Solutions guideline to HW1 for MATH3270A

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Any questions about the solutions, please email Ms. Rong ZHANG at
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Question 1

1 mark each if the order, linearity and autonomy are all correct. Otherwise zero marks.

Solution:

(a) 2nd order; linear; non-automonous;

(b) 4th order; nonlinear; non-automonous;

(b) 2nd order; linear; non-automonous; (In fact, $y'' + \ln t = 0$)

(d) 2nd order; nonlinear; non-automonous;

(e) 1st order; linear; automonous; (In fact, $y' = 0$)

Question 2

Solution:

(a) As $m > 0$, the IVP can be written as

$$v' + \frac{\gamma}{m}v = g, \quad v(0) = 0$$

which is separable ODE or using the method of integrating factors. The solution formula can be given directly. More precisely, multiply the equation by $e^{\frac{\gamma}{m}t}$, and we have

$$(e^{\frac{\gamma}{m}t}v)' = e^{\frac{\gamma}{m}t}g.$$

Taking the integral w.r.t t yields

$$e^{\frac{\gamma}{m}t}v = \frac{mg}{\gamma}e^{\frac{\gamma}{m}t} + C$$

with arbitrary constant C. Combining with the initial condition gives

$$0 = v(0) = \frac{mg}{\gamma} + C.$$

Thus, the solution is

$$v(t) = \frac{mg}{\gamma} - \frac{mg}{\gamma}e^{-\frac{\gamma}{m}t}.$$

Then, from the formula, as $t \to \infty$,

$$v(t) \to \frac{mg}{\gamma}.$$

(b) If the object reaches 90% of its limiting velocity, then

$$v = \frac{mg}{\gamma} - \frac{mg}{\gamma}e^{-\frac{\gamma}{m}t} = 0.9\frac{mg}{\gamma}$$

Here $m = 10, \gamma = 2, g = 9.8$ and so

$$t = 5\ln 10.$$

The distance the object has fallen during $(0, 5\ln 10)$ is

$$\int_0^{5\ln 10} v dt = \int_0^{5\ln 10} 49 - 49e^{-\frac{t}{5}} dt = 49(5\ln 10 - \frac{9}{2}).$$

(c) If $v \neq \sqrt{\frac{mg}{\gamma}}$, then

$$\frac{dv}{\frac{mg}{\gamma} - v^2} = \frac{\gamma}{m}dt.$$

Observe that

$$\frac{1}{\frac{mg}{\gamma} - v^2} = \frac{A}{\sqrt{\frac{mg}{\gamma}} - v} + \frac{B}{v + \sqrt{\frac{mg}{\gamma}}}, \quad A = B = \frac{1}{2}\frac{1}{\sqrt{\frac{gm}{\gamma}}},$$

and so after taking integral gives

$$-\frac{1}{2}\sqrt{\frac{\gamma}{mg}}\ln\left|\frac{v - \sqrt{\frac{mg}{\gamma}}}{v + \sqrt{\frac{mg}{\gamma}}}\right| = \frac{\gamma}{m}t + c.$$

Rearranging gives

$$\frac{v - \sqrt{\frac{mg}{\gamma}}}{v + \sqrt{\frac{mg}{\gamma}}} = Ce^{-2\sqrt{\frac{\gamma g}{m}}t}$$

where C is a constant. It follows from the initial condition that

$$\frac{v - \sqrt{\frac{mg}{\gamma}}}{v + \sqrt{\frac{mg}{\gamma}}} = -e^{-2\sqrt{\frac{\gamma g}{m}}t}$$

Rearranging again gives

$$v = \sqrt{\frac{mg}{\gamma}}\frac{1 - e^{-2\sqrt{\frac{\gamma g}{m}}t}}{1 + e^{-2\sqrt{\frac{\gamma g}{m}}t}}.$$

Then, as $t \to \infty$,

$$v \to \sqrt{\frac{mg}{\gamma}}.$$

Question 3

(a) Define $M(t,y) = (\frac{2ty}{t^2+1} - 2t)$ and $N(t,y) = \ln(t^2+1) - 2$. Note that $\partial_t(-(2-\ln(t^2+1))) = \partial_y(\frac{2ty}{t^2+1})$, so that the equation is exact. Then, integrating $N(t,y)$ w.r.t. y gives

$$\Psi(t,y) = (\ln(t^2+1) - 2)y + g(t),$$

for some function g depending only on t. Taking the derivative w.r.t. t gives

$$\partial_t \Psi(t,y) = \frac{2ty}{t^2+1} + g'(t) = M(t,y) \quad \Rightarrow \quad g'(t) = -2t.$$

