 4a] + \cos(\beta t)[-b\beta^2 - 4b] + \cos(\alpha t)[-3a\alpha] + \sin(\beta t)[3b\beta] \\
&= 2\sin(t).
\end{aligned}$$

Since the RHS only involves $\sin(t)$*, we can already set*

$$\alpha = 1, \quad \beta = 1.$$

This simplifies the above calculation to

$$\sin(t)[-5a + 3b] + \cos(t)[-5b - 3a] = 2\sin(t).$$

Since there is no term involving the cosine on the RHS, we must have

$$-5a + 3b = 2, \quad -5b - 3a = 0 \quad \Rightarrow \quad a = -\frac{5}{17}, \quad b = \frac{3}{17}.$$

Therefore, the general solution y to the ODE $y'' - 3y' - 4y = 2\sin(t)$ can be expressed as

$$\boxed{y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{5}{17}\sin(t) + \frac{3}{17}\cos(t)}.$$

Remark 3.3. *What if we only consider Y as a function of sine? Suppose we have $Y(t) = a\sin(\alpha t)$ for undetermined coefficients a and α. Plugging this into the ODE gives*

$$\begin{aligned}
Y'' - 3Y' - 4Y &= -a\alpha^2 \sin(\alpha t) - 3a\alpha \cos(\alpha t) - 4a \sin(\alpha t) \\
&= \sin(\alpha t)[-a\alpha^2 - 4a] + \cos(\alpha t)[-3a\alpha] = 2\sin(t).
\end{aligned}$$

Again we choose $\alpha = 1$, but now we have

$$-5a \sin(t) - 3a \cos(t) = 2\sin(t).$$

Since the RHS does not contain any cosine, we must have $a = 0$, but if $a = 0$, then $Y(t) = a\sin(t) = 0$. This leads to a contradiction, which means that our ***guess*** *$Y(t) = a\sin(\alpha t)$ is not sufficient. Therefore we need to include a cosine into the guess.*

One more example but now $r(t)$ is a polynomial.

Example 3.16. *Solve*

$$y'' - 3y' - 4y = t^2 + t + 1.$$

We know the complementary solution is $y_c = c_1 e^{4t} + c_2 e^{-t}$. Since $r(t)$ is a polynomial of degree 2, a possible guess is that the particular solution Y is also a polynomial *of the same degree, that is* $Y(t) = At^2 + Bt + C$ *for some undetermined coefficients* A, B, C. *Then, plugging into the equation gives*

$$\begin{aligned} Y'' - 3Y' - 4Y &= 2A - 3(2At + B) - 4(At^2 + Bt + C) \\ &= -4At^2 - (4B + 6A)t + (2A - 3B - 4C) = t^2 + t + 1 \end{aligned}$$

Comparing coefficients immediately gives

$$A = \frac{-1}{4}, \quad B = \frac{1}{8}, \quad C = \frac{-7}{32},$$

and so the general solution y *to the ODE* $y'' - 3y' - 4y = t^2 + t + 1$ *can be expressed as*

$$\boxed{y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{4}t^2 + \frac{1}{8}t - \frac{7}{32}}.$$

Example 3.17. *Solve*

$$y'' - 3y' - 4y = e^{-t}.$$

Since $r(t)$ *is an exponential, we try* $Y(t) = Ae^{-t}$ *and determine the value of* A. *However, it turns out that*

$$\boxed{Y'' - 3Y' - 4Y = A(1 + 3 - 4)e^{-t} = 0}.$$

So ***<u>no choice</u>*** *of* A *would satisfy the non-homogeneous ODE. What went wrong here?*

If you recall, a fundamental set of solutions to the homogeneous ODE $y'' - 3y' - 4y = 0$ *is* $y_1 = e^{4t}$ *and* $y_2 = e^{-t}$. *That is, the guess function* $Y(t) = Ae^{-t}$ *actually is a solution to the homogeneous problem, and consequently, it cannot be a solution to the non-homogeneous problem!*

