MATH3720A - Lecture Notes

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2 First order equations

2.1 Two linear ODE example

We begin our study with two examples of first order linear ODEs.

2.1.1 Example 1

For given real constants a, b, t_0, y_0, solve

$$\begin{cases} \frac{dy}{dt} = ay + b, \\ y(t_0) = y_0. \end{cases}$$

This is a linear and autonomous ODE. Let us consider the case $a = 0$. Then the ODE becomes

$$y' = b, \quad y(t_0) = y_0.$$

Integrating yields the general solution

$$y(t) = bt + c, \quad c \in \mathbb{R},$$

and the initial condition gives the particular solution

$$\boxed{y(t) = y_0 + b(t - t_0)}.$$

For the case $a \neq 0$, we rearrange the ODE into another form:

$$y' = ay + b = a(y + b/a) \Rightarrow \frac{1}{y + \frac{b}{a}} \frac{dy}{dt} = a.$$

If there exists a function $H(y)$ such that $H'(y) = (y + b/a)^{-1}$, then the ODE becomes (via the Chain rule)

$$H'(y) \frac{dy}{dt} = \frac{d}{dt} H(y(t)) = a.$$

It turns out that

$$H(y) = \ln(y + b/a),$$

and so we have

$$\ln(y(t) + b/a) = at + c, \quad c \in \mathbb{R}.$$

Taking exponential then leads to the general solution

$$\boxed{y(t) = \kappa \exp(at) - \frac{b}{a}}, \quad \kappa := \exp(c).$$

Using the initial condition $y(t_0) = y_0$ we obtain the particular solution

$$\boxed{y(t) = (y_0 + b/a)\exp(a(t - t_0)) - \frac{b}{a}}.$$

In summary we find that

$$y(t) = \begin{cases} y_0 + b(t - t_0) & \text{for } a = 0, \\ (y_0 + b/a)\exp(a(t - t_0)) - \frac{b}{a} & \text{for } a \neq 0. \end{cases}$$

This example shows that the explicit formula for the solution can <u>**depend**</u> on the values of the given coefficients. <u>**Always**</u> keep this in mind before starting to solve the ODE.

2.1.2 Example 2

For a given function $p(t)$, find the general solution to

$$\frac{dy}{dt} = p(t)y.$$

Note that $y(t) \equiv 0$ is one solution! Suppose that $y(t_*) \neq 0$ for some $t_* \in I$, then we can rearrange the ODE into the form

$$\frac{1}{y}\frac{dy}{dt} = p(t) \Rightarrow \frac{d}{dt}\ln(y(t)) = p(t).$$

Integrating yields

$$\ln(y(t)) = \int p(t)\, dt + c, \quad c \in \mathbb{R},$$

and taking exponential gives the general solution

$$\boxed{y(t) = \kappa \exp\left(\int p(t)\, dt\right)}, \quad \kappa := \exp(c).$$

Remark 2.1. *In the above examples, it should have been*

$$\ln|y(t)| = \int p(t)\, dt + c.$$

Once we take the exponential we find that

$$|y(t)| = \exp\left(\int p(t)\, dt\right)\exp(c).$$

Setting

$$\kappa = \begin{cases} \exp(c) & \textit{if } y(t) > 0, \\ -\exp(c) & \textit{if } y(t) < 0, \end{cases} \Rightarrow y(t) = \kappa \exp\left(\int p(t)\, dt\right).$$

The sign of the constant κ does not matter once we use the initial condition to determine its value.

2.2 Linear first order equations - method of integrating factors

In the above Example 2, we obtain that the general solution to the ODE

$$\frac{dy}{dt} = p(t)y$$

is

$$y(t) = \kappa \exp\left(\int p(t)\, dt\right),$$

for some arbitrary non-negative constant κ. We now study the general linear first order ODE:

$$\begin{cases} \frac{dy}{dt} = p(t)y + q(t), \\ y(t_0) = y_0 \end{cases} \tag{2.1}$$

for some given functions $p(t)$, $q(t)$ and constants t_0 and y_0. One example is the equation for the motion of the falling object: $mv' = mg - \gamma v$, where we set $y = v$, $p = -\gamma/m$ and $q = g$. The method we use is called the **method of integrating factors**.

Idea: Multiply the ODE (2.1) by a function $\mu(t)$, leading to

$$\mu(t)\frac{dy}{dt} - \mu(t)p(t)y(t) = \mu(t)q(t). \tag{2.2}$$

Suppose

$$\boxed{\mu(t)\frac{dy}{dt} - \mu(t)p(t)y(t) = \frac{d}{dt}\left(\mu(t)y(t)\right)}, \tag{2.3}$$

then, the multiplied ODE (2.2) becomes

$$\frac{d}{dt}(\mu(t)y(t)) = \mu(t)q(t) \Rightarrow \boxed{\mu(t)y(t) = \int \mu(t)q(t)\, dt + c}, \quad c \in \mathbb{R}. \tag{2.4}$$

If in addition, $\mu(t)$ is **non-zero**, we can divide by $\mu(t)$ and end up with the general solution

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)\, dt + c\right]}. \tag{2.5}$$

Definition 2.1. *If such a function $\mu(t)$ exists satisfying (2.3), then we call $\mu(t)$ the* ***integrating factor****.*

But does such a function $\mu(t)$ exists? If it doesn't then this is a useless method. What is the equation satisfied by μ? From (2.3) we see that

$$\begin{aligned}
&\mu(t)y'(t) - \mu(t)p(t)y(t) = \frac{d}{dt}(\mu y) = \mu'(t)y(t) + \mu(t)y'(t) \\
\Rightarrow\ & y(t)\left(\frac{d\mu}{dt} + p(t)\mu(t)\right) = 0.
\end{aligned}$$

The above equation is satisfied if $y(t) = 0$ or $\mu'(t) + p(t)\mu(t) = 0$. The first case $y(t) = 0$ is not desirable, since if the initial condition y_0 is non-zero, we have a contradiction. Therefore, we consider the second case and obtain the equation

$$\boxed{\frac{d\mu}{dt} = -p(t)\mu} \tag{2.6}$$

as the ODE for μ. But this type of equation has been encountered before. From Example 2, we see that the general solution is

$$\mu(t) = \kappa \exp\left(-\int p(t)\,dt\right), \quad \kappa \in \mathbb{R}_{\geq 0}. \tag{2.7}$$

