Solutions guideline to HW2 for MATH3270A

October 20, 2017

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Question 1

(a) If the new ODE

$$\mu(f(t)g(y))M(t,y) + \mu(f(t)g(y))N(t,y)y' = 0$$

is to be an exact equation, then we must have

$$(\mu(f(t)g(y))M(t,y))_y = (\mu(f(t)g(y))N(t,y))_t.$$

Computing this gives

$$\begin{aligned} f(t)g'(y)\mu'(z)M(t,y) + \mu(z)M_y &= g(y)f'(t)\mu'(z)N(t,y) + \mu(z)N_t \\ \Rightarrow \mu'(z)\left(f(t)g'(y)M(t,y) - g(y)f'(t)N(t,y)\right) &= \mu(z)\left(N_t(t,y) - M_y(t,y)\right) \\ \text{also allow } \mu'(z) = \mu(z)\left(\frac{N_t(t,y) - M_y(t,y)}{f(t)g'(y)M(t,y) - g(y)f'(t)N(t,y)}\right) \end{aligned}$$

(b) Define

$$M(t,y) = \frac{\sin y}{y} - 3e^{-t}\sin t, \quad N(t,y) = \frac{\cos y + 3e^{-t}\cos t}{y}.$$

Computing the derivatives gives

$$M_y = \frac{\cos y}{y} - \frac{\sin y}{y^2}, \quad N_t = -\frac{3}{y}e^{-t}\cos t - \frac{3}{y}e^{-t}\sin t.$$

Then, since $f(t) = e^t$, $g(y) = y$, we have

$$\begin{aligned} &f(t)g'(y)M(t,y) - g(y)f'(t)N(t,y) \\ &= e^t(\sin y)/y - 3\sin t - e^t\cos y - 3\cos t. \end{aligned}$$

Hence,

$$\begin{aligned} \frac{N_t - M_y}{f(t)g'(y)M - g(y)f'(t)N} &= -\frac{1}{y}\frac{3e^{-t}(\cos t + \sin t) + \cos y - (\sin y)/y}{e^t((\sin y)/y - \cos y) - 3\sin t - 3\cos t} \\ &= \frac{1}{y}\frac{1}{e^t}\frac{(\sin y)/y - \cos y - 3e^{-t}(\cos t + \sin t)}{((\sin y)/y - \cos y) - 3e^{-t}(\sin t + \cos t)} \\ &= (ye^t)^{-1} = \frac{1}{z}. \end{aligned}$$

Therefore

$$\mu'(z) = \frac{\mu(z)}{z}.$$

Question 2

(a) Multiplying the differential inequality by a non-negative function μ gives

$$\mu(t)z' - p(t)\mu(t)z \leq 0.$$

Following the ideas of integrating factors, suppose μ is such that

$$\mu(t)z' - p(t)\mu(t)z = \frac{d}{dt}(\mu(t)z(t)).$$

This means that $\mu(t) = e^{-\int p(t)\, dt} = e^{-P(t)}$. Therefore, we see that

$$\frac{d}{dt}(e^{-P(t)}z(t)) \leq 0,$$

i.e., $F(t, z(t)) = e^{-P(t)}z(t)$. Upon integrating both sides of the inequality yields

$$e^{-P(s)}z(s) \leq e^{-P(t_0)}z(t_0) \Rightarrow z(s) \leq z(t_0)e^{P(s)-P(t_0)}.$$

(b) The difference $z = y_1 - y_2$ satisfies the IVP

$$z' = f(t, y_1) - f(t, y_2), \quad z(t_0) = y_0 - y_0 = 0.$$

Multiplying the above by z gives

$$\frac{1}{2}\frac{d}{dt}|z|^2 = z(f(ty_1) - f(t, y_2)).$$

Then, using the assumption of f, we see that

$$|z(f(t, y_1) - f(t, y_2))| \leq |z|\, L\, |y_1 - y_2| = L\, |z|^2,$$

and so

$$\frac{d}{dt}|z|^2 \leq 2L\, |z|^2.$$

Setting $p(t) = 2L$ so that $P(t) = 2Lt$, then from part (a) we obtain

$$|z(s)|^2 \leq |z(t_0)|^2\, e^{2L(s-t_0)}. \tag{1}$$

Since $z(t_0) = y_1(t_0) - y_2(t_0) = y_0 - y_0 = 0$, we have

$$|z(s)|^2 \leq 0 \Rightarrow z(s) = 0 \quad \forall s \geq t_0.$$

The implication is that $y_1(t) = y_2(t)$ for all $t \geq t_0$, and we have uniqueness of solutions to the ODE.