In this case, where the assumed form of the particular solution Y *is a duplicate of one of the solutions to the homogeneous problem, we can consider a new guess for* Y *which looks like*

$$Y(t) = Ate^{-t},$$

for undetermined constant A. *This is similar to the fundamental set of solutions* $(e^{-\frac{b}{2a}t}, te^{-\frac{b}{2a}t})$ *for the ODE* $ay'' + by' + cy = 0$ *when* $b^2 = 4ac$. *Trying this new guess yields*

$$Y'' - 3Y' - 4Y = -5Ae^{-t} = e^{-t}.$$

This means that we should take

$$A = -\frac{1}{5} \quad \Rightarrow \quad Y(t) = -\frac{1}{5}te^{-t}.$$

Thus a general solution y *to the ODE* $y'' - 3y' - 4y = e^{-t}$ *is*

$$\boxed{y(t) = c_1 e^{4t} + c_2 e^{-t} - \frac{1}{5}te^{-t}}.$$

Based on the above examples, we summarise: Given an ODE

$$ay'' + by' + cy = r(t).$$

1. First compute the complementary solution $y_c = c_1 y_1 + c_2 y_2$ to the homogeneous problem $ay'' + by' + cy = 0$.
2. If $r(t)$ is an exponential function $e^{\alpha t}$ and is not a multiple of y_1 or y_2, then try $Y(t) = Ae^{\alpha t}$ for some constant A.
3. If $r(t)$ is a multiple of y_1 (or y_2), then try $Y(t) = (At^2 + Bt)e^{\alpha t}$ for some constants A and B.
4. If $r(t)$ is a polynomial of degree k, then try $Y(t) = \sum_{i=0}^{k} a_i t^i$ for constants $a_0, \dots, q_k$.
5. If $r(t)$ is a linear combination of $\sin(\alpha t)$ and $\cos(\alpha t)$, then try $Y(t) = A\sin(\alpha t) + B\cos(\alpha t)$ for some constants A, B.

It turns out that the same principle extends to the case where $r(t)$ is a product of exponentials, cosine and sines, and polynomials. Let us now outline a general procedure.

Theorem 3.8. *Suppose Y_1 is a solution to*

$$ay'' + by' + cy = k(t),$$

and Y_2 is a solution to

$$ay'' + by' + cy = l(t).$$

Then the sum $Y_1 + Y_2$ is a solution to

$$ay'' + by' + cy = k(t) + l(t).$$

The above theorem shows that even if we have a complicated expression (involving exponentials, cosine, sine and polynomials) for the non-homogeneous term $r(t)$, if $r(t)$ can be written as a sum $r_1 + r_2 + \cdots + r_m$, where each of the r_i are simpler so that the corresponding particular solution Y_i can be found, and it turns out that $Y_1 + \cdots + Y_m$ is a solution for the original problem involving $r(t)$.

In the following, we define

$$P_n(t) = a_0 + a_1 t + a_2 t^2 + \cdots + a_n t^n$$

for given functions $a_0, \dots, a_n$.

Case 1: $r(t) = P_n(t)$**.** In the case the ODE becomes

$$ay'' + by' + cy = a_0 + a_1 t + \cdots + a_n t^n.$$

A possible guess

$$\boxed{Y(t) = \gamma t^s Q_n(t)}, \tag{3.15}$$

where γ is an undetermined constant, $Q_n(t) = A_0 + A_1 t + \cdots + A_n t^n$ is a polynomial with undetermined coefficients $A_0, \ldots, A_n$, and $s \in \{0, 1, 2\}$ is an exponent determined by the following criterion:

$$s = \begin{cases} 0 & \text{if } c \neq 0, \\ 1 & \text{if } c = 0, b \neq 0, \\ 2 & \text{if } b = c = 0. \end{cases}$$

Case 2: $r(t) = P_n(t)e^{\alpha}t$**.** A possible guess is

$$\boxed{Y(t) = \gamma t^s Q_n(t) e^{\alpha t}}, \tag{3.16}$$

where γ is an undetermined constant, $Q_n(t) = A_0 + A_1 t + \cdots + A_n t^n$ is a polynomial with undetermined coefficients $A_0, \ldots, A_n$, and $s \in \{0, 1, 2\}$ is an exponent determined by the following criterion:

$$s = \begin{cases} 0 & \text{if } \alpha \neq r_1,\ \alpha \neq r_2, \\ 1 & \text{if } \alpha = r_1,\ \alpha \neq r_2 \text{ (or vice versa)}, \\ 2 & \text{if } r_1 = r_2 = \alpha, \end{cases}$$

where r_1 and r_2 are the roots to the characteristic equation $ar^2 + br + c = 0$. In fact s is the **multiplicity** of α as a root of the characteristic equation.