Take note of the minus sign! The question is what should we take the value of κ to be? Let us first substitute the formula (2.7) into the multiplied ODE (2.2):

$$\begin{aligned}
&\kappa \exp\left(-\int p(t)\,dt\right)\frac{dy}{dt} - \kappa \exp\left(-\int p(t)\,dt\right)p(t)y(t) = \kappa \exp\left(-\int p(t)\,dt\right)q(t) \\
\Rightarrow\ & \kappa\frac{d}{dt}\left(e^{-\int p(t)\,dt}y(t)\right) = \kappa e^{-\int p(t)\,dt}q(t).
\end{aligned}$$

It turns out that κ appears on both sides of the equation, and thus we can cancel out κ. In effect, we can choose $\kappa = 1$, which we will do so from now on. This implies that we take the integrating factor $\mu(t)$ to be

$$\boxed{\mu(t) = \exp\left(-\int p(t)\,dt\right)}, \tag{2.8}$$

and the general solution $y(t)$ to the ODE $y' = p(t)y + q(t)$ is given as

$$\boxed{y(t) = e^{\int p(t)\,dt}\left[\int e^{-\int p(t)\,dt}q(t)\,dt + c\right]}. \tag{2.9}$$

The particular solution and the constant c can be computed with the initial condition $y(t_0) = y_0$, which we will not do here.

Example 2.1. *Derive the general solution to the ODE*

$$t\frac{dy}{dt} + 2y = 4t^2.$$

Step 1. *Write the ODE in the form $y' = p(t)y + q(t)$ and identify p and q:*

$$t\frac{dy}{dt} + 2y = 4t^2 \Rightarrow \frac{dy}{dt} = -\frac{2}{t}y + 4t \Rightarrow p(t) = -\frac{2}{t}, \quad q(t) = 4t.$$

Step 2. *Compute the integrating factor $\mu(t)$:*

$$\mu(t) = \exp\left(-\int p(t)\, dt\right) = \exp\left(\int \frac{2}{t}\, dt\right) = t^2.$$

Step 3. *Plug into the formula* (2.9)

$$y(t) = \frac{1}{t^2}\left[\int t^2 \times 4t\, dt + c\right] = t^2 + \frac{c}{t^2}.$$

Remark 2.2. *The general solution $y(t) = t^2 + \frac{c}{t^2}$, for $c \neq 0$, is not defined at the point $t = 0$. So far in the course, we have not really discussed the interval of definition $I \subset \mathbb{R}$. In this case, the general solution is defined only for $t \in (-\infty, 0) \cup (0, \infty) = \mathbb{R} \setminus \{0\}$. If the graph of $y(t)$ is sketched we see that the graph has two parts, one to the left of the y-axis and one to the right of the y-axis. Which part we take depends on the initial condition.*

If we consider an initial condition $y(t_0) = y_0$, where $t_0 > 0$, then we choose the right part - since we can determine the arbitrary constant c in the general solution only in the interval $(0, \infty)$. In this case the interval of definition is $I = (0, \infty)$. Similarly, if $t_0 < 0$, then we choose the left part as the solution, with $I = (-\infty, 0)$. This example serves as a reminder that the solution $y(t)$ to ODEs may not be defined for all values of $t \in \mathbb{R}$, and the initial condition plays a role in determining the interval of definition.

2.3 Separable equations

The theory of first order linear ODEs is complete with the method of integrating factors. We now turn to a subclass of ODEs that can be **non-linear**.

Example 2.2. *Solve the following first order non-linear, non-autonomous ODE*

$$\begin{cases} \frac{dy}{dt} = \frac{\sin(t)}{1-y^2}, \\ y(t_0) = y_0. \end{cases}$$

Idea: *Bring the "y" to the LHS. Rearranging the ODE gives*

$$(1 - y^2)\frac{dy}{dt} = \sin(t).$$

Recognise that the LHS can be expressed as $\frac{d}{dt}H(y(t))$ by the Chain rule. In fact $H(y) = y - \frac{1}{3}y^3$. Hence, the general solution is

$$y(t) - \frac{1}{3}y(t)^3 = -\cos(t) + c, \quad c \in \mathbb{R}.$$

Using the initial condition, the particular solution is

$$y(t) - \frac{1}{3}y(t)^3 = \cos(t_0) - \cos(t) + y_0 - \frac{1}{3}y_0^3.$$

One thing to observe is that ***there is no explicit expression*** *for $y(t)$ (due to the non-linear function $y(t)^3$). We call this an* ***implicit solution*** *to the ODE. This is a typical characteristic of non-linear ODEs.*

Definition 2.2 (Separable equation)**.** *A first order ODE $y' = f(t,y)$ is* ***separable*** *if it can be written in the form*

$$M(t) + N(y)\frac{dy}{dt} = 0 \tag{2.10}$$

for some functions M and N.

The key to solving separable equations is to recognise that $N(y)\frac{dy}{dt}$ can be written as $\frac{d}{dt}(n(y(t))$ by the Chain rule if the anti-derivative n of N exists. Suppose there exist functions m and n such that

$$m' = M, \quad n' = n.$$

Then (2.10) can be written as

$$\frac{d}{dt}m(t) + \frac{d}{dt}n(y(t)) = 0.$$

Integrating yields the general (implicit) solution

$$\boxed{m(t) + n(y(t)) = c}, \quad c \in \mathbb{R}. \tag{2.11}$$

For the initial data $y(t_0) = y_0$ we compute to find the particular (implicit) solution

$$\boxed{n(y(t)) - n(y_0) = m(t_0) - m(t)}. \tag{2.12}$$

Example 2.3. *Let us return to the ODE $y' = p(t)y$ which has been discussed in Example 2. This is a separable equation with*

$$y' = p(t)y \Rightarrow -p(t) + \frac{1}{y}\frac{dy}{dt} = 0 \Rightarrow M(t) = -p(t), \quad N(y) = \frac{1}{y}.$$