Question 3

(a) Let α_1 and α_2 be constants such that

$$\alpha_1 f(t) + \alpha_2 g(t) = 0 \quad \forall t \in \mathbb{R}.$$

Then differentiating the above with respect to t gives

$$\alpha_1 f'(t) + \alpha_2 g'(t) = 0 \quad \forall t \in \mathbb{R}.$$

The two equations can be expressed together as

$$\begin{pmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Since the Wronskian is non-zero, the matrix is invertible, and thus $\alpha_1 = \alpha_2 = 0$. Hence, f and g are linearly independent.

Also allow the following argument: If f and g are linearly dependent, then there are constants $a, b \neq 0$ such that

$$af(t) + bg(t) = 0 \quad \forall t \in \mathbb{R}.$$

Hence $f(t) = -\frac{b}{a} g(t)$ with $f'(t) = -\frac{b}{a} g'(t)$ and so

$$g'(t)f(t) - f'(t)g(t) = -\frac{b}{a} g(t)g'(t) + \frac{b}{a} g'(t)g(t) = 0 \quad \forall t \in \mathbb{R}.$$

On the other hand the Wronskian $W(f,g)[t]$ is non-zero for some $t \in I$, but

$$W(f,g)[t] = g'(t)f(t) - g(t)f'(t) = 0$$

which yields a contradiction.

(b) (i) For $t > 0$, $f(t) = t^3$ and so $f'(t) = 3t^2$. For $t < 0$, $f(t) = -t^3$ and so $f'(t) = -3t^2$. For $t = 0$, consider the definition of the derivative and take

$$\lim_{h \to 0^+} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^+} h^2 = 0, \quad \lim_{h \to 0^-} \frac{f(h) - f(0)}{h} = \lim_{h \to 0^-} -h^2 = 0.$$

Hence $f'(0) = 0$ and the derivative of f is

$$f'(t) = \begin{cases} 3t^2 & \text{if } t > 0, \\ 0 & \text{if } t = 0, \\ -3t^2 & \text{if } t < 0. \end{cases}$$

(ii) The Wronskian of $f(t) = t^2 |t|$ and $g(t) = t^3$ for $t > 0$ is

$$W(f,g)[t] = g'(t)f(t) - f'(t)g(t) = 3t^2 \times t^3 - 3t^2 \times t^3 = 0.$$

For $t < 0$ the Wronskian is

$$W(f,g)[t] = -3t^2 \times t^3 - (-3t^2) \times t^3 = 0.$$

For $t = 0$ the Wronskian is

$$W(f,g)[0] = 0.$$

(iii) Let α_1 and α_2 be such that

$$\alpha_1 t^2 |t| + \alpha_2 t^3 = 0 \quad \forall t \in \mathbb{R}.$$

For $t > 0$ the above becoms

$$(\alpha_1 + \alpha_2)t^3 = 0,$$

but since $t > 0$ we must have $\alpha_1 + \alpha_2 = 0$. Similarly for $t < 0$ we deduce

$$(-\alpha_1 + \alpha_2)t^3 = 0,$$

and so $\alpha_2 - \alpha_1 = 0$. This implies that $\alpha_1 = \alpha_2 = 0$.
Also allow for example:

$$t = 1 \Rightarrow \alpha_1 t^2 |t| + \alpha_2 t^3 = \alpha_1 + \alpha_2 = 0,$$
$$t = -1 \Rightarrow \alpha_1 t^2 |t| + \alpha_2 t^3 = \alpha_1 - \alpha_2 = 0.$$

Hence, it must hold that $\alpha_1 = \alpha_2 = 0$.