Remark 3.4. *Recall Example 3.17, where e^{-t} was a solution to the homogeneous problem, and the non-homogeneous term was $r(t) = 2e^{-t}$. In this case we have $r_2 = \alpha = -1$ and $r_1 = 4$, therefore it is suggested to try a particular solution Y of the form*

$$Y(t) = \gamma t e^{-t}.$$

Case 3: $r(t) = e^{\alpha t} P_n(t) \cos(\beta t)$ **or** $e^{\alpha t} P_n(t) \sin(\beta t)$**.** In this case, using the Euler formula:

$$\boxed{\cos(\beta t) = \frac{1}{2}(e^{\beta i t} + e^{-\beta i t}), \quad \sin(\beta t) = \frac{1}{2}\frac{1}{i}(e^{\beta i t} - e^{-\beta i t})},$$

the ODE becomes (for the case where $r(t) = e^{\alpha t} P_n(t) \sin(\beta t)$)

$$ay'' + by' + cy = \frac{1}{2i} P_n(t) \left(e^{(\alpha + \beta i)t} - e^{(\alpha - \beta i)t} \right).$$

A possible guess is

$$\boxed{Y(t) = \gamma t^s e^{\alpha t}(Q_n(t)\cos(\beta t) + R_n(t)\sin(\beta t))}, \tag{3.17}$$

where γ is an undetermined constant, $Q_n(t) = A_0 + A_1 t + \cdots + A_n t^n$, $R_n(t) = B_0 + B_1 t + \cdots + B_n t^n$ are polynomials with undetermined coefficients $A_0, \ldots, A_n, B_0, \ldots, B_n$, and $s \in \{0, 1, 2\}$ is an exponent determined by the following criterion:

$$s = \begin{cases} 0 & \text{if } r_1 \text{ (or } r_2) \neq \alpha + i\beta, \\ 1 & \text{if } r_1 = \alpha + i\beta \text{ (and thus } r_2 = \alpha - i\beta). \end{cases}$$

In light of Theorem 3.8 if we encounter an ODE of the form

$$ay'' + by' + cy = P_n^{(1)}(t) + P_n^{(2)}(t)e^{\alpha t} + P_n^{(3)}(t)e^{\alpha t}\cos(\beta t),$$

where $P_n^{(1)}$, $P_n^{(2)}$ and $P_n^{(3)}$ are given polynomials of degree n, then we can first compute

$$\begin{aligned}
&Y_1 = \gamma_1 t^{s_1} Q_n^{(1)}(t) \\
&\quad \text{solution to } ay'' + by' + cy = P_n^{(1)}(t), \\
&Y_2 = \gamma_2 t^{s_2} Q_n^{(2)}(t) e^{\alpha t} \\
&\quad \text{solution to } ay'' + by' + cy = P_n^{(2)}(t)e^{\alpha t}, \\
&Y_3 = \gamma_3 t^{s_3} e^{\alpha t}(Q_n^{(3)}(t)\cos(\beta t) + R_n^{(3)}(t)\sin(\beta t)) \\
&\quad \text{solution to } ay'' + by' + cy = P_n^{(3)}(t)e^{\alpha t}\cos(\beta t),
\end{aligned}$$

so that $Y_1 + Y_2 + Y_3$ is a particular solution to the original problem.

3.5.2 Why it works

We briefly outline a proof for the method of undetermined coefficients.