Hence, by the formula (2.11) *the general solution is*

$$-\int p(t)\,dt + \ln(y(t)) = c \Rightarrow y(t) = \exp\left(-\int p(t)\,dt\right)\exp(c).$$

2.4 Transformation methods

2.4.1 Nonlinear to Linear

Let n be a real number, $n \neq 0, 1$, and $p(t), q(t)$ be given functions. The **Bernoulli equation** is a first order non-linear ODE of the form

$$\boxed{\frac{dy}{dt} + p(t)y = q(t)y^n}. \tag{2.13}$$

As this is a non-linear ODE we cannot use integrating factors. Moreover, it doesn't seem like the equation is separable. Let us move the nonlinearity y^n to the derivative by multiplying the whole equation with y^{-n}:

$$y^{-n}\frac{dy}{dt} + p(t)y^{1-n} = q(t). \tag{2.14}$$

Now recognise that

$$\frac{d}{dt}\left(y^{1-n}\right) = (1-n)y^{-n}\frac{dy}{dt}.$$

So (2.14) can be simplified to

$$\frac{d}{dt}y^{1-n} + (1-n)p(t)y^{1-n} = (1-n)q(t). \tag{2.15}$$

Then, considering a **new variable** $v(t) = y^{1-n}(t)$, (2.15) becomes

$$\boxed{\frac{dv}{dt} + P(t)v = Q(t)}, \quad P(t) = (1-n)p(t), \quad Q(t) = (1-n)q(t), \tag{2.16}$$

which is a **linear** ODE for the variable v, and we can use integrating factors to solve. Let $\mu(t)$ be the integrating factor for (2.16), then the general solution is

$$v(t) = \frac{1}{\mu(t)}\left[\int Q(t)\mu(t)\,dt + c\right] \Rightarrow \boxed{y(t) = \left(\frac{1}{\mu(t)}\left[\int Q(t)\mu(t)\,dt + c\right]\right)^{\frac{1}{1-n}}}.$$

The take-away message is that sometimes we can **transform** a non-linear ODE to a linear ODE, and using integrating factors to obtain the solution. Always try to look for suitable transformations!

2.4.2 Homogeneous equations

Definition 2.3 (Homogeneous first order equation)**.** *A first order ODE $\frac{dy}{dt} = f(t,y)$ is called **homogeneous** if the function f only depends on the **ratio** $\frac{y}{t}$. That is, we can express*

$$f(t,y) = F(y/t) \text{ for some function } F.$$

Example 2.4. *Consider the ODE*

$$\frac{dy}{dt} = \frac{y - 4t}{t - y} = f(t, y).$$

Dividing numerator and denominator by t leads to

$$f(t, y) = \frac{y - 4t}{t - y} = \frac{y/t - 4}{1 - y/t} = F(y/t), \text{ where } F(s) = \frac{s - 4}{1 - s}.$$

So how do we solve an ODE of the form $\frac{dy}{dt} = F(y/t)$? The answer is to use a transformation. Define a new variable $v = y/t \Leftrightarrow y = vt$. Then, the RHS of the ODE becomes just $F(v)$. For the LHS, by the product rule

$$\begin{aligned} y(t) = tv(t) &\Rightarrow \frac{dy}{dt} = t\frac{dv}{dt} + v \\ &\Rightarrow \boxed{t\frac{dv}{dt} + v(t) = F(v)}. \end{aligned}$$

Note that the initial condition $y(t_0) = y_0$ also transforms:

$$y(t_0) = y_0 \Rightarrow \boxed{t_0 v(t_0) = y_0},$$

and it is important to see that if $y_0 \neq 0$ then we cannot choose $t_0 = 0$, otherwise we get a contradiction.

The transformed ODE in the variable v is now

$$\frac{dv}{dt} = \frac{F(v) - v}{t} \Rightarrow \boxed{\frac{1}{F(v) - v}\frac{dv}{dt} = \frac{1}{t}},$$

which is a **separable equation**!

Example 2.5. *Returning to the example where we solve the ODE $\frac{dy}{dt} = \frac{y-4t}{t-y}$. Using a transformation $y = tv$ we find that v satisfies*

$$\frac{1}{F(v) - v}\frac{dv}{dt} = \frac{1}{t} \Rightarrow \frac{1 - v}{(v - 2)(v + 2)}\frac{dv}{dt} = \frac{1}{t}.$$

Using partial fractions the coefficient can be simplified to

$$\frac{1 - v}{(v - 2)(v + 2)} = -\frac{1}{4}\frac{1}{v - 2} - \frac{3}{4}\frac{1}{v + 2}.$$

Then, integrating gives the general solution

$$\begin{aligned} &-\frac{1}{4}\ln(v - 2) - \frac{3}{4}\ln(v + 2) = \ln t + c \\ \Rightarrow &-\frac{1}{4}\ln(y(t)/t - 2) - \frac{3}{4}\ln(y(t)/t + 2) = \ln t + c. \end{aligned}$$

2.5 Exact equations

We recall that an ODE is **separable** if it can be expressed in the form

$$M(t) + N(y)\frac{dy}{dt} = 0,$$

for some functions M and N such that their anti-derivative exist. What if M and N depend on both t and y? That is, we encounter an ODE of the form

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0. \tag{2.17}$$

Example 2.6. *The ODE*

$$2t + y^2 + 2ty\frac{dy}{dt} = 0$$

is a non-linear, non-autonomous ODE with $M(t,y) = 2t + y^2$ *and* $N(t,y) = 2ty$.