Hence, $g(t) = -t^2$ and the function Ψ is

$$\Psi(t,y) = (\ln(t^2+1) - 2)y - t^2.$$

The general solution to the ODE is

$$(\ln(t^2+1) - 2)y - t^2 = C \quad \Rightarrow \quad y = \frac{C + t^2}{\ln(t^2+1) - 2},$$

and by the initial condition we have $C = -25$, therefore

$$y = \frac{t^2 - 25}{\ln(t^2+1) - 2}.$$

Rewrite the equation as

$$y' = -\frac{2t}{(t^2+1)(\ln(t^2+1) - 2)}y + \frac{2t}{\ln(t^2+1) - 2} =: p(t)y + q(t)$$

and note that $p(t)$ and $q(t)$ are continuous on $(-\infty, -\sqrt{e^2-1})$ or $(-\sqrt{e^2-1}, +\sqrt{e^2-1})$ or $(\sqrt{e^2-1}, \infty)$. As $5 \in (\sqrt{e^2-1}, \infty)$, so the existence interval of the solution to above IVP is

$$(\sqrt{e^2-1}, \infty).$$

(b) Setting $M = ye^{2ty}, N = bte^{2ty}$, then $M_y = e^{2ty} + 2tye^{2ty}, N_t = be^{2ty} + 2btye^{2ty}$. The equation is exact if and only if $M_y = N_t$, that is,

$$b = 1.$$

In the case $b \neq 1$, we compute that

$$\frac{N_t - M_y}{tM - yN} = \frac{e^{2ty}(b-1)(1+2ty)}{(1-b)yte^{2ty}} = -\frac{1+2ty}{ty} = F(ty).$$

Setting $z = ty$, the integrating factor μ is a function of z that satisfies

$$\frac{d\mu}{dz} = F(z)\mu(z) = -\frac{1+2z}{z}\mu(z)$$
$$\Rightarrow \mu(z) = z^{-1}e^{-2z} = (ty)^{-1}e^{-2ty}.$$

(c) (i) Define $M = y, N = 2ty - ye^{-2y}$, then $M_y = 1, N_t = 2y$. Note that

$$\frac{N_t - M_y}{M} = \frac{2y-1}{y}$$

depends only on y, we may assume that $\mu = \mu(y)$ such that $(\mu M)_y = (\mu N)_t$. Then

$$\frac{d\mu}{dy} = \frac{2y-1}{y}\mu \ \Rightarrow\ \mu = \frac{1}{y}e^{2y}.$$

Multiplying the original equation by $\mu(y)$ gives

$$e^{2y} + (2te^{2y} - 1)y' = 0$$

which is exact. Then there exists a function $\Psi(t, y)$ such that

$$\Psi_t = e^{2y}, \quad \Psi_y = 2te^{2y} - 1.$$

Solving $\Psi_t = e^{2y}$ yields $\Psi = te^{2y} + h(y)$ with arbitrary function $h(y)$. Then it follows from $\Psi_y = 2te^{2y} - 1$ that $2te^{2y} - 1 = 2te^{2y} + h'(y)$, that is, $h'(y) = -1$ and then $h(y) = -y$. Finally, the general solution is given by

$$te^{2y} - y = C$$

with arbitrary constant C.

(ii) Define $M = 1, N = \frac{t}{y} - \cos y$, then $M_y = 0, N_t = \frac{1}{y}$. Note that $\frac{N_t - M_y}{M} = \frac{1}{y}$ depends only on y, then the integrating factor is given by

$$\mu(y) = e^{\int \frac{1}{y} dy} = y.$$

Multiplying the original equation by $\mu(y)$ gives

$$y + (t - y\cos y)y' = 0$$

which is exact. Then there exists a function $\Psi(t, y)$ such that

$$\Psi_t = y, \quad \Psi_y = t - y\cos y.$$

Solving $\Psi_t = y$ yields $\Psi = ty + h(y)$ with arbitrary function $h(y)$. Then it follows from $\Psi_y = t - y\cos y$ that $t - y\cos y = t + h'(y)$, that is, $h'(y) = -y\cos y$ and then $h(y) = -y\sin y - \cos y$. Finally, the general solution is given by

$$ty - y\sin y - \cos y = C$$

with arbitrary constant C.

Question 4

(a) Denote the volume of the container by V (gallons), here $V = 10$. The incoming chemical supply is modeled by $\gamma(t)$ (grams per gallon) with rate r (gallons per minute) flowing in, so the amount of chemical flowing in is $r\gamma(t)$ grams per minute at time t.