Case 1: $r(t) = P_n(t)$. In this case the ODE is

$$ay'' + by' + cy = a_0 + a_1 t + \cdots + a_n t^n.$$

It is likely that a possible particular solution is also a polynomial, and so we try

$$Y(t) = A_0 + A_1 t + \cdots + A_n t^n$$

for undetermined coefficients $A_0, \ldots A_n$. Substituting this into the ODE yields

$$\begin{aligned} &a[n(n-1)A_n t^{n-2} + \cdots + 2A_2] + b[nA_n t^{n-1} + \cdots + A_1] + c[A_n t^n + \ldots A_1 t + A_0] \\ &\quad = a_0 t^n + \cdots + a_1 t + a_0. \end{aligned} \tag{3.18}$$

Equating coefficients of powers of t leads to the following sequence of equations

$$\begin{aligned}
cA_n &= a_n, \\
cA_{n-1} + bnA_n &= a_{n-1}, \\
cA_{n-2} + b(n-1)A_{n-1} + an(n-1)A_n &= a_{n-2}, \\
&\vdots \\
cA_0 + bA_1 + 2aA_2 &= a_0.
\end{aligned}$$

Note that for each equation, only three terms appear on the LHS. Let's consider the case $c \neq 0$, so that we immediately have

$$A_n = \frac{a_n}{c},$$

and the remaining coefficients $A_{n-1}, \ldots, A_0$ can then be worked out sequentially. This means we can **completely** determine the coefficients $A_n, \ldots, A_0$ for the particular solution.

If $c = 0$, but $b \neq 0$, then the polynomial on the LHS of (3.18) is of degree $n-1$, and since the polynomial on the RHS of (3.18) is of degree n, we cannot satisfy the equation (3.18). Therefore, one way to match the degree is to consider $aY''(t) + bY'(t)$ is a polynomial of degree n, which implies $Y(t)$ must be a polynomial of degree $n+1$. Hence, we assume

$$Y(t) = t(A_n t^n + \cdots + A_1 t + A_0). \tag{3.19}$$

Note that there is no term in the polynomial that is a constant, but recognise that a constant is a solution to the ODE when $c = 0$. By computing similar to (3.18), substituting the new guess (3.19) into the ODE and matching order by order, we arrive at

$$\begin{aligned} b(n+1)A_n &= a_n, \\ a(n+1)nA_n + bnA_{n-1} &= a_{n-1}, \\ &\vdots \\ 2aA_1 + bA_0 &= a_0. \end{aligned}$$

Once again, as $b \neq 0$, we have $A_n = \frac{a_n}{bn}$, and the other coefficients can be determined in a systematic way. If $c = b = 0$ then we consider

$$Y(t) = t^2(A_n t^n + \cdots + A_1 t + A_0).$$

Since we require $aY''(t) = a_n t^n + \cdots + a_0$ and the only way to make $Y''(t)$ to be a polynomial of degree n is to consider $Y(t)$ as a polynomial of degree $n+2$.

Case 2: $r(t) = e^{\alpha t} P_n(t)$. The problem of determining a particular solution to the ODE

$$ay'' + by' + cy = P_n(t)e^{\alpha t}$$

can be reduced to Case 1 by considering a substitution. Let

$$Y(t) = e^{\alpha t} u(t),$$

and by substituting this into the ODE we obtain

$$\begin{aligned} &e^{\alpha t}(a[u'' + 2\alpha u' + \alpha^2 u] + b[u' + \alpha u] + cu) = e^{\alpha t} P_n(t) \\ \Rightarrow\ &au'' + (2a\alpha + b)u' + (a\alpha^2 + b\alpha + c)u = P_n(t). \end{aligned} \tag{3.20}$$

Determining a particular solution u to (3.20) is similar to Case 1, i.e., we should take

$$u(t) = \begin{cases} A_n t^n + \cdots + A_0 & \text{if } a\alpha^2 + b\alpha + c \neq 0, \\ t(A_n t^n + \cdots + A_0) & \text{if } a\alpha^2 + b\alpha + c = 0,\ 2a\alpha + b \neq 0, \\ t^2(A_n t^n + \cdots + A_0) & \text{if } a\alpha^2 + b\alpha + c = 0,\ 2a\alpha + b = 0. \end{cases}$$