Idea: Suppose there is a function $\Psi(t,y)$ such that

$$\boxed{\frac{\partial \Psi}{\partial y}(t,y) = N(t,y), \quad \frac{\partial \Psi}{\partial t}(t,y) = M(t,y)}, \tag{2.18}$$

where $\frac{\partial \Psi}{\partial t}$ denotes the **partial derivative** of Ψ with respect to t. Then the ODE (2.17) can be expressed as

$$M(t,y) + N(t,y)\frac{dy}{dt} = 0 \Rightarrow \frac{\partial \Psi}{\partial t}(t,y) + \frac{\partial \Psi}{\partial y}(t,y)\frac{dy}{dt} = \frac{d}{dt}\Psi(t,y(t)) = 0.$$

This is similar to the ideas behind separable equations. Then, integrating gives the general (implicit) solution

$$\boxed{\Psi(t,y(t)) = c}, \quad c \in \mathbb{R}, \tag{2.19}$$

and if the initial condition is $y(t_0) = y_0$, then the particular (implicit) solution is

$$\boxed{\Psi(t,y(t)) = \Psi(t_0,y_0)}. \tag{2.20}$$

Example 2.7. *Back to the example* $(2t + y^2) + (2ty)\frac{dy}{dt} = 0$. *If a function* $\Psi(t,y)$ *exists, then*

$$\frac{\partial \Psi}{\partial t} = 2t + y^2, \quad \frac{\partial \Psi}{\partial y} = 2ty.$$

One possible choice is

$$\Psi_a(t,y) = t^2 + ty^2 + a, \quad a \in \mathbb{R}.$$

Then, the general (implicit) solution to the ODE is $\Psi_a(t,y) = c$, $c \in \mathbb{R}$. *In fact we could have chosen* $a = 0$ *and the general solution then becomes*

$$t^2 + ty(t)^2 = c, \quad c \in \mathbb{R}.$$

Definition 2.4 (Exact equation)**.** *A first order ODE $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is an* ***exact equation*** *if there exists a function $\Psi(t,y)$ such that*

$$\boxed{\frac{\partial \Psi}{\partial t}(t,y) = M(t,y), \quad \frac{\partial \Psi}{\partial y}(t,y) = N(t,y)}. \tag{2.21}$$

The general solution $y(t)$ to the ODE is given implicitly as $\Psi(t,y(t)) = c$, $c \in \mathbb{R}$.

Questions: 1) What are the ways to **determine** if an ODE of the form $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is **exact**? 2) How do we find the function $\Psi(t,y)$?

To answer this, let us assume the following: For fixed constants $\alpha, \beta, \gamma, \delta$ with $(\alpha, \beta) \subset I$, suppose $M, N, M_y = \frac{\partial M}{\partial y}$ and $N_t = \frac{\partial N}{\partial t}$ are **continuous** in the rectangle $R := (\alpha,\beta) \times (\gamma, \delta)$, and suppose Ψ is two-times differentiable function in R with continuous derivatives. We state a theorem answering the above questions.

Theorem 2.1. *Under the above assumptions, it holds that*

$$\boxed{M(t,y) + N(t,y)\frac{dy}{dt} = 0 \text{ is exact } \Leftrightarrow M_y(t,y) = N_t(t,y) \text{ for each } (t,y) \in R}. \tag{2.22}$$

As a consequence the function $\Psi(t,y)$ defined as

$$\boxed{\Psi(t,y) = \int_{t_0}^{t} M(s,y)\, ds + \int_{y_0}^{y} N(t,r)\, dr - \int_{y_0}^{y} \frac{\partial}{\partial r} \int_{t_0}^{t} M(s,r)\, ds\, dr}, \tag{2.23}$$

for constants $t_0 \in (\alpha, \beta)$, $y_0 \in (\gamma, \delta)$, satisfies $\Psi_t(t,y) = M(t,y)$, $\Psi_y(t,y) = N(t,y)$ if and only if $M_y(t,y) = N_t(t,y)$ for each $(t,y) \in R$.

The proof of the theorem has two parts. Let us first show ($\Rightarrow$) of (2.22). If $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ is exact, by the symmetry of second order derivatives, necessarily it holds that

$$\frac{\partial}{\partial t} N = \frac{\partial}{\partial t}\frac{\partial \Psi}{\partial y} = \frac{\partial}{\partial y}\frac{\partial \Psi}{\partial t} = \frac{\partial}{\partial y} M \Rightarrow M_y = N_t.$$

For the reverse direction ($\Leftarrow$) of (2.22), suppose $M_y = N_t$ holds and let us construct the function Ψ. Since $\Psi_t = M$, integrating from $t_0 \in (\alpha, \beta)$ to $t > t_0$ gives

$$\frac{\partial \Psi}{\partial t} = M \Rightarrow \boxed{\Psi(t,y) = \int_{t_0}^{t} M(s,y)\, ds + h(y)} \tag{2.24}$$

with some function $h(y)$ acting as the constant of integration. What are the conditions on h so that $\frac{\partial \Psi}{\partial y} = N$?

Lets now differentiate the formula for Ψ with respect to y:

$$N(t,y) = \frac{\partial \Psi}{\partial y}(t,y) = \frac{\partial}{\partial y} Q(t,y) + h'(y), \quad Q(t,y) := \int_{t_0}^{t} M(s,y)\, ds$$

$$\Rightarrow \boxed{h'(y) = N(t,y) - \frac{\partial}{\partial y} Q(t,y)}. \tag{2.25}$$

We now claim that, under the condition $M_y = N_t$, the RHS of (2.25) does not depend on t. Indeed, just differentiating the RHS with respect to t gives

$$\frac{\partial}{\partial t}\left(N(t,y) - \frac{\partial}{\partial y}\int_{t_0}^{t} M(s,y)\,ds\right) = N_t(t,y) - \frac{\partial}{\partial y}\frac{\partial}{\partial t}\int_{t_0}^{t} M(s,y)\,ds$$
$$= N_t(t,y) - \frac{\partial}{\partial y}M(t,y) = (N_t - M_y)(t,y) = 0.$$

In the above, we used the formula

$$\frac{\partial}{\partial t}\int_{t_0}^{t} M(s,y)\,ds = M(t,y).$$

So, under the hypothesis $M_y = N_t$, it turns out that the RHS of (2.25) depends only on y. So we have an equation of the form $h'(y) = f(y)$ for some function f. Integrating from y_0 to y gives

$$h(y) = \int_{y_0}^{y}\left(N(t,r) - \frac{\partial}{\partial r}\int_{t_0}^{t} M(s,r)\,ds\right)dr + b, \quad b \in \mathbb{R}.$$

Plugging this into (2.24) and as discussed before we can choose $b = 0$ then yields the formula (2.23).