Since the amount of chemical is $Q(t)$ grams at time t, the density of chemical at time t is $\frac{Q(t)}{V}$ grams per gallon. And the mixture (fresh+chemical) flows out at a rate r gallons per minute. Thus the amount of chemical flowing out is $r\frac{Q(t)}{V}$ grams per minute.

Then the"velocity" of the chemical at time t is $Q'(t)$, which means the variation of chemical mass at time t. Note that at time t,

$$\text{variation of chemical mass} = \text{flow in} - \text{flow out}.$$

Thus,

$$Q'(t) = r\gamma(t) - r\frac{Q(t)}{V}.$$

(b) The above equation is a linear ODE, the general solution is given by

$$Q(t) = e^{-\frac{r}{V}t}\Big(C + r\int \gamma(t)e^{\frac{r}{V}t}dt\Big)$$

with arbitrary constant C.

(c) Take $V = 10, Q(0) = 0, \gamma(t) = 2 + \sin 2t$, then from the above formula, we have

$$0 = Q(0) = C$$

This leads to the particular solution

$$Q(t) = re^{-\frac{q}{V}t}\int_0^t \gamma(s)e^{\frac{r}{V}s}\, ds.$$

Using integration by parts, we have

$$\begin{aligned}\int e^{\alpha t}\sin(2t)\, dt &= \frac{1}{\alpha}e^{\alpha t}\sin(2t) - \int \frac{2}{\alpha}e^{\alpha t}\cos(2t)\, dt \\ &= \frac{1}{\alpha}e^{\alpha t}\sin(2t) - \frac{2}{\alpha^2}e^{\alpha t}\cos(2t) - \int \frac{4}{\alpha^2}e^{\alpha t}\sin(2t)\, dt \\ \Rightarrow \int e^{\alpha t}\sin(2t)\, dt &= \frac{\alpha^2}{\alpha^2+4}\left(\frac{1}{\alpha}e^{\alpha t}\sin(2t) - \frac{2}{\alpha^2}e^{\alpha t}\cos(2t)\right).\end{aligned}$$

So that

$$\begin{aligned}Q(t) =& re^{-\frac{q}{10}t}\int_0^t (2+\sin 2s)e^{\frac{r}{10}s}ds \\ =& 20(1-e^{-\frac{r}{10}t}) + \frac{10r^2}{r^2+400}(\sin 2t - \frac{20}{r}\cos 2t + \frac{20}{r}e^{-\frac{r}{10}t}) \\ =& \frac{10r^2}{r^2+400}(\sin 2t - \frac{20}{r}\cos 2t) - 20\frac{r^2+400-10r}{r^2+400}e^{-\frac{r}{10}t} + 20\end{aligned}$$

Question 5

(a) The logistic equation is a separable ODE, and can be rewritten as

$$\frac{dp}{pr(1-p/K)} = dt.$$

Using partial fractions, we have

$$\frac{1}{p(1-p/K)} = \frac{A}{p} + \frac{B}{1-p/K}, \quad A = 1, \quad B = \frac{1}{K}.$$

Then, integrating gives

$$\frac{1}{r}\ln\left|\frac{p}{p-K}\right| = t + c$$

that is,

$$\frac{p}{p-K} = Ce^{rt}$$

where C is an arbitrary constant. The initial condition $p(0) = p_0$ implies that

$$\frac{p_0}{p_0 - K} = C.$$

Hence the particular solution is given by

$$p = \frac{Kp_0e^{rt}}{p_0e^{rt} - p_0 + K} = \frac{Kp_0}{p_0 + (K - p_0)e^{-rt}}.$$

(b) If $p_0 = 0$, then $p = 0$. If $p_0 > 0$, then $p \to K$ as $t \to +\infty$.

(c) The equilibrium solutions to the ODE are $p = 0$ and $p = K$. To see that $p = 0$ is a solution, one can use L'Hopital's rule

$$\lim_{p\to 0} p\ln(K/p) = \lim_{p\to 0}\frac{\ln(K/p)}{1/p} = \lim_{p\to 0}\frac{p/K \times -(K/p^2)}{-1/p^2} = \lim_{p\to 0} p = 0.$$

Let $u = \ln\frac{p}{K}$, then $u' = \frac{p'}{p}$. The original ODE becomes

$$u' = -ru,$$

thus

$$u = Ce^{-rt}$$

where C is an arbitrary constant. Hence using that $p = Ke^u$, we have

$$p = K(\exp(C))^{e^{-rt}} =: Kc^{e^{-rt}}$$

and the initial condition implies $p_0 = Kc$. Thus, the particular solution is

$$p = K(\frac{p_0}{K})^{e^{-rt}}.$$

If $p_0 = 0$, then $p = 0$, and if $p_0 > 0$, then $p \to K$ as $t \to +\infty$.