In particular, observe that

$$a\alpha^2 + b\alpha + c \neq 0 \Leftrightarrow \alpha \text{ not a root of chara. equ.}$$

and so $\alpha \neq r_1, r_2$. Meanwhile, if $a\alpha^2 + b\alpha + c = 0$ we have that α is one of the roots r_1 (or r_2), but not both, since if $\alpha = r_1 = r_2$, then we know the only possibility is

$$r_1 = r_2 = \alpha = -\frac{b}{2a}.$$

But since $2a\alpha + b \neq 0$, this implies that the above cannot happen. For the last case, the conditions $a\alpha^2 + b\alpha + c = 0$ and $2a\alpha + b = 0$ means that $\alpha = -\frac{b}{2a}$ is a repeated root of the characteristic equation.

Case 3: $r(t) = e^{\alpha t} P_n(t)\cos(\beta t)$ or $e^{\alpha t} P_n(t)\sin(\beta t)$. The two cases are similar, and so let us consider only the case $r(t) = e^{\alpha t} P_n(t)\sin(\beta t)$. We consider

$$Y(t) = e^{\alpha t}(Q(t)\cos(\beta t) + R(t)\sin(\beta t)),$$

for some functions Q and R, and upon differentiating

$$\begin{aligned} Y'(t) &= \alpha e^{\alpha t}(Q(t)\cos(\beta t) + R(t)\sin(\beta t)) + e^{\alpha t}\beta(-Q(t)\sin(\beta t) + R(t)\cos(\beta t)) \\ &\quad + e^{\alpha t}(Q'(t)\cos(\beta t) + R'(t)\sin(\beta t)), \\ Y''(t) &= \alpha^2 e^{\alpha t}(Q(t)\cos(\beta t) + R(t)\sin(\beta t)) + 2e^{\alpha t}\alpha\beta(-Q(t)\sin(\beta t) + R(t)\cos(\beta t)) \\ &\quad + 2\alpha e^{\alpha t}(Q'(t)\cos(\beta t) + R'(t)\sin(\beta t)) + \beta^2 e^{\alpha t}(-Q(t)\cos(\beta t) - R(t)\sin(\beta t)) \\ &\quad + 2\beta e^{\alpha t}(-Q'(t)\sin(\beta t) + R'(t)\cos(\beta t)) + e^{\alpha t}(Q''(t)\cos(\beta t) + R''(t)\sin(\beta t)). \end{aligned}$$

Plugging the above expression into the ODE yields

$$\begin{aligned} &e^{\alpha t} P_n(t)\sin(\beta t) = aY'' + bY' + cY \\ &= e^{\alpha t}\cos(\beta t)\left[(a\alpha^2 - a\beta^2 + b\alpha + c)Q + (2\alpha a + b)(\beta R + Q') + 2a\beta R' + aQ''\right] \\ &+ e^{\alpha t}\sin(\beta t)\left[(a\alpha^2 - a\beta^2 + b\alpha + c)R + (2a\alpha + b)(-\beta Q + R') - 2a\beta Q' + aR''\right]. \end{aligned}$$

Equating coefficients means that

$$\begin{aligned} (a\alpha^2 - \beta^2 + b\alpha + c)Q + (2\alpha a + b)(\beta R + Q') + 2a\beta R' + aQ'' &= 0, \\ (a\alpha^2 - a\beta^2 + b\alpha + c)R + (2a\alpha + b)(-\beta Q + R') - 2a\beta Q' + aR'' &= P_n. \end{aligned} \tag{3.21}$$

Observe that, $\alpha + i\beta$ is a root of the characteristic equation if and only if

$$a(\alpha + i\beta)^2 + b(\alpha + i\beta) + c = \left[a\alpha^2 - a\beta^2 + b\alpha + c\right] + i(2a\alpha + b)\beta = 0.$$