Check: Does the function $\Psi(t,y)$ defined in (2.23) satisfies $\Psi_t = M$ and $\Psi_y = N$? - Exercise.

Example 2.8. *Solve the ODE*

$$(y\cos(t) + 2te^y) + (\sin(t) + t^2e^y - 1)\frac{dy}{dt} = 0.$$

Set

$$M(t,y) = y\cos(t) + 2te^y, \quad N(t,y) = \sin(t) + t^2e^y - 1,$$

and computing the partial derivatives gives

$$M_y = \cos(t) + 2te^y, \quad N_t = \cos(t) + 2te^y \Rightarrow \text{ ODE is exact!.}$$

By Theorem 2.1 there exists a function $\Psi(t,y)$ *such that*

$$\Psi_t = M = y\cos(t) + 2te^y, \quad \Psi_y = N = \sin(t) + t^2e^y - 1.$$

Integrating Ψ_t *with respect to* t *gives*

$$\Psi(t,y) = \int M(t,y)\,dt + h(y) = y\sin(t) + t^2e^y + h(y).$$

Differentiating with respect to y *shows that*

$$\frac{\partial\Psi}{\partial y} = \sin(t) + t^2e^y + h'(y) = N(t,y).$$

Comparing gives the relation

$$h'(y) = -1 \Rightarrow h(y) = -y \Rightarrow \Psi(t,y) = y\sin(t) + t^2e^y - y.$$

Therefore, the general (implicit) solution to the ODE is

$$y(t)\sin(t) + t^2e^{y(t)} - y(t) = c, \quad c \in \mathbb{R}.$$

Remark 2.3. *What about separable equations* $M(t) + N(y)\frac{dy}{dt} = 0$*? The condition* $M_y = N_t$ *holds trivially since* $M_y = 0 = N_t$*. Then, from the formula* (2.23)*, the function* $\Psi(t,y)$ *reads as*

$$\begin{aligned}\Psi(t,y) &= \int_{t_0}^{t} M(s)\,ds + \int_{y_0}^{y} N(r)\,dr - \int_{y_0}^{y} \frac{\partial}{\partial r}\int_{t_0}^{t} M(s)\,ds\,dr \\ &= \int_{t_0}^{t} M(s)\,ds + \int_{y_0}^{y} N(r)\,dr \\ &= m(t) + n(y) + \text{ constant },\end{aligned}$$

which agrees with (2.11)*. Note that since* M *depends only on* s*,*

$$\frac{\partial}{\partial r}\int_{t_0}^{t} M(s)\,ds = 0.$$

2.6 Exact equations with integrating factor

We begin with an example

Example 2.9. *The non-linear ODE* $(3ty + y^2) + (t^2 + ty)\frac{dy}{dt} = 0$ *is* ***not exact****! Since for* $M(t,y) = 3ty + y^2$ *and* $N(t,y) = t^2 + ty$*, the partial derivatives are*

$$M_y = 3t + 2y \neq N_t = 2t + y.$$

If there was a function $\Psi(t,y)$ *such that* $\Psi_t = M$ *and* $\Psi_y = N$*, then integrating* $\Psi_t = M$ *with respect to* t *leads to*

$$\Psi(t,y) = \int 3ty + y^2\,dt + h(y) = \frac{3}{2}t^2y + ty^2 + h(y),$$

for some function $h(y)$*. Then, differentiating the above express with respect to* y *leads to*

$$\Psi_y = \frac{3}{2}t^2 + y^2 + h'(y)$$

and compare with $N(t,y) = t^2 + ty$ *there is no possibility to satisfy the relation* $\Psi_y = N$*.*

So how to we solve a non-exact ODE?

Idea: Similar to the way we treated the first order linear ODEs, consider multiplying with a "integrating factor μ" and hope things are better. We obtain after multiplying a new ODE

$$\mu M(t,y) + \mu N(t,y)\frac{dy}{dt} = 0. \tag{2.26}$$

If (2.26) is an exact equation, then by Theorem 2.1 necessarily the following relation must be satisfied:

$$\boxed{\frac{\partial}{\partial t}(\mu N) = \frac{\partial}{\partial y}(\mu M)}. \tag{2.27}$$

Lets investigate two cases.

Case 1. μ is just a function of t, i.e., $\mu = \mu(t)$. Then (2.27) simplifies to

$$N(t,y)\frac{d\mu}{dt} + \mu(t)N_t(t,y) = \mu(t)M_y(t,y). \tag{2.28}$$

If $N(t,y) \neq 0$ for $(t,y) \in (\alpha,\beta) \times (\gamma,\delta) = R$, then we obtain an ODE for μ:

$$\frac{d\mu}{dt} = \mu(t)\left(\frac{M_y - N_t}{N}\right)(t,y) =: \mu(t)K(t,y). \tag{2.29}$$

Further suppose the factor $K(t,y)$ **<u>depends only on</u>** <u>t</u>, then (2.29) is a first order **<u>linear</u>** ODE in $\mu(t)$ which can be solved by the method of integrating factors.

Case 2. μ is just a function of y, i.e., $\mu = \mu(y)$. Then (2.27) simplifies to

$$M(t,y)\frac{d\mu}{dy} + \mu(t)M_y(t,y) = \mu(t)N_t(t,y). \tag{2.30}$$

If $M(t,y) \neq 0$ for $(t,y) \in (\alpha,\beta) \times (\gamma,\delta) = R$, then we obtain an ODE for μ:

$$\frac{d\mu}{dy} = \mu(y)\left(\frac{N_t - M_y}{M}\right)(t,y) =: \mu(t)H(t,y). \tag{2.31}$$

Further suppose the factor $H(t,y)$ **<u>depends only on</u>** <u>y</u>, then (2.29) is a first order **<u>linear</u>** ODE in $\mu(y)$ (where the independent variable is now y), and again can be solved by the method of integrating factors.