(c) If y_2 is a function satisfying $W(y_1, y_2)[t] \neq 0$, and y_1 is a solution. Then, differentiating the ODE $y_1 y_2' - y_1' y_2 = ce^{-\int p\, dt}$ gives

$$y_1' y_2' + y_1 y_2'' - y_1'' y_2 - y_1' y_2' = y_1 y_2'' - y_1'' y_2 = -p(t)ce^{-\int p(t)\, dt}.$$

Adding and subtracting zero in a clever way and using that y_1 is a solution to the ODE:

$$y_1(y_2'' + p(t)y_2' + q(t)y_2) - p(t)y_2' y_1 - q(t)y_2 y_1$$
$$+ p(t)y_1' y_2 + q(t)y_1 y_2 = -p(t)ce^{-\int p(t)\, dt}$$

Simplifying gives

$$y_1(y_2'' + p(t)y_2' + q(t)y_2) = p(t)\left(-ce^{-\int p(t)\, dt} + y_2' y_1 - y_1' y_2\right) = 0$$

due to Abel's theorem. Hence

$$y_1(y_2'' + p(t)y_2' + q(t)y_2) = 0, \tag{1}$$

and since y_1 is a non-zero function, this means y_2 must satisfy the ODE $y'' + p(t)y' + q(t)y = 0$.

Also allow for obtaining the answer by differentiating the Wronskian and using Abel's theorem:

$$\frac{d}{dt}W(y_1, y_2)[t] = y_2'' y_1 - y_1'' y_2 = -cp(t)e^{-\int p(t)\, dt} = -p(t)W(y_1, y_2)[t].$$

Then, using that y_1 is a solution to the ODE

$$y_2'' y_1 - y_1'' y_2 = y_2'' y_1 + y_2(p(t)y_1' + q(t)y_1) = -p(t)W(y_1, y_2)[t] = -p(t)(y_2' y_1 - y_1' y_2),$$

and so

$$y_1(y_2'' + p(t)y_2' + q(t)y_2) = 0.$$

Since y_1 is non-zero, y_2 must satisfy the ODE.

Question 4

(a) Multiplying the ODE for u with u and the ODE for v with v gives

$$u'u = \frac{d}{dt}\frac{1}{2}u^2 = vu, \quad v'v = \frac{d}{dt}\frac{1}{2}v^2 = -p(t)v^2 - q(t)uv.$$

Upon adding gives

$$\frac{d}{dt}(u^2 + v^2) = -2p(t)v^2 + 2(1 - q(t))uv.$$

Also allow students to compute

$$\frac{d}{dt}(u^2 + v^2) = 2uu' + 2vv' = 2uv - 2p(t)v^2 - 2q(t)uv.$$

(b) By Young's inequality

$$2\left|(1 - q(t))uv\right| \le 2\left|uv\right|(1 + |q(t)|) \le (1 + Q)(u^2 + v^2),$$
$$\left|-2p(t)v^2\right| \le 2\left|p(t)\right| v^2 \le 2Pv^2.$$

(c) One obtains the differential inequality

$$\frac{d}{dt}z(t) \le (1 + Q + 2P)z, \quad z(t) = u^2 + v^2.$$

From answers to Q2, one finds

$$z(t) \le e^{(1+Q+2P)t}z(0) \Rightarrow (u^2(t) + v^2(t)) \le (u_0^2 + v_0^2)e^{(1+Q+2P)t}.$$

If there is a t_* such that $|u(t)| \to \infty$ as $t \to t_*$, then from the above inequality this implies that

$$(u_0^2 + v_0^2)e^{(1+Q+2P)t} \ge u^2(t) \to \infty \text{ as } t \to t_*.$$

However,

$$\lim_{t \to t_*}(u_0^2 + v_0^2)e^{(1+Q+2P)t} = (u_0^2 + v_0^2)e^{(1+Q+2P)t_*} < \infty$$

as $t_* < \infty$, therefore we have a contradiction. Hence, no such t_* can exist.

If y is a solution to the second order linear ODE $y'' + p(t)y' + q(t)y = 0$, set $u(t) = y(t)$ and $v(t) = y'(t)$, then

$$u' = v, \quad v' = y'' = -p(t)y' - q(t)y = -p(t)v - q(t)u.$$

Hence we deduce from (b) and (c) that

$$y^2(t) + (y')^2(t) \le (y_0^2 + y_1^2)e^{(1+Q+2P)t}.$$

Therefore the conclusion is that any solution y to IVP with bounded coefficients cannot blow up in finite time.