Using the fact that a complex number is zero if and only if the real and imaginary parts are zero, we have

$$\alpha + i\beta \text{ is a root} \quad \Leftrightarrow \quad a(\alpha^2 - \beta^2) + b\alpha + c = 0, \quad (2a\alpha + b)\beta = 0.$$

As the RHS of (3.21) are polynomials, it is likely that taking Q and R to be polynomials would give a particular solution. The question is what is the degree. Consider the case where $\alpha + i\beta$ is not a root of the characteristic equation. Then, $(a\alpha^2 - a\beta^2 + b\alpha + c)$ and $(2a\alpha + b)\beta$ are both not zero, then from the second equation of (3.21) we have that the degree on the LHS would be the degree of R or Q (which ever is higher). This is due to the fact that taking derivatives of a polynomial reduces the degree. Therefore, for convenience, let's take Q and R to have the **same degree** as the polynomial P_n, i.e.,

$$Q(t) = A_n t^n + \cdots + A_0, \quad R(t) = B_n t^n + \cdots + B_0$$

However, if $\alpha + i\beta$ is a root of the characteristic equation, then (3.21) simplifies to

$$\begin{aligned} (2\alpha a + b)Q' + 2a\beta R' + aQ'' &= 0, \\ (2a\alpha + b)R' - 2a\beta Q' + aR'' &= P_n, \end{aligned}$$

and from the second equation, we see that the degree of the LHS would be the degree of R' or Q' (which ever is higher). This motivates us to take

$$Q(t) = t(A_n t^n + \cdots + A_1 t + A_0), \quad R(t) = t(B_n t^n + \cdots + B_1 t + B_0),$$

in order to match the degree with the RHS.

3.6 Variation of parameters

The method of undetermined coefficients is a straightforward method, but requires that the non-homogeneous term $r(t)$ to be in a special form. If we encounter an ODE

$$y'' - 3y' + 2y = \frac{e^{3t}}{e^t + 1}$$

then the method of undetermined coefficients does not apply. Therefore, we need a more general method that in principle can be applied to any equation. One such method is the **variation of parameters**.

We now outline a general theory. Consider a general ODE

$$y'' + p(t)y' + q(t)y = r(t), \tag{3.22}$$

and suppose (y_1, y_2) forms a fundamental set of solutions to the homogeneous equation

$$y'' + p(t)y' + q(t)y = 0.$$

For constant functions $p(t) = p_*$ and $q(t) = q_*$, we know how to derive a fundamental set of solutions. The idea is as follows. Consider for some functions $u_1(t)$, $u_2(t)$ such that the new function

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t) \tag{3.23}$$

solves the non-homogeneous equation. We now determine what equations u_1 and u_2 have to satisfy.

Differentiating (3.23) yields

$$y' = u_1'y_1 + u_1y_1' + u_2'y_2 + u_2y_2'.$$

In order to simplify the computations later, let us impose a condition

$$\boxed{u_1'y_1 + u_2'y_2 = 0}.$$

Then the derivative becomes

$$y' = u_1y_1' + u_2y_2'. \tag{3.24}$$

Differentiating again leads to

$$y'' = u_1'y_1' + u_1y_1'' + u_2'y_2' + u_2y_2''. \tag{3.25}$$

Substituting (3.23)-(3.25) into the non-homogeneous ODE then gives

$$\begin{aligned} y'' + p(y)y' + q(t)y &= u_1(y_1'' + p(t)y_1' + q(t)y_1) + u_2(y_2'' + p(t)y_2' + q(t)y_2) \\ &\quad + u_1'y_1' + u_2'y_2' \\ &= u_1'y_1' + u_2'y_2' = r(t). \end{aligned}$$

Hence, we obtain two conditions for u_1 and u_2:

$$\boxed{u_1'y_1 + u_2'y_2 = 0, \quad u_1'y_1' + u_2'y_2' = r(t)},$$

which can be conveniently summarised in matrix notion

$$\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix} \begin{pmatrix} u_1' \\ u_2' \end{pmatrix} = \begin{pmatrix} 0 \\ r \end{pmatrix}$$