Take away message. If we encounter an ODE $M(t,y) + N(t,y)\frac{dy}{dt} = 0$ that is not an exact equation, that is $M_y \neq N_t$, then try computing

(1) $K(t,y) = \frac{M_y - N_t}{N}(t,y)$; or

(2) $H(t,y) = \frac{N_t - M_y}{M}(t,y)$.

If K is only a function of t then solving for the integrating factor $\mu(t)$ that satisfies

$$\frac{d\mu}{dt} = \mu(t)K(t),$$

and multiplying with the non-exact ODE, the new ODE $\mu(t)M(t,y) + \mu(t)N(t,y)\frac{dy}{dt} = 0$ becomes an **<u>exact</u>** equation. Similarly, if H is only a function of y, then solving for the integrating factor $\mu(y)$ that satisfies

$$\frac{d\mu}{dy} = \mu(y)H(y),$$

and multiplying with the non-exact ODE, the new ODE $\mu(y)M(t,y) + \mu(y)N(t,y)\frac{dy}{dt} = 0$ becomes an **<u>exact</u>** equation.

Check. If $K(t,y) = \frac{M_y - N_t}{N}(t,y)$ is only a function of t, then for the new ODE $\mu(t)M(t,y) + \mu(t)N(t,y)\frac{dy}{dt} = 0$ let us compute

$$
\begin{aligned}
&\frac{\partial}{\partial y}(\mu(t)M(t,y)) = \mu(t)M_y(t,y), \\
&\frac{\partial}{\partial t}(\mu(t)N(t,y)) = N(t,y)\frac{d\mu}{dt} + \mu(t)N_t(t,y), \\
\Rightarrow\ &\frac{\partial}{\partial y}(\mu(t)M(t,y)) - \frac{\partial}{\partial t}(\mu(t)N(t,y)) = \mu(t)(M_y - N_t)(t,y) - N(t,y)\frac{d\mu}{dt} = 0.
\end{aligned}
$$

Example 2.10. *Returning to the example ODE $(3ty + y^2) + (t^2 + ty)\frac{dy}{dt} = 0$, which is not an exact equation. Computing*

$$
M_y = 3t + 2y, \quad N_t = 2t + y, \quad K = \frac{M_y - N_t}{N} = \frac{t + y}{t^2 + ty} = \frac{1}{t}, \quad H = \frac{N_t - M_y}{M} = \frac{-t - y}{3ty + y^2}.
$$

We see that K is only a function of t but H is not just a function of y. So we expect the integrating factor μ to be a function of t only, which solves the ODE

$$
\frac{d\mu}{dt} = \frac{\mu(t)}{t} \Rightarrow \mu(t) = t\exp(c), \quad c \in \mathbb{R}.
$$

Multiplying this integrating factor with the ODE yields

$$
t(3ty + y^2) + t(t^2 + ty)\frac{dy}{dt} = 0,
$$

which is now an exact equation with function $\Psi(t,y)$ given as

$$
\Psi(t,y) = t^3 y + \frac{1}{2}t^2 y^2.
$$

So the general (implicit) solution to the ODE is

$$
t^3 y(t) + \frac{1}{2}t^2 y^2(t) = c, \quad c \in \mathbb{R}.
$$

So far for non-exact ODEs of the form $M(t,y) + N(t,y)\frac{dy}{dt} = 0$, the suggestion is to check whether $K(t,y)$ is only a function of t or $H(t,y)$ is only a function of y. If either one is true then we can apply the method of integrating factors to obtain an exact equation. But **<u>what if neither is true</u>**?

The key requirement in the analysis of exact equations is the relation

$$
\frac{\partial}{\partial t}(\mu N) = \frac{\partial}{\partial y}(\mu M).
$$

If $\mu = \mu(t,y)$ is a function of t and y, computing using the product rule and chain rule yields

$$
\boxed{M(t,y)\mu_y - N(t,y)\mu_t = \mu(t,y)\left(N_t - M_y\right)(t,y)}. \tag{2.32}
$$

The above is so-called a **Partial Differential Equation** (PDE) since it involves the partial derivatives of μ with respect to t and y. In general the analysis for PDEs is much more involved than ODEs, in particular a PDE may not have a solution (**non-existence**) and even if a solution exists, there may be many (often infinitely many) of them (**non-uniqueness**). So the general situation seems to be impenetrable, but looking back at transformation methods and how we dealt with homogeneous equations, we can use similar methods to treat the case if μ is a function of $z = ty$. Using the chain rule

$$\frac{\partial}{\partial t}\mu(z) = \mu'(z)\frac{\partial z}{\partial t} = y\mu'(ty),$$
$$\frac{\partial}{\partial y}\mu(z) = \mu'(z)\frac{\partial z}{\partial y} = t\mu'(ty).$$

Then, in (2.32) we now have the relation

$$(tM(t,y) - yN(t,y))\mu'(z) = \mu(z)(N_t - M_y)(t,y)$$
$$\Rightarrow \mu'(z) = \mu(z)\left(\frac{N_t - M_y}{tM - yN}\right)(t,y) =: \mu(z)L(t,y).$$

If the factor L is a function only of $z = ty$, i.e., $L = L(z) = L(ty)$, then we can deduce an integrating factor μ as a function of $z = ty$. Repeating our procedure this would then yield an exact equation.

Exercise: Show that if L is only a function of z, then the new ODE $\mu(ty)M(t,y) + \mu(ty)N(t,y)\frac{dy}{dt} = 0$ is an exact equation.

2.7 Linear vs Nonlinear ODEs - a comparison

So far let us summarise the methods we have learnt:

Type	Method	Explicit/Implicit solution
$y' = p(t)y + q(t)$	Integrating factor	$y(t) = \mu(t)^{-1}(\int \mu(t)q(t)\,dt + c)$
$M(t) + N(y)y' = 0$	Separable equation	$m(t) + n(y(t)) = c$
$y' + p(t)y = q(t)y^n$	$v := y^{1-n}$	$y(t) = (\mu^{-1}(\int Q(t)\mu(t)\,dt + c)^{1/(1-n)}$
$y' = F(y/t)$	$v = y/t$	$1/(F(v) - v)\frac{dv}{dt} = \frac{1}{t}$
$M(t,y) + N(t,y)y' = 0$	Exact equation	$\Psi(t, y(t)) = c$

While ODEs which are first order and linear have been completely solved, for non-linear ODE there are a variety of methods, but still a general theory is missing. The mathematical theory we want to develop consists of the following: What are the **conditions** for a general first order (possibly non-linear) ODE $\frac{dy}{dt} = f(t,y)$ to have a solution. If there is a solution is it the only one? Together these two questions form the issue of **existence and uniqueness** of solutions to first order ODEs. Let us first discuss why these are important properties to study.