Question 5

(a) The characteristic equation for the ODE $\theta'' + w^2\theta = 0$ is

$$r^2 + w^2 = 0 \Rightarrow r = \pm iw.$$

Hence, the general solution is given as

$$\theta(t) = A\cos(wt) + B\sin(wt)$$

for constants A, B. If we write

$$\theta(t) = \sqrt{A^2+B^2}\left(\frac{A}{\sqrt{A^2+B^2}}\cos(wt) + \frac{B}{\sqrt{A^2+B^2}}\sin(wt)\right)$$

and consider ϕ such that

$$\cos(\phi) = \frac{A}{\sqrt{A^2+B^2}}, \quad \sin(\phi) = \frac{B}{\sqrt{A^2+B^2}} \Rightarrow \tan(\phi) = \frac{B}{A},$$

we see that

$$\theta(t) = \sqrt{A^2+B^2}(\cos(wt)\cos(\phi) + \sin(wt)\sin(\phi)) = \sqrt{A^2+B^2}(\cos(wt-\phi)).$$

(b) The characteristic equation is

$$r^2 + \lambda r + w^2 = 0.$$

(i) In this case $\lambda^2 > 4w^2$, we have two real roots and so

$$\theta(t) = Ae^{r_1 t} + Be^{r_2 t}, \quad r_1 = -\frac{\lambda}{2} + \frac{\sqrt{\lambda^2-4w^2}}{2}, \quad r_2 = -\frac{\lambda}{2} - \frac{\sqrt{\lambda^2-4w^2}}{2}.$$

Since $\lambda > 0$ and $\lambda^2 > 4w^2$, we have $r_1, r_2 < 0$ and so the solution $\theta(t)$ decays to zero as $t \to \infty$.

(ii) In this case we have a repeated root and so

$$\theta(t) = (A + Bt)e^{\frac{-\lambda}{2}t}.$$

Since $\lambda > 0$, the solution $\theta(t)$ decays to zero as $t \to \infty$.

(iii) In this case we have a complex conjugate pair of roots and so

$$\theta(t) = e^{-\frac{\lambda}{2}t}(A\cos(\mu t) + B\sin(\mu t)), \quad \mu = \frac{\sqrt{4w^2-\lambda^2}}{2}$$

Similarly, as $\lambda > 0$, we will see oscillations but with smaller and smaller amplitude as $t \to \infty$ and so $\theta(t)$ decays to zero as $t \to \infty$.

(c) (i) Writing the ODE into standard form we have

$$x'' + \frac{k}{M}x' + \frac{h}{M}x = 0, \quad \lambda := \frac{k}{M}, \quad w := \sqrt{\frac{h}{M}}.$$

Computing for the critical value $\lambda_c = 2w = 10$. Hence

$$\frac{\lambda}{\lambda_c} = 0.0125 \approx 1\%.$$

(ii) The new ODE reads as

$$x'' + \frac{k - 300N}{M} x' + \frac{h}{M} x = 0.$$

The critical value for damping is $\lambda_c = 2\sqrt{h/M}$. If the bridge is no longer damped for N_0 pedestrians, then

$$\lambda = \frac{k - 300N_0}{M} < \lambda_c = 2\sqrt{\frac{h}{M}}.$$

Rearranging gives

$$N_0 > \frac{k}{300} - \frac{2\sqrt{hM}}{300}.$$

[2 marks for correct answer and working]

(iii) With the values of M, N, h and k, we see that

$$x'' - 0.025x' + 25x = 0.$$

The roots of the characteristic equation is

$$r = 0.0125 \pm i\frac{\sqrt{100 - (0.025)^2}}{2} \approx 0.0125 \pm (4.9999)i.$$

Then, the solution to the ODE is

$$x(t) = Me^{0.0125t} \cos((4.9999)t - \phi),$$

for some constants M and ϕ. Since $4.9999/2\pi \approx 0.79577 \approx 0.8$, it holds that the solution has oscillations with amplitude growing like $e^{t/80}$ and frequency approximately 0.8 hertz.

(d) (i) Multiple the ODE by u' leads to

$$\frac{m}{2}\frac{d}{dt}(u')^2 + \frac{k}{2}\frac{d}{dt}(u)^2 = \frac{d}{dt}(K(t) + P(t)) = 0 \Rightarrow K(t) + P(t) = K(0) + P(0) \quad \forall t \geq 0.$$

Also allow if students differentiated

$$\frac{d}{dt}(K(t) + P(t)) = mu''u' + kuu' = (mu'' + ku)u' = 0,$$

and then integrating gives the statement.

(ii) Multiplying the ODE with u' gives

$$\frac{d}{dt}(K(t) + P(t)) + \lambda(u')^2 = 0 \Rightarrow \frac{d}{dt}(K(t) + P(t)) = -\lambda(u')^2 \leq 0.$$

Consequently, if u is not constant in time, the total energy is not conserved and decreases to zero as time goes to infinity.