Notice that the matrix is invertible (and hence a solution (u_1', u_2') to the above problem exists) if the determinant is non-zero. But the determinant is the Wronskian $W(y_1, y_2)[t]$ which is non-zero since (y_1, y_2) is a fundamental set of solutions. Therefore, we can compute

$$\boxed{u_1'(t) = -\frac{y_2r}{W(y_1, y_2)}(t), \quad u_2'(t) = -\frac{y_1r}{W(y_1, y_2)}(t)}. \tag{3.26}$$

Integrating gives

$$u_1(t) = -\int \frac{y_2r}{W(y_1, y_2)}(t)\, dt + d_1, \quad u_2(t) = \int \frac{y_1r}{W(y_1, y_2)}(t)\, dt + d_2, \tag{3.27}$$

for constants $d_1, d_2 \in \mathbb{R}$, and the general solution to the non-homogeneous equation is

$$\boxed{y(t) = (c_1 + d_1)y_1 + (c_2 + d_2)y_2 - y_1 \int \frac{y_2 r}{W(y_1, y_2)}(t)\, dt + y_2 \int \frac{y_1 r}{W(y_1, y_2)}(t)\, dt}.$$

In fact, we can always take $d_1 = d_2 = 0$ in (3.27).

Let us summarise with a theorem.

Theorem 3.9. *Let $I \subset \mathbb{R}$ be an open interval, p, q, r continuous on I. If (y_1, y_2) is a fundamental set of solutions to the homogeneous equation $y'' + p(t)y' + q(t)y = 0$, then a particular solution to the non-homogeneous equation $y'' + p(t)y' + q(t)y = r(t)$ is*

$$Y(t) = -y_1 \int \frac{y_2 r}{W(y_1, y_2)}(t)\, dt + y_2 \int \frac{y_1 r}{W(y_1, y_2)}(t)\, dt,$$

and the general solution to the non-homogeneous equation is

$$y(t) = c_1 y_1(t) + c_2 y_2(t) + Y(t)$$

for constants $c_1, c_2 \in \mathbb{R}$.

Remark 3.5. *This method is able to treat rather general second order ODEs (since $p(t)$ and $q(t)$ need not be constants). However, it is not easy to find a fundamental set of solutions (if $p(t)$ and $q(t)$ are not constant functions). Furthermore, another difficulty lies in the evaluation of the integrals:*

$$-\int \frac{y_2 r}{W(y_1, y_2)}(t)\, dt, \quad \int \frac{y_1 r}{W(y_1, y_2)}(t)\, dt$$

which may not be possible if r, y_1, y_2 are complicated functions.

Going back to the example (3.22), let us first look at the homogeneous problem

$$y'' - 3y' + 2y = 0,$$

which we know the general solution (complementary solution) is given as

$$y_c(t) = c_1 e^t + c_2 e^{2t}.$$

We now compute for u_1 and u_2, where we use

$$y_1 = e^t, \quad y_2 = e^{2t}, \quad r = \frac{e^{3t}}{e^t + 1}, \quad W(y_1, y_2)[t] = e^{3t},$$

and from (3.26), we see that

$$u_1'(t) = -\frac{e^{2t}}{e^t + 1}, \quad u_2'(t) = \frac{e^t}{e^t + 1}.$$

Integrating gives

$$u_1(t) = \ln(e^t + 1) - e^t, \quad u_2(t) = \ln(e^t + 1).$$

Hence, a candidate particular solution is

$$\tilde{Y}(t) = u_1 y_1 + u_2 y_2 = e^t \ln(e^t + 1) + e^{2t} \ln(e^t + 1) - e^{2t}.$$

But since e^{2t} satisfies the homogeneous equation, we can forget about the last term in $\tilde{Y}$ and thus set

$$Y(t) = e^t \ln(e^t + 1) + e^{2t} \ln(e^t + 1).$$

The general solution to the ODE (3.22) is

$$y(t) = c_1 e^t + c_2 e^{2t} + e^t \ln(e^t + 1) + e^{2t} \ln(e^t + 1).$$

Remark 3.6. *When computing u_1 and u_2, note that the particular solution $u_1 y_1 + u_2 y_2$ can be simplified.*