Existence of solutions. An ODE is often derived as a **model** of some physical phenomenon. We can observe, independently of how we model, that some quantity changes in time. It is important to stress that models are only an **approximation** of the true phenomenon that is occurring, since in the real-world there are many processes either too complex to model or that are still unknown to modellers. Therefore to obtain a tractable description, certain simplifying assumptions have to be made (recall the ODE for population dynamics). Nevertheless, once a model has been proposed, one should first check if a solution to the model exists. If no solution exists then the model is **not consistent** with reality and **modifications** should be made.

Uniqueness of solutions. If the model (or ODE) has at least one solution, the next question is **is it the only solution**. A related concept is **predictability** of the model. If there is only one possible solution to the model, then you have **completely determined** the behaviour of the solution. If more than one solution to the model exists, then one has to ask if your solution is the one that is observed in reality. In effect the predictive power of the model is decreased, as you cannot be sure if the solution you are using is really the one used by nature.

2.7.1 Existence and Uniqueness of solutions

Let us first state the mathematical result for linear ODEs.

Theorem 2.2 (Existence and Uniqueness for first order linear ODEs)**.** *Let I be an open interval of $\mathbb{R}$ with $(\alpha,\beta) \subset I$. Suppose functions p and q are **continuous** on (α,β). Then, for any $t_0 \in (\alpha,\beta)$, $y_0 \in \mathbb{R}$, there **exists** a **unique** function $y(t)$ that satisfies the linear differential equation*

$$\frac{dy}{dt} = p(t)y + q(t)$$

for each $t \in (\alpha,\beta)$ with $y(t_0) = y_0$.

In particular, the existence and uniqueness of solutions is **guaranteed** by checking that the functions p and q are **continuous** in the interval of definition I. If the interval I contains points t_* where p or q are **discontinuous**, then there may be no solution to the ODE or there may not be a unique solution.

Proof. Repeating the ideas from the method of integrating factors, we first look at the function

$$\mu(t) = \exp\left(-\int p(t)\, dt\right). \tag{2.33}$$

Since p is continuous in (α,β) one can show that $\mu(t)$ is also continuous and non-zero (due to the exponential) for $t \in (\alpha,\beta)$. Therefore the reciprocal $1/\mu(t)$ makes sense and the integral $\int \mu(t)q(t)\, dt$ is well-defined and differentiable. In particular the formula for the general solution

$$y(t) = \frac{1}{\mu(t)}\left[\int \mu(t)q(t)\, dt + c\right] \tag{2.34}$$

is well-defined for $t \in (\alpha, \beta)$. This shows **existence**. For **uniqueness** we have to uniquely determine the constant of integration c. Since the equation (2.33) defines the integrating factor $\mu(t)$ up to a multiplicative factor that depends on the lower limit of the integration, in choosing this lower limit to be t_0, that is

$$\mu(t) := \exp\left(\int_{t_0}^{t} -p(s)\, ds\right) \Rightarrow \mu(t_0) = 1.$$

Then, modifying (2.34) to

$$y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)q(s)\, ds + c\right],$$

to satisfy the initial condition $y(t_0) = y_0$ we must have $c = y_0$. Therefore the unique solution to the IVP is

$$\boxed{y(t) = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)q(s)\, ds + y_0\right]}.$$

□

Interval of definition. Theorem 2.2 asserts that the unique solution to the linear ODE with initial condition $y(t_0) = y_0$ exists throughout any interval about $t = t_0$ in which the functions p and q are **continuous**. At points of discontinuity (for p or q) we may expect the solution to tend to $\pm\infty$ and then ceases to exist.

What about for non-linear ODEs? It turns out that the criterion is rather similar:

Theorem 2.3 (Existence and Uniqueness for first order non-linear ODEs)**.** *Let $\alpha, \beta, \gamma, \delta$ be fixed constants, and define the rectangle $R = (\alpha, \beta) \times (\gamma, \delta)$. Given constants t_0, y_0 such that $t_0 \in (\alpha, \beta)$ and $y_0 \in (\gamma, \delta)$ and suppose the function f and its partial derivative $\frac{\partial f}{\partial y}$ are **continuous** in R. Then, there exists a constant $h > 0$ such that for any $t \in (t_0 - h, t_0 + h)$ and $t \in (\alpha, \beta)$, there **exists** a **unique** solution $y(t)$ to the differential equation*

$$\frac{dy}{dt} = f(t, y)$$

for each $t \in (t_0 - h, t_0 + h) \cap (\alpha, \beta)$ with $y(t_0) = y_0$.

Observations:

- If $f(t, y) = p(t)y + q(t)$, then $\frac{\partial f}{\partial y} = p(t)$ and the assumptions are the same as those in Theorem 2.2, namely p and q are continuous.
- We have existence and uniqueness for a possibly smaller interval $(t_0 - h, t_0 + h)$ than the linear case.
- It turns out that if we only assume f is continuous, then we still have **existence** of solutions, but in general **uniqueness** is not guaranteed.

We defer the proof to later sections, and look at some examples first.

2.7.2 Examples

Example 2.11. *For the IVP*

$$ty' + 2y = 4t^2, \quad y(1) = 2,$$

find an interval for which a unique solution exists.

In the standard form for linear ODEs, we find that

$$\frac{dy}{dt} = -\frac{2}{t}y + 4t^2 \Rightarrow p(t) = -\frac{2}{t}, \quad q(t) = 4t^2.$$

From this, $q(t)$ is continuous for all $t \in \mathbb{R}$, but $p(t)$ is continuous only in $\mathbb{R} \setminus \{0\}$. Since the interval $(0, \infty)$ contains $t_0 = 1$, a unique solution to the IVP exists only for $t \in (0, \infty)$.

Consequently, if we change the initial condition to $y(-1) = 2$, then Theorem 2.2 asserts the existence of a unique solution to the IVP for $t \in (-\infty, 0)$.

Example 2.12 (Size of the rectangle R)**.** *For non-linear ODEs, Theorem 2.3 requires that f and $\frac{\partial f}{\partial y}$ to be continuous in a rectangle R which contains the point (t_0, y_0). Consider the ODE*

$$\frac{dy}{dt} = \frac{3t^2 + 4t + 2}{2(y-1)}.$$

Then observe that

$$f(t,y) = \frac{3t^2 + 4t + 2}{2(y-1)}, \quad \frac{\partial f}{\partial y}(t,y) = -\frac{3t^2 + 4t + 2}{(y-1)^2}$$

are continuous everywhere expect on the line $y = 1$. If our initial poinit (t_0, y_0) does not intersect the line $y = 1$, then we can always draw a rectangle R around the point (t_0, y_0) for which f and $\frac{\partial f}{\partial y}$ are continuous in R. Then Theorem 2.3 says that there is a unique solution to the ODE in some interval about $t = t_0$ with $y(t_0) = y_0$.

One may now be tempted to think that the rectangle R can be extended infinitely in both the positive and negative t directions, which means the solution $y(t)$ may be defined for all $t \in \mathbb{R}$. It turns out that by solving the ODE (it is a separable equation) we have the general (implicit) solution

$$y^2(t) - 2y(t) = t^3 + 2t^2 + 2t + c.$$

As this is a quadratic in y, a simple calculation shows that

$$y(t) = 1 \pm \sqrt{t^3 + 2t^2 + 2t + 1 + c},$$

and this means that the solution is valid as long as the function $f(t) = t^3+2t^2+2t+1+c$ is non-negative. For example, if $c = 3$ then $g(t) = t^3 + 2t^2 + 2t + 4 = (t+2)(t^2+2)$ has a zero at $t = -2$. When $t < -2$, $g(t)$ is negative and so the square root $\sqrt{g(t)}$ is not defined. Therefore one has to ***be careful*** *about the interval of definition before claiming that the solution $y(t)$ exists in a much larger interval about $t = t_0$ simply because the rectangle R can be extended in such a way that the functions f and $\frac{\partial f}{\partial y}$ remain continuous.*

Example 2.13 (Non-uniqueness). *We now give an example where the solution to an ODE may be non-unique if the assumptions of Theorem 2.3 are not satisfied. Consider the IVP*

$$\frac{dy}{dt} = y^{1/3}, \quad y(0) = 0$$

for $t \geq 0$. *The function* $f(y) = y^{1/3}$ *is continuous everywhere in* $[0, \infty)$, *but the partial derivative* $\frac{\partial f}{\partial y} = \frac{1}{3} y^{-2/3}$ *does not exist at* $y = 0$, *and so it is not continuous at* $y = 0$. *While Theorem 2.3 does not apply, a similar result can be used to show that there* ***exists*** *at least one solution* $y(t)$ *to the IVP.*

Since the ODE is separable, we obtain as a particular solution

$$y(t) = \left[\frac{2}{3}t\right]^{3/2} \textit{ for } t \geq 0.$$

But note that the function $y_1(t) \equiv 0$ *is also another solution, so is the function* $y_2(t) = -\left[\frac{2}{3}t\right]^{3/2}$, *and for arbitrary positive* t_0 *the family of functions*

$$y_{t_0}(t) = \begin{cases} 0 & \textit{if } 0 \leq t \leq t_0, \\ \pm\left[\frac{2}{3}(t - t_0)\right]^{3/2} & \textit{if } t \geq t_0, \end{cases}$$

also solves the IVP. In particular we have found an ***infinite*** *family of solutions to the IVP.*

It is ***important*** *to note that this* ***does not*** *contradict Theorem 2.3, since the condition "*$\frac{\partial f}{\partial y}$ *is continuous in* R*" is not satisfied and so the theorem is* ***not applicable*** *to this IVP. Nevertheless, if we consider another initial condition* (t_0, y_0) *such that* $y_0 \neq 0$, *then Theorem 2.3 guarantees there is a unique solution to the IVP with* $y(t_0) = y_0$.

Example 2.14 (Application of uniqueness). *Consider the IVP*

$$\frac{dy}{dt} = \sin(\exp(\exp(t))y), \quad y(0) = 0.$$

This is a (highly) non-linear, non-separable equation. But observe that $y(t) \equiv 0$ *is a solution. Can we say this is the* ***only*** *solution? If so, then we have completely solved the IVP. Let us check if the assumptions of Theorem 2.3 are fulfilled. Setting* $f(t, y) = \sin(\exp(\exp(t))y)$ *with* $\frac{\partial f}{\partial y} = \cos(\exp(\exp(t))y)\exp(\exp(t))$, *we see that both are continuous on* $\mathbb{R}^2$. *Therefore Theorem 2.3 says that there* ***exists*** *a* ***unique*** *solution to the IVP for* $t \in (-h, h)$ *for some constant* $h > 0$. *In particular, if* y_1 *is any other solution to the IVP for* $t \in (-h, h)$, *it must be equal to the solution* $y_2 \equiv 0$. *Thus, the only solution to the IVP is* $y(t) \equiv 0$ *for all* $t \in (-h, h)$.

Example 2.15 (Blow up). *Consider the ODE* $\frac{dy}{dt} = y^2$. *One solution is the function* $y(t) \equiv 0$. *For non-zero solutions to the ODE, using the fact that the ODE is a separable equation we obtain*

$$y(t) = -\frac{1}{t + c}, \quad c \in \mathbb{R}.$$

For the initial condition $y(0) = 1$, *we compute* $c = -1$ *and so*

$$y(t) = \frac{1}{1-t}.$$

Note that $y(t) \to \infty$ *as* $t \to 1$ *(a behaviour which we call* ***blow up*** *as the solution becomes unbounded. Thus the interval of definition for the solution is* $I = [0, 1